A Thesis Entitled

SEMI-SYMMETRIC CONTACT MANIFOLDS

Submitted to the

Faculty of Science and Technology

KUVEMPU UNIVERSITY

For the Award of the Degree of

Doctor of Philosophy

in

MATHEMATICS

by

Smt. Vidyavathi K. R.

Research Supervisor

Prof.C.S. Bagewadi

Retd. Professor (Emeritus Fellow-UGC)

Department of P.G. Studies and Research in Mathematics,
Jnana Sahyadri, Shankaraghatta - 577 451,
Shivamogga, Karnataka, India.

April - 2017

Dedicated To My Beloved Mother

DECLARATION

I hereby declare that the thesis entitled **Semi-Symmetric Contact Manifolds**, submitted to the Faculty of Science and Technology, Kuvempu University for the award of the degree of Doctor of Philosophy in Mathematics is the result of the research work carried out by me in the Department of Mathematics, Kuvempu University under the guidance of **Prof.C.S. Bagewadi**, Retd. Professor (Emeritus Fellow-UGC), Department of P.G. Studies and Research in Mathematics, Kuvempu University, Jnana Sahyadri, Shankaraghatta.

I further declare that this thesis or part thereof has not been previously formed the basis of the award of any degree, associateship etc., of any other University or Institution.

Place: Jnana Sahyadri

Date: 10-04-2017

Vidyavathi K.R.

Vidyavathi K. R.

CERTIFICATE

This is to certify that the thesis entitled **Semi-Symmetric Contact Manifolds**, submitted to the Faculty of Science and Technology, Kuvempu University for the award of the degree of Doctor of Philosophy in Mathematics by **Vidyavathi K. R.** is the result of bonafide research work carried out by her under my guidance in the Department of P.G. Studies and Research in Mathematics, Kuvempu University, Jnana Sahyadri, Shankaraghatta.

This thesis or part thereof has not been previously formed the basis of the award of any degree, associateship etc., of any other University or Institution.

Place: Jnana Sahyadri

Date: 10-04-2017

Prof.C.S. Bagewadi

Research Supervisor

ACKNOWLEDGEMENT

It is a matter of pleasure and pride that I must gratitude to the various persons who helped me a lot directly and indirectly to bring out my Ph.D research work entitled "**Semi-symmetric Contact Manifolds**". A long list of people to thank for their wholehearted support, blessings and good wishes.

First and foremost, I sincerely express my deep sense of everlasting profound gratitude to my esteemed "Guru" and supervisor Prof.C.S. Bagewadi, Retd. Professor (Emeritus Fellow-UGC), Department of P.G. studies and Research in Mathematics, Kuvempu University, for his invaluable and patient guidance, immense knowledge, sustained interest, moral support, encouragement and useful critiques for my research work. Completion of this thesis could not have been accomplished without his support.

I owe my deepest gratitude to Dr.S.K. Narasimhamurthy, Professor and Chairman, Department of P.G. studies and Research in Mathematics, Kuvempu University, for his kind co-operation, valuable advice and encouragement throughout my research work.

I take this opportunity to express heartfelt gratitude to Dr.B.J. Gireesha and Dr. Venkatesha for their concern, constructive suggestions and valuable advice. Their involvement has triggered and nourished my intellectual maturity.

I also thankful to Dr. S. Nagaraj for giving valuable suggestions quite often during my work.

I would like to offer my thanks to Dr.S.R. Ashoka, R.T. Naveen Kumar, M.M. Praveena, Sushilabai Adigond, M.S. Siddesha, N. Srikantha, M. Archana, K. R. Thippeswamy, M. K. Roopa, R. Dhanalakshmi. My sincere thanks to all colleagues and research friends who directly or indirectly helped me in the completion of this thesis.

My spacial thanks to Dr.R. Archana and family members, Smt. Omkaramma and family members and Ms.S.D. Usha for their love and affection.

I cannot express enough thanks to my caring, loving and supportive father Sri. Rajashekharappa, mother Smt. Savithramma and dear sister Smt. Vijayalakshmi, for their understanding, endless patience, affection and encouragement. I express my deep appreciation to my husband Sri. Harsha, for his emotional support and love. With their support and constant patience, I could reach to this stage of completion. I am thankful to my brother-in-law Sri. Madhu, father-in-law Sri. Thippeswamy, mother-in-law Smt. Vijayalakshmi and my dear brothers Sri. Abhishek, Sri. Darshan, Sri. Harsha and to all my family members for their support and best wishes.

I acknowledge the financial support received from the BCM-cell, Kuvempu University for doing my Ph. D. Lastly I would like to thank Smt. Ambika, Sri. Basavaraj, Smt. Annapurnamma for their help and the library of Kuvempu University for providing useful information for research work.

ABOVE ALL I AM THANKFUL TO GOD.

Place: Jnana Sahyadri

Date: [Vidyavathi K. R.]

CONTENTS

PREFACE

Preface

Differential geometry is a mathematical discipline that uses the techniques of differential calculus, integral calculus, linear algebra and multilinear algebra to study problems in geometry. The theory of curves, planes and space formed the basis for initial development of differential geometry during the 18th and the 19th centuries. Riemann's revolutionary ideas generalized the geometry of surfaces which have been studied earlier by Gauss and Lobachevsky. Later this lead to an exact definition of the modern concept of an abstract Riemannian manifold. The development of the subject during 20th century has turned Riemannian geometry into one of the most important parts of modern Mathematics. Levi-Civitae and Ricci developed the concept of parallel translation in the classical language of tensors. This approach received a tremendous impetus from Einstein's work on relativity. Cartan initiated research and methods that were independent of a particular coordinate system.

In 1930, Schouten and Van Dantzing tried to transfer the results of Differential Geometry of spaces with Riemannian metric and affine connection to the case of spaces with complex structure. These spaces were also found independently by Kaehler in 1933 and are now called as Kaehler spaces which are even dimensional. Also using the complex structure and differential 1-form on a manifold, a great deal of work is carried out on these manifolds from 1960 onwords. These are known as Contact manifolds and are odd dimensional. Contact geometry has been seen to underly many physical phenomena and are related to many other mathematical structures. More recently Contact structures have been seen to have relations with Riemannian geometry, low dimensional topology and provide an interesting class of subelliptic operators.

A differentiable $(2n+1)$-dimensional manifold M^{2n+1} is said to be a contact manifold or to have a contact structure, if it carries a global differential 1-form η such that

$$\eta \wedge (d\eta)^n \neq 0$$

everywhere on M^{2n+1} where the exponent denotes the n^{th} exterior power and η is a contact form.

One can obtain different structures like Sasakian, K-Contact, Kenmotsu, trans-Sasakian, para-Sasakian et al. by providing additional conditions to the contact structure. In 1958, Boothby and Wang initiated the study of odd dimensional manifolds with contact structure and almost contact structure. Sasaki and Hatekeyama reinvestigated them using tensor calculus in 1961. Almost contact metric structures and Sasakian structure, nearly Sasakian et. al., were proposed by Sasaki in 1960. Later Kenmotsu defined a class of almost contact Riemannian manifold called Kenmotsu manifold, in 1985 Oubina introduced the trans-Sasakian manifolds. The geometry of these manifolds was studied extensively by many geometers like Blair, Yano, Kon, Sasaki, Kobayashi, Gray, Harada, Hatakeyama, Okumara, Goldberg, Endo, Chen, Ozgur, Takahashi, Rastogi, Amur, Bagewadi, De, Maralabhavi, Tripati, Shaikh, Venkatesha, Nagaraja, Roy Sengupta, Tanno, Hasan Shahid, Bhattacharya, Prakasha et al.

Our thesis deals with semi-symmetric contact manifolds. A Riemannian manifold $(\mathbb{M}, g)$ is called locally symmetric if its curvature tensor R is parallel i.e., $\nabla R = 0$, where ∇ denotes Levi-Civita connection. As a generalization of symmetry, the notion of semi-symmetry is given by $R \cdot R = 0$. Every symmetric manifold is semi-symmetric but the converse is not true. A detail explanation about contact manifolds is given above. The thesis is partitioned into six chapters.

The first chapter is all about basic concepts, it includes the definitions and preliminaries which are used in following chapters. The first section deals regarding about almost contact metric manifolds, definitions and notions of Sasakian manifolds, generalized Sasakian space forms, Kenmotsu manifolds, para-Kenmotsu manifolds and $(LCS)_n$-manifolds. The second section carries the definition, notions and example of almost $C(\alpha)$ manifolds. The section three devotes to S-manifolds. The Ricci soliton, η-Ricci soliton and conformal Ricci soliton are includes in next section. The last section follows the notions of semi-symmetric and pseudo-symmetric contact manifolds.

Chapter-2 is devoted to the study of almost $C(\alpha)$ manifolds. Introduction is the first section of this chapter. In the section-2 we study flat C-Bochner curvature tensor in almost $C(\alpha)$ manifold and showed that it is an η-Einstein manifold. From section-3 to section-6 we proved that semi-symmetric and pseudo-symmetric almost $C(\alpha)$ manifolds with conditions $B \cdot S = 0, B \cdot R = 0, B \cdot S = L_S Q(g, S), B \cdot R = L_R Q(g, R)$ are Einstein manifolds, where B is the C-Bochner curvature tensor. From section-7 to section-12 we study Ricci soliton in almost $C(\alpha)$ manifolds for some semi-symmetric and pseudo-symmetric conditions such as $R \cdot R = 0$, $\bar{M} \cdot R = 0$, $R \cdot \bar{M} = 0$, $R \cdot R = L_1 Q(g, R)$, $\bar{M} \cdot R = L_2 Q(g, R)$, $R \cdot \bar{M} = L_3 Q(g, \bar{M})$ and it has shown that, Ricci soliton in above cases is shrinking, steady and expanding accordingly as Kenmotsu, co-Kaehler and Sasakian manifold. Here $\bar{M}$ is M-projective curvature tensor. In later sections we discuss conformal Ricci soliton in n-dimensional almost $C(\alpha)$ manifolds for conditions $R\cdot W_2 = 0$, $W_2\cdot R = 0$, $R\cdot W_2 = L_{W_2}Q(g, W_2)$, $W_2\cdot R = L_RQ(g, R)$, $W_2\cdot S = 0$ and $W_2\cdot S = L_SQ(g, S)$, where W_2 is the W_2-curvature tensor. Finally last section concludes the results which we obtained for almost $C(\alpha)$ manifolds.

Chapter-3 deals with the study of Ricci solitons in S-manifolds. The first section of the chapter concerned with introduction of S-manifolds. From section-2 to section-9 we have worked on S-manifolds admitting semi-symmetric and pseudo-symmetric conditions such as $R\cdot R = 0$, $R\cdot C = 0$, $C\cdot R = 0$, $C\cdot C = 0$, $R\cdot R = L_5Q(g, R)$, $R\cdot C = L_6Q(g, C)$, $C\cdot R = L_7Q(g, R)$ and $C\cdot C = L_8Q(g, C)$, where C is the concircular curvature tensor and L_5, L_6, L_7, L_8 are some functions on M and proved that these manifolds are Einstein and Ricci soliton for these manifolds is shrinking. Section-10 devotes to irrotational τ-curvature tensor in S-manifolds and it has been shown that, if the τ-curvature tensor is irrotational then the manifold is η-Einstein. And we discuss about Ricci soliton. Finally the last section is the conclusion of above results.

Chapter-4 is related to Sasakian manifold and generalized Sasakian space forms. First section of this chapter includes introduction to Sasakian manifolds and generalized Sasakian space forms. Second section contains Ricci-generalized pseudo-symmetric Sasakian manifold which is Einstein and Ricci soliton for this manifold is shrinking. In sections third, fourth and fifth we proved that, pseudo-projective Ricci-generalized pseudo-symmetric Sasakian manifold, quasi-conformal Ricci-generalized pseudo-symmetric Sasakian manifold, concircular Ricci-generalized pseudo-symmetric Sasakian manifold respectively are Einstein and also Ricci soliton of such manifolds is shrinking. In sixth section we consider $(0,6)$-tensor $R \cdot R = 0$ in generalized Sasakian space form and show that it is Einstein and considering the Ricci soliton (g, V, λ), where V as conformal killing vector field and λ as a scalar, we have shown that Ricci soliton is shrinking if $f_1 < f_3$, steady if $f_1 = f_3$ and expanding $f_1 > f_3$. In section-7 to section-10, we worked on pseudo-symmetric, quasi-conformal semi-symmetric, quasi-conformal pseudo-symmetric generalized Sasakian space forms by considering $(0,6)$-tensors $R \cdot R = L_R Q(g, R)$, $R \cdot \tilde{C} = 0$, $R \cdot \tilde{C} = L_{\tilde{C}} Q(g, \tilde{C})$ and $\tilde{C} \cdot \tilde{C} = 0$. Last section is conclusion of this chapter.

Chapter-5 is all about Kenmotsu manifolds admitting semi-symmetric metric connection and conformal Ricci soliton in para-Kenmotsu manifolds. First section is introductory about Kenmotsu manifolds, para-Kenmotsu manifolds and semi-symmetric metric connection. Section second is concerned to semi-symmetric metric connection on Kenmotsu manifolds and it is proved that, Kenmotsu manifold admitting semi-symmetric metric connection is an η-Einstein manifold. Sections third, fourth and fifth are devoted to Ricci soliton in semi-symmetric, pseudo-projective semi-symmetric, pseudo-symmetric Kenmotsu manifolds with respect to semi-symmetric metric connection and proved that Ricci solitons in these manifolds is expanding with respect to Levi-Civita connection. Later in sections sixth, seventh, eighth and ninth we worked on Ricci soliton para-Kenmotsu manifold satisfying the conditions $R \cdot C = L_C Q(g, C)$, $C \cdot R = L_R Q(g, R)$, $R \cdot \bar{P} = L_{\bar{P}} Q(g, \bar{P})$ and $\bar{P} \cdot R = L_R Q(g, R)$ admitting conformal Ricci soliton.

Chapter-6 deals with $(LCS)_n$-manifolds. Introduction is the first section of this chapter. In second section we proved that, an $(LCS)_n \eta$-Ricci soliton $(M, (g, \xi, \lambda, \mu))$ is an

η-Einstein manifold. Section third deals with η-Ricci soliton on pseudo-projective pseudo-symmetric $(LCS)_n$-manifolds. Fourth section devotes to η-Ricci soliton on $(LCS)_n$-manifold admitting the pseudo-symmetric condition $\bar{P} \cdot R = L_R Q(g, R)$. In sections fifth, sixth, seventh we worked on Ricci soliton in $(LCS)_n$-manifolds for irrotational conditions using pseudo-projective curvature tensor, quasi-conformal curvature tensor and conformal curvature tensor respectively. Last section concludes the chapter.

Finally, the thesis ends with a list of bibliography and publications.

CHAPTER 1

Chapter 1

Preliminaries

This chapter is introductory and consists of basic concepts and definitions of almost contact metric manifolds, Ricci solitons, semi-symmetric and pseudo-symmetric manifolds, which are used in the future chapters.

1.1 Almost contact metric manifolds

Definition 1.1.1. A differentiable manifold $(\mathbb{M}, g)$ is said to be an almost contact metric manifold, if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η on $\mathbb{M}$ such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \tag{1.1.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.1.2}$$

$$g(X, \xi) = \eta(X), \tag{1.1.3}$$

where X, Y are vector fields defined on TM.

If on a almost contact metric structure $(\mathbb{M}, \phi, \xi, \eta, g)$ the exterior derivative of 1-form η satisfies

$$d\eta(X, Y) = g(\phi X, Y),$$

then the structure $(\mathbb{M}, \phi, \xi, \eta, g)$ is said to define a contact metric structure and the manifold is named as contact metric manifold.

Definition 1.1.2. An almost contact metric manifold $(\mathbb{M}, g)$ is said to be a Sasakian manifold, if and only if

$$(\nabla_X \phi)(Y) = g(X,Y)\xi - \eta(Y)X, \tag{1.1.4}$$

where ∇ is Levi-Civita connection of the Riemannian metric g. From the above equation it follows that

$$\nabla_X \xi = -\phi X, \tag{1.1.5}$$

$$(\nabla_X \eta)Y = g(X, \phi Y). \tag{1.1.6}$$

For a Sasakian manifold, the Riemannian curvature tensor R, Ricci tensor S satisfy the following conditions

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{1.1.7}$$

$$R(X,\xi)Y = \eta(Y)X - g(X,Y)\xi, \tag{1.1.8}$$

$$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X, \tag{1.1.9}$$

$$\eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{1.1.10}$$

$$S(X,\xi) = 2n\eta(X), \tag{1.1.11}$$

$$S(\phi X, \phi Y) = S(X,Y) - 2n\eta(X)\eta(Y). \tag{1.1.12}$$

Generalized Sasakian space form: A Sasakian manifold with constant ϕ-sectional curvature is a Sasakian space form and it has a specific form of its curvature tensor. The notion of generalized Sasakian space forms was introduced by Alegre et al. [2] with several examples. A generalized Sasakian space form is an almost contact metric manifold $\mathbb{M}(\phi,\xi,\eta,g)$ whose curvature tensor is given by.

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X-g(X,Z)Y\}+f_2\{g(X,\phi Z)\phi Y-g(Y,\phi Z)\phi X+2g(X,\phi Y)\phi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X+g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi\}. \end{aligned} \tag{1.1.13}$$

where f_1, f_2, f_3 are differentiable functions and X, Y, Z are vector fields on $\mathbb{M}$. In such case we will write the manifold as $\mathbb{M}(f_1, f_2, f_3)$. This kind of manifold appears as a natural generalization of the Sasakian space forms: $f_1 = \frac{C+3}{4}$ and $f_2 = f_3 = \frac{C-1}{4}$, where C denotes constant ϕ-sectional curvature. The ϕ-sectional curvature of generalized Sasakian space form $\mathbb{M}(f_1, f_2, f_3)$ is $f_1 + 3f_2$. Moreover cosymplectic space forms and Kenmotsu space forms are also considered as particular types of generalized Sasakian space forms.

Again for a $(2n+1)$-dimensional generalized Sasakian space form, we have [67]

$$S(X,Y) = (2nf_1+3f_2-f_3)g(X,Y)+(3f_2-(2n-1)f_3)\eta(X)\eta(Y), \tag{1.1.14}$$

$$R(X,Y)\xi = (f_1-f_3)[\eta(Y)X-\eta(X)Y], \tag{1.1.15}$$

$$R(\xi,X)Y = (f_1-f_3)[g(X,Y)\xi-\eta(Y)X], \tag{1.1.16}$$

$$S(X,\xi) = 2n(f_1-f_3)\eta(X), \tag{1.1.17}$$

$$QX = (2nf_1+3f_2-f_3)X+(3f_2-(2n-1)f_3)\eta(X)\xi. \tag{1.1.18}$$

Definition 1.1.3. An almost contact metric manifold $(\mathbb{M}, g)$ is said to be a Kenmotsu manifold if the following relations hold on $\mathbb{M}$

$$(\nabla_X\phi)Y = g(\phi X,Y)\xi-\eta(Y)X, \tag{1.1.19}$$

$$\nabla_X\xi = X-\eta(X)\xi. \tag{1.1.20}$$

In a Kenmotsu manifold $(\mathbb{M}, g)$, besides above relations the following conditions also hold [50]

$$\begin{aligned} R(X,Y)Z &= g(X,Z)Y - g(Y,Z)X, & (1.1.21)\\ R(X,Y)\xi &= \eta(X)Y - \eta(Y)X, & (1.1.22)\\ S(X,\xi) &= -(n-1)\eta(X). & (1.1.23) \end{aligned}$$

Para-Kenmotsu manifold: Let $\mathbb{M}$ be an n-dimensional differentiable manifold equipped with structure (ϕ, ξ, η, g), where ϕ is a tensor field of type $(1,1)$, ξ is a vector field, η is a 1-form and g be the metric satisfying

$$\begin{aligned} \phi^2 X = X - \eta(X)\xi, \quad \eta(\xi) &= 1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, & (1.1.24)\\ g(\phi X, \phi Y) &= -g(X,Y) + \eta(X)\eta(Y), & (1.1.25)\\ g(X,\xi) &= \eta(X), & (1.1.26) \end{aligned}$$

for all vectors $X, Y \in TM$.

Then the manifold is said to admit an almost paracontact structure and the manifold is refereed to as almost paracontact metric manifold.

Definition 1.1.4. An almost paracontact metric manifold is said to be an para-Kenmotsu manifold if the following relations hold on $\mathbb{M}$

$$\begin{aligned} (\nabla_X \phi)Y &= g(\phi X, Y)\xi - \eta(Y)\phi X, & (1.1.27)\\ \nabla_X \xi &= X - \eta(X)\xi. & (1.1.28) \end{aligned}$$

In a para-Kenmotsu manifold, below mentioned conditions also hold

$$R(X,Y)Z = g(X,Z)Y - g(Y,Z)X, \tag{1.1.29}$$

$$R(\xi,X)Y = \eta(Y)X - g(X,Y)\xi, \tag{1.1.30}$$

$$S(X,Y) = -(n-1)g(X,Y), \tag{1.1.31}$$

$$S(X,\xi) = -(n-1)\eta(X). \tag{1.1.32}$$

$(LCS)_n$**-manifolds:** An n-dimensional Lorentzian manifold $\mathbb{M}$ is a smooth connected paracompact Hausdarff manifold with a Lorentzian metric g, that is, $\mathbb{M}$ admits a smooth symmetric tensor field g of type $(0,2)$, a unit timelike concircular vector field ξ called it as the characteristic vector field of the manifold, a non-zero 1-form η such that

$$g(X,\xi) = \eta(X), \qquad g(\xi,\xi) = -1, \tag{1.1.33}$$

and the equation of the following form holds.

$$(\nabla_X \eta)(Y) = \alpha\{g(X,Y) + \eta(X)\eta(Y)\}, \qquad \alpha \neq 0, \tag{1.1.34}$$

that is

$$(\nabla_X \xi) = \alpha[X + \eta(X)\xi], \tag{1.1.35}$$

for all vector fields X, Y where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function satisfies

$$(\nabla_X \alpha) = X\alpha = d\alpha(X) = \rho\eta(X), \tag{1.1.36}$$

ρ being a certain scalar function given by $\rho = -(\xi\alpha)$. If we put

$$\phi X = \frac{1}{\alpha}\nabla_X \xi, \tag{1.1.37}$$

then from (1.1.34) and (1.1.37) we have

$$\phi X = X + \eta(X)\xi, \tag{1.1.38}$$

for which it follows that ϕ is a symmetric $(1,1)$ tensor and called the structure tensor of the manifold. Thus the Lorentzian manifold $\mathbb{M}^n$ together with the unit timelike concircular vector field ξ, its associated 1-form η and $(1,1)$ tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly $(LCS)_n$ manifold) [74]. Especially if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure of Matsumoto [53]. In a $(LCS)_n$-manifold, the following relations hold.

$$\eta(\xi) = -1, \quad \phi\xi = 0,, \quad g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{1.1.39}$$

$$\phi^2 X = X + \eta(X)\xi, \tag{1.1.40}$$

$$(\nabla_X \phi)(Y) = \alpha\{g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}, \tag{1.1.41}$$

$$R(X,Y)\xi = (\alpha^2 - \rho)\{n(Y)X - \eta(X)Y\}, \tag{1.1.42}$$

$$R(\xi,Y)Z = (\alpha^2 - \rho)\{g(Y,Z)\xi - \eta(Z)Y\}, \tag{1.1.43}$$

$$\eta(R(X,Y)Z) = (\alpha^2 - \rho)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}. \tag{1.1.44}$$

1.2 Almost $C(\alpha)$ manifolds

Definition 1.2.1. An almost contact metric manifold is named as an almost $C(\alpha)$ manifold, if the Riemannian curvature tensor R gratifies the undermentioned relation [6],[7]

$$\begin{aligned} R(X,Y)Z &= R(\phi X, \phi Y)Z - \alpha[g(Y,Z)X - g(X,Z)Y \\ &- g(\phi Y, Z)\phi X + g(\phi X, Z)\phi Y], \end{aligned} \tag{1.2.1}$$

where X, Y, Z are vector fields on TM and α is a real number.

Remark 1.2.1. A $C(1)$-curvature tensor is a Sasakian curvature tensor, a $C(0)$-curvature tensor is a co-Kaehler or CK-curvature tensor and $C(-1)$-curvature tensor is a Kenmotsu curvature tensor.

For an almost $C(\alpha)$ manifold the following relations holds,

$$R(X,Y)\xi = R(\phi X, \phi Y)\xi - \alpha[\eta(Y)X - \eta(X)Y], \quad (1.2.2)$$

$$R(\xi,X)Y = -\alpha[g(X,Y)\xi - \eta(Y)X], \quad (1.2.3)$$

$$R(\xi,X)\xi = -\alpha[\eta(X)\xi - X], \quad (1.2.4)$$

$$(\nabla_X\phi)Y = g(X,Y)\xi - \eta(Y)X, \quad (1.2.5)$$

$$\nabla_X\xi = -\phi X. \quad (1.2.6)$$

On an almost $C(\alpha)$ manifold, we also have [4]

$$QX = AX + B\eta(X)\xi, \quad (1.2.7)$$

where Q is the Ricci operator, i.,e $g(QX,Y) = S(X,Y)$ for all vector fields on the tangent space of $\mathbb{M}$.

$$\eta(QX) = (A+B)\eta(X), \quad (1.2.8)$$

$$S(X,Y) = Ag(X,Y) + B\eta(X)\eta(Y), \quad (1.2.9)$$

$$r = -4n^2\alpha, \quad (1.2.10)$$

$$S(X,\xi) = (A+B)\eta(X), \quad (1.2.11)$$

$$S(\xi,\xi) = A+B. \quad (1.2.12)$$

where $A = -\alpha(2n-1)$ and $B = -\alpha$.

Example for 3-dimensional almost $C(\alpha)$ manifold: Consider the 3-dimensional manifold $\mathbb{M} = \{(x,y,z)/(x,y,z) \in R^3, z \neq 0\}$. Let $\{E_1, E_2, E_3\}$ be linearly independent at each point of $\mathbb{M}$ is given by

$$E_1 = 2(\tfrac{\partial}{\partial y} - x\tfrac{\partial}{\partial z}), \quad E_2 = 2\tfrac{\partial}{\partial x}, \quad E_3 = \tfrac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by $g(E_1, E_2) = g(E_2, E_3) = g(E_1, E_3) = 0$, $g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1$, where g is given by

$$g = \tfrac{1}{4}[(1 - 4x^2)dx \otimes dx + dy \otimes dy + 4dz \otimes dz].$$

The (ϕ, ξ, η) is given by $\eta = dz + xdy$, $\xi = E_3 = \partial/\partial z$, $\phi E_1 = E_2$, $\phi E_2 = -E_1$, $\phi E_3 = 0$. The linearity property of ϕ and g yields that $\eta(E_3) = 1$, $\phi^2 U = -U + \eta(U)E_3$, $g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W)$, for any vector fields U, W on $\mathbb{M}$. By the definition of Lie bracket, we have

$$[E_1, E_2] = 2E_3, \quad [E_1, E_3] = 0, \quad [E_2, E_3] = 0.$$

Let ∇ be the Levi-Civita connection with respect to the above metric g by the Koszula formula.

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ &- g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

Then,

$$\begin{aligned} &\nabla_{E_1} E_1 = 0, \quad \nabla_{E_1} E_2 = E_3, \quad \nabla_{E_1} E_3 = -E_2, \\ &\nabla_{E_2} E_1 = -E_3, \quad \nabla_{E_2} E_2 = 0, \quad \nabla_{E_2} E_3 = -E_1, \\ &\nabla_{E_3} E_1 = -E_2, \quad \nabla_{E_3} E_2 = E_1, \quad \nabla_{E_3} E_3 = 0. \end{aligned}$$

The tangent vectors X and Y to $\mathbb{M}$ are expressed as linear combination of E_1, E_2, E_3, that is, $X = \sum_{i=1}^{3} a_i E_i$ and $Y = \sum_{i=1}^{3} b_i E_i$, where a_i, b_i are scalars. Clearly (ϕ, ξ, η, g) and X, Y, satisfy (1.1.2), (1.2.1), (1.2.5) and (1.2.6). Thus $\mathbb{M}$ is a almost $C(\alpha)$ manifold.

1.3 S-manifold

Let $\mathbb{M}$ be a $(2n+\mathsf{S})$-dimensional manifold with an f-structure of rank $2n$. If there exists global vector fields $\xi_\alpha, \alpha = (1, 2, 3 \ldots \mathsf{S})$ on $\mathbb{M}$ such that;

$$f^2 = -I + \sum \xi_\alpha \otimes \eta_\alpha, \quad \eta_\alpha(\xi_\beta) = \delta^\alpha_\beta, \tag{1.3.1}$$

$$f\xi_\alpha = 0, \quad \eta_\alpha \circ f = 0, \tag{1.3.2}$$

$$g(X, \xi_\alpha) = \eta_\alpha(X), \quad g(X, fY) = -g(fX, Y), \tag{1.3.3}$$

where η_α are the dual 1-forms of ξ_α, we say that the f-structure has complemented frames. For such a manifold there exists a Riemannian metric g such that

$$g(X, Y) = g(fX, fY) + \sum_\alpha \eta_\alpha(X)\eta_\alpha(Y), \tag{1.3.4}$$

for any vector fields X and Y on $\mathbb{M}$.

An f-structure f is normal, if it has complemented frames and

$$[f, f] + 2\textstyle\sum_\alpha \xi_\alpha \otimes d\eta_\alpha = 0,$$

where $[f, f]$ is Nijenhuis torsion of f.

Let F be the fundamental 2-form defined by $F(X, Y) = g(X, fY), X, Y \in TM$. A normal f-structure for which the fundamental form F is closed, $\eta_1 \wedge \eta_2 \wedge \ldots \eta_{\mathsf{S}} \wedge (d\eta_\alpha)^n \neq 0$ for any α, and $d\eta_1 = d\eta_2 = \ldots = d\eta_{\mathsf{S}} = F$ is called to be an S-structure. A smooth manifold endowed with an S-structure will be called an S-manifold. These manifolds introduced by Blair [20].

We have to remark that if we take $\mathsf{S} = 1$, S-manifolds are natural generalizations of Sasakian manifolds. In the case $\mathsf{S} \geq 2$ some interesting examples are given [20], [39].

If $\mathbb{M}$ is an S-manifold, then the following relations holds true [20];

$$\nabla_X \xi_\alpha = -fX, \qquad X \in T(M), \alpha = 1, 2 \ldots \mathsf{S} \tag{1.3.5}$$

$$(\nabla_X \eta)(Y) = -g(fX, Y), \tag{1.3.6}$$

$$(\nabla_X f)Y = \sum_\alpha \{g(fX, fY)\xi_\alpha + \eta_\alpha(Y) f^2 X\}, \quad X, Y \in T(M), \tag{1.3.7}$$

where ∇ is the Riemannian connection of g. Let Ω be the distribution determined by the projection tensor-f^2 and let N be the complementry distribution which is determined by $f^2 + I$ and spanned by $\xi_1 \ldots \xi_{\mathsf{S}}$. It is clear that if $X \in \Omega$, then $\eta_\alpha(X) = 0$ for any α, and if $X \in N$, then fX=0. A plane section π on $\mathbb{M}$ is called an invariant f-section if it is determined by a vector $X \in \Omega(x)$, $x \in \mathbb{M}$, such that $\{X, fX\}$ is an orthonormal pair spanning the section. The sectional curvature of π is called the f-sectional curvature. If $\mathbb{M}$ is an S-manifold of constant f-sectional curvature k, then its curvature tensor has the form

$$\begin{aligned} R(X, Y, Z, W) &= \sum_{\alpha,\beta} \{g(fX, fW)\eta_\alpha(Y)\eta_\beta(Z) - g(fX, fZ)\eta_\alpha(Y)\eta_\beta(W) \\ &+ g(fY, fZ)\eta_\alpha(X)\eta_\beta(W) - g(fY, fW)\eta_\alpha(X)\eta_\beta(Z)\} \\ &+ \frac{1}{4}(k + 3\mathsf{S})\{g(fX, fW)g(fY, fZ) - g(fX, fZ)g(fY, fW)\} \\ &+ \frac{1}{4}(k - \mathsf{S})\{F(X, W)F(Y, Z) - F(X, Z)F(Y, W) \\ &- 2F(X, Y)F(Z, W)\}, \end{aligned} \tag{1.3.8}$$

where $X, Y, Z, W \in TM$. Such a manifold $N(K)$ will be called an S-space form. The Euclidean space $E^{2n+\mathsf{S}}$ and the hyperbolic space $H^{2n+\mathsf{S}}$ are examples of S-space forms.

Now contracting equation(1.3.8) we get Ricci tensor given by

$$S(Y,Z) = b_1 g(Y,Z) + b_2 \eta_\alpha(Y)\eta_\alpha(Z), \tag{1.3.9}$$

$$S(Y,\xi_\alpha) = b_3 \eta_\alpha(Y), \tag{1.3.10}$$

where $b_1 = \left[\frac{4\mathsf{S}+(k+3\mathsf{S})(2n-1)+3(k-\mathsf{S})}{4}\right]$, $b_2 = \left[\frac{(2n+\mathsf{S}-2)(4-k-3\mathsf{S})-3(k-\mathsf{S})}{4}\right]$ and $b_3 = \left[\frac{\mathsf{S}^2(13-6n-k-3\mathsf{S})+2\mathsf{S}(7n-5)+k(2-\mathsf{S})+2nk(1-\mathsf{S})}{4}\right]$. The equation (1.3.8) yields the following conditions

$$R(X,Y)\xi_\alpha = \mathsf{S}\sum_\alpha\{\eta_\alpha(Y)X - \eta_\alpha(X)Y\}, \tag{1.3.11}$$

$$R(\xi_\alpha,X)Y = \mathsf{S}\sum_\alpha\{g(X,Y)\xi_\alpha - \eta_\alpha(Y)Z\}, \tag{1.3.12}$$

$$\eta_\alpha(R(X,Y)Z) = \mathsf{S}\sum_\alpha\{g(Y,Z)\eta_\alpha(X) - g(X,Z)\eta_\alpha(Y)\}. \tag{1.3.13}$$

1.4 Ricci soliton

In differential geometry, the Ricci flow is an intrinsic geometric flow. It is a process that deforms the metric of a Riemannian manifold in a way formally analogous to the diffusion of heat, smoothing out irregularities in the metric. The Ricci flow, named after Gregorio Ricci-Curbastro, was first introduced by Richard S. Hamilton in 1981 and is also referred to as the Ricci-Hamilton flow. It is given by the following geometric evolution equation.

$$\frac{\partial g}{\partial t} = -2Ric(g).$$

Here g is a Riemannian metric and $Ric(g)$ is Ricci tensor depending on time t. A Riemannian metric g on a smooth manifold is Einstein if its Ricci tensor is a constant multiple of g.

The concept of Ricci solitons was introduced by Hamilton [37], [38]. They are natural generalizations of Einstein metrics. Ricci solitons also correspond to selfsimilar solutions of Hamilton's Ricci flow and often arise as limits of dilations of singularities in the Ricci flow

Definition 1.4.1. A Ricci soliton is a natural generalization of an Einstein metric and is defined on a Riemannian manifold $(\mathbb{M}, g)$. A Ricci soliton is a triple (g, V, λ) with g a Riemannian metric, V a vector field and λ a real scalar such that

$$(\mathcal{L}_V g)(U, V) + 2S(U, V) + 2\lambda g(U, V) \quad = \quad 0, \tag{1.4.1}$$

where S is a Ricci tensor of M and L_V denotes the Lie derivative operator along with vector field V.

As a generalization of Ricci solitons, the notion of η-Ricci solitons was introduced by Cho and Kimura [26]. This notion has also been studied in [23] for Hopf hypersurfaces in complex space forms.

Definition 1.4.2. An η-Ricci soliton is a tuple (g, V, λ, μ), where λ and μ are real scalars gratifying the equation

$$(\mathcal{L}_V g)(U, V) + 2S(U, V) + 2\lambda g(U, V) + 2\mu\eta(U)\eta(V) \quad = \quad 0, \tag{1.4.2}$$

in particular if $\mu = 0$, then (g, V, λ) is Ricci soliton.

The author Fischer introduced [33] a variation of the classical Ricci flow equation that modifies the unit volume constraint of that equation to a scalar curvature constraint. The resulting equations are named as the conformal Ricci flow equations.

$$\begin{aligned} \frac{\partial g}{\partial t} + 2\left(Ric(g) + \frac{1}{n} g\right) &= -\rho g, \\ R(g) &= -1, \end{aligned}$$

where $R(g)$ is the scalar curvature of the manifold and ρ is scalar non-dynamical field and n is the dimension of manifold. The conformal Ricci flow equations are analogous to the Navier-Stokes equations of fluid mechanics and because of this analogy, the timedependent scalar field ρ is called a conformal pressure.

Definition 1.4.3. The conformal Ricci soliton equation is given by [30]

$$(\mathcal{L}_V g)(U,V) + 2S(U,V) = \left[2\lambda - \left(\rho + \frac{2}{n}\right)\right] g(U,V), \tag{1.4.3}$$

and is the generalization of the Ricci soliton equation and it also gratifies the conformal Ricci flow equation, where ρ is scalar non-dynamical field and n is the dimension of manifold.

1.5 Semi-symmetric manifolds

A Riemannian manifold $(\mathbb{M}, g)$ is called locally symmetric if its curvature tensor R is parallel [24] i.e., $\nabla R = 0$, where ∇ denotes the Levi-Civita connection. As a proper generalization of locally symmetric manifold the notion of semi-symmetric manifold was defined by

$$(R(X,Y)\cdot R)(U,V,W) = 0, \;\; X,Y,U,V \; and \; W \;\in\; TM,$$

and studied by the authors [80], [60]. A complete intrinsic classification of these manifolds was given by Szabo [78].

The $(0,6)$-tensor $R \cdot R$ obtained by the action of the curvature operator $R(X,Y)$ on the $(0,4)$-curvature tensor R, and is given by [87]

$$\begin{aligned}(R\cdot R)(U,V,W,Z;X,Y) &= -R(R(X,Y)U,V,W,Z) - R(U,R(X,Y)V,W,Z) \\ &- R(U,V,R(X,Y)W,Z) - R(U,V,W,R(X,Y)Z).\end{aligned} \tag{1.5.1}$$

whereby $U, V, W, Z, X, Y \in TM$.

The tensor $R \cdot R$ has the following algebraic properties.

- $(R \cdot R)(U, V, W, Z; X, Y) = -(R \cdot R)(V, U, W, Z; X, Y) = -(R \cdot R)(U, V, Z, W; X, Y)$.
- $(R \cdot R)(U, V, W, Z; X, Y) + (R \cdot R)(U, W, Z, V; X, Y) + (R \cdot R)(U, Z, V, W; X, Y) = 0$.
- $(R \cdot R)(U, V, W, Z; X, Y) = -(R \cdot R)(U, V, W, Z; Y, X)$.
- $(R \cdot R)(U, V, W, Z; X, Y) + (R \cdot R)(W, Z, X, Y; U, V) + (R \cdot R)(X, Y, U, V; W, Z) = 0$.

The simplest (0, 6)-tensor having the same symmetry properties as $R \cdot R$ may well be the Tachibana tensor $Q(g, R)$ defined by [87]

$$\begin{aligned} Q(g, R)(U, V, W, Z; X, Y) &= R((X \wedge Y)U, V, W,)Z + R(U, (X \wedge Y)V, W, Z) \\ &+ R(U, V, (X \wedge Y)W, Z) + R(U, V, W, (X \wedge Y)Z) \end{aligned} \quad (1.5.2)$$

For a $(0, k)$-tensor field T on M, $k \geq 1$, and a symmetric $(0, 2)$ tensor fields g and S on $\mathbb{M}$, we define the $(0, k+2)$ tensor fields $R \cdot T$, $Q(g, T)$ and $Q(S, T)$ by

$$\begin{aligned} (R \cdot T)(X_1, \ldots, X_k, X, Y) &= -T(R(X, Y)X_1, X_2, \ldots, X_k) - T(X_1, R(X, Y)X_2, \ldots, X_k) \\ &- , \ldots, -T(X_1, X_2, \ldots, X_{k-1}, R(X, Y)X_k), \\ Q(g, T)(X_1, \ldots, X_k, X, Y) &= -T((X \wedge_g Y)X_1, X_2, \ldots, X_k) - T(X_1, (X \wedge_g Y)X_2, \ldots, X_k) \\ &- , \ldots, -T(X_1, X_2, \ldots, X_{k-1}, (X \wedge_g Y)X_k), \\ Q(S, T)(X_1, \ldots, X_k, X, Y) &= -T((X \wedge_S Y)X_1, X_2, \ldots, X_k) - T(X_1, (X \wedge_S Y)X_2, \ldots, X_k) \\ &- , \ldots, -T(X_1, X_2, \ldots, X_{k-1}, (X \wedge_S Y)X_k), \end{aligned}$$

where $(X \wedge_g Y)$ and $(X \wedge_S Y)$ are the endomorphism given by

$$(X \wedge_g Y)Z = g(Y, Z)X - g(X, Z)Y, \quad (X \wedge_S Y)Z = S(Y, Z)X - S(X, Z)Y.$$

Definition 1.5.1. A Riemannian manifold is said to be pseudo symmetric (in the sense of Deszcz [63], [31]) if

$$R \cdot R = L_R Q(g, R)$$

holds on the set $U_R = \{x \in M; R - \frac{r}{n(n-1)} G \neq 0 \text{ at } x\}$, where G is the $(0,4)$-tensor defined by $G(X_1, X_2, X_3, X_4) = g((X_1 \wedge X_2)X_3, X_4)$ and L_R is some function on $\mathbb{M}$.

Definition 1.5.2. A Riemannian manifold is said to be Ricci generalized pseudo symmetric (in the sense of Deszcz [63], [31]) if

$$R \cdot R = L_R Q(S, R),$$

holds on the set $U_R = \{x \in M; Q(S, R) \neq 0 \text{ at } x\}$, and L_R is some function on $\mathbb{M}$.

Definition 1.5.3. In a differentiable manifold $\mathbb{M}$, the Ricci tensor S satisfies the condition

$$S = ag + b\eta \otimes \eta,$$

where a and b are some functions on C^∞, then the manifold $\mathbb{M}$ is coined to be an η-Einstein manifold. If in particular $a = 0$, then the manifold becomes a special type of η-Einstein manifold.

.

CHAPTER 2

Publications based on this Chapter

- *Ricci Soliton of Almost $C(\alpha)$ Manifolds,* Bull. Cal. Math. Soc., 107, (6), 483-494, (2015).
- *C-Bochner Curvature Tensor in Almost $C(\alpha)$ Manifolds,* (Communicated).
- *A Study on Conformal Ricci Soliton in Almost $C(\alpha)$ Manifolds Admitting W_2-Curvature Tensor,* (Communicated).

Chapter 2

On Almost $C(\alpha)$ Manifolds

2.1 Introduction

In his study on Betti numbers of Kaehler manifolds, Bochner introduced a tensor which plays similar role of the Weyl tensor on Riemannian manifolds. Thus a conformally flat manifold is an extension of a real space form. So a Bochner flat Kaehler manifold has to be an extension in the same sense of a complex space form. By using additional structures to Kaehler manifolds one can also study classes of odd dimensional manifolds or almost contact metric manifolds; in particular Sasakian, co-Kaehlerian/cosymplectic and Kenmotsu manifolds. Janssens and Vanhecke [48] using decomposition theory defined Bochner curvature for a class of almost contact metric manifolds known as C-Bochner curvature tensor. The elements of this class are called as almost $C(\alpha)$ manifolds, where α is a real number.

Further Olszak and Rosca [62] investigated such manifold. Again Kharitonova [52] studied conformally flat almost $C(\alpha)$ manifolds. The authors [4], [5], [6] have studied the geometry of Ricci tensor, quasi conformal curvature tensor of almost $C(\alpha)$ manifolds and conharmonically flat, ξ-conharmonically flat, concircularly flat and ξ-concircularly flat almost $C(\alpha)$ manifolds. In [7] the authors studied the flatness of the pseudo-projective, quasi-conformal curvature tensor, ξ-pseudo-projective, ξ-quasi-conformal curvature tensor in an almost $C(\alpha)$ manifolds. Further Eisenhat problem was applied to study Ricci solitons in almost $C(\alpha)$ manifolds [16]. Motivated by the above work we study C-Bochner semi-symmetric and pseudo-symmetric almost $C(\alpha)$ manifolds. Further we study Ricci solitons, conformal Ricci solitons of these manifolds.

2.2 Flat C-Bochner curvature tensor in almost $C(\alpha)$-manifold

The C-Bochner curvature tensor is given by [45]

$$\begin{aligned} B(U,V)Z &= R(U,V)Z + \frac{1}{2n+4}[g(U,Z)QV - S(V,Z)U - g(V,Z)QU + S(U,Z)V \\ &+ g(\phi U,Z)Q\phi V - S(\phi V,Z)\phi U - g(\phi V,Z)Q\phi U + S(\phi U,Z)\phi V \\ &+ 2S(\phi U,V)\phi Z + 2g(\phi U,V)Q\phi Z + \eta(V)\eta(Z)QU - \eta(V)S(U,Z)\xi \\ &+ \eta(U)S(V,Z)\xi - \eta(U)\eta(Z)QV] - \frac{D+2n}{2n+4}[g(\phi U,Z)\phi V - g(\phi V,Z)\phi U \\ &+ 2g(\phi U,V)\phi Z] + \frac{D}{2n+4}[\eta(V)g(U,Z)\xi - \eta(V)\eta(Z)U + \eta(U)\eta(Z)V \\ &- \eta(U)g(V,Z)\xi] - \frac{D-4}{2n+4}[g(U,Z)V - g(V,Z)U], \end{aligned} \tag{2.2.1}$$

where $D = \frac{r+2n}{2n+2}$ and r is the scalar curvature. In view of (1.2.1), (1.2.2), (1.2.3) and (1.2.4) in (2.2.1) we get the following

$$B(U,V)\xi = R(\phi U,\phi V)\xi + \frac{2(\alpha+1)}{n+2}[\eta(U)V - \eta(V)U], \tag{2.2.2}$$

$$B(\xi,V)Z = \frac{2(\alpha+1)}{n+2}[\eta(Z)V - g(V,Z)\xi], \tag{2.2.3}$$

$$B(U,\xi)Z = \frac{2(\alpha+1)}{n+2}[g(U,Z)\xi - \eta(Z)U], \tag{2.2.4}$$

$$B(\xi,V)\xi = \frac{2(\alpha+1)}{n+2}[V - \eta(V)\xi]. \tag{2.2.5}$$

Let us consider an almost $C(\alpha)$ manifold which has flat C-Bochner curvature tensor i.e., $B(U,V)Z = 0$, then from (1.2.1) and (2.2.1) we have

$$\begin{aligned} 0 &= R(\phi U,\phi V)Z - \alpha[g(V,Z)U - g(U,Z)V - g(\phi V,Z)\phi U + g(\phi U,Z)\phi V] \\ &+ \frac{1}{2n+4}[g(U,Z)QV - S(V,Z)U - g(V,Z)QU + S(U,Z)V \\ &+ g(\phi U,Z)Q\phi V - S(\phi V,Z)\phi U - g(\phi V,Z)Q\phi U + S(\phi U,Z)\phi V + 2S(\phi U,V)\phi Z \\ &+ 2g(\phi U,V)Q\phi Z + \eta(V)\eta(Z)QU - \eta(V)S(U,Z)\xi + \eta(U)S(V,Z)\xi - \eta(U)\eta(Z)QV] \\ &- \frac{D+2n}{2n+4}[g(\phi U,Z)\phi V - g(\phi V,Z)\phi U + 2g(\phi U,V)\phi Z] + \frac{D}{2n+4}[\eta(V)g(U,Z)\xi \\ &- \eta(V)\eta(Z)U + \eta(U)\eta(Z)V - \eta(U)g(V,Z)\xi] - \frac{D-4}{2n+4}[g(U,Z)V - g(V,Z)U]. \end{aligned} \tag{2.2.6}$$

Take inner product of (2.2.6) with W, we get

$$\begin{aligned}
0 &= (R(\phi U,\phi V)Z,W) - \alpha[g(V,Z)(U,W) - g(U,Z)g(V,W) - g(\phi V,Z)g(\phi U,W)\\
&+ g(\phi U,Z)g(\phi V,W)] + \frac{1}{2n+4}[g(U,Z)g(QV,W) - S(V,Z)g(U,W)\\
&- g(V,Z)g(QU,W) + S(U,Z)g(V,W) + g(\phi U,Z)g(Q\phi V,W) - S(\phi V,Z)g(\phi U,W)\\
&- g(\phi V,Z)g(Q\phi U,W) + S(\phi U,Z)g(\phi V,W) + 2S(\phi U,V)g(\phi Z,W)\\
&+ 2g(\phi U,V)g(Q\phi Z,W) + \eta(V)\eta(Z)g(QU,W) - \eta(V)\eta(W)S(U,Z)\\
&+ \eta(U)\eta(W)S(V,Z) - \eta(U)\eta(Z)g(QV,W)] - \frac{D+2n}{2n+4}[g(\phi U,Z)g(\phi V,W)\\
&- g(\phi V,Z)g(\phi U,W) + 2g(\phi U,V)g(\phi Z,W)] + \frac{D}{2n+4}[\eta(V)\eta(W)g(U,Z)\\
&- \eta(V)\eta(Z)g(U,W) + \eta(U)\eta(Z)g(V,W) - \eta(U)\eta(W)g(V,Z)]\\
&- \frac{D-4}{2n+4}[g(U,Z)g(V,W) - g(V,Z)g(U,W)]. \qquad (2.2.7)
\end{aligned}$$

Let $\{e_1, e_2, \ldots, e_{2n+1}\}$ be an orthonormal basis of the tangent space at each point of the manifold, putting $V = Z = e_i$ in (2.2.7) and taking summation over i $(1 \le i \le 2n+1)$ we can get

$$\begin{aligned}
0 &= S(\phi U,W) - \alpha[(n-1)g(U,W) - g(\phi U,\phi W)] + \frac{1}{2n+4}[-(n-3)s(U,W)\\
&- rg(U,W) - 6S(\phi U,\phi W) + (r - 2(A+B))\eta(U)\eta(W)] + \frac{3(D+2n)}{2n+4}[g(\phi U,\phi W)]\\
&+ \frac{D}{2n+4}[-(n-2)\eta(U)\eta(W) - g(U,W)] - \frac{D-4}{2n+4}[-(n-1)g(U,W)]. \qquad (2.2.8)
\end{aligned}$$

Using (1.1.2), (1.2.9) in (2.2.8) we get

$$\begin{aligned}
0 &= S(\phi U,W) - \alpha[(n-2)g(U,W) + \eta(U)\eta(W)] + \frac{1}{2n+4}[-(n-3)S(U,W)\\
&- (r+6A)g(U,W) + (6A + r - 2(A+B))\eta(U)\eta(W)] + \frac{3(D+2n)}{2n+4}[g(U,W)\\
&- \eta(U)\eta(W)] + \frac{D}{2n+4}[-(n-2)\eta(U)\eta(W) - g(U,W)]\\
&- \frac{D-4}{2n+4}[-(n-1)g(U,W)]. \qquad (2.2.9)
\end{aligned}$$

Interchanging X and W in (2.2.9)

$$\begin{aligned} 0 &= S(\phi W,U)-\alpha[(n-2)g(U,W)+\eta(U)\eta(W)]+\frac{1}{2n+4}[-(n-3)S(U,W)\\ &- (r+6A)g(U,W)+(6A+r-2(A+B))\eta(U)\eta(W)]+\frac{3(D+2n)}{2n+4}[g(U,W)\\ &- \eta(U)\eta(W)]+\frac{D}{2n+4}[-(n-2)\eta(U)\eta(W)-g(U,W)]\\ &- \frac{D-4}{2n+4}[-(n-1)g(U,W)]. \end{aligned} \tag{2.2.10}$$

Add (2.2.9) and (2.2.10) we get the value of Ricci tensor

$$\begin{aligned} S(U,W) &= \frac{1}{n-3}[-2\alpha(3n^2+6n-7)+2(D+3n)+(n-1)(D-4)]g(U,W)\\ &+ \frac{1}{n-3}[-2\alpha(2n^2+5n-1)-3(D+2n)-D(n-2)]\eta(U)\eta(W). \end{aligned} \tag{2.2.11}$$

We can state the following:

Theorem 2.2.1. *A C-Bochnerly flat almost $C(\alpha)$ manifold is η-Einstein manifold.*

2.3 Almost $C(\alpha)$ manifolds satisfying $B\cdot S=0$

Let us consider an almost $C(\alpha)$ manifold $\mathbb{M}$ with $B\cdot S=0$. Then we get

$$(B(U,V)\cdot S)(X,Y)=0,$$

$$S(B(U,V)X,Y)+S(X,B(U,V)Y)=0. \tag{2.3.1}$$

Putting $U=Y=\xi$ in (2.3.1), using (2.2.2) and (2.2.3) we get

$$S(B(\xi,V)X,\xi)+\frac{2(\alpha+1)}{n+2}[S(V,X)-(A+B)\eta(V)\eta(X)]=0. \tag{2.3.2}$$

Using (2.2.3) and (1.2.12) in (2.3.2) we get

$$\frac{2(\alpha+1)}{n+2}[S(V,X)-(A+B)g(V,X)]=0. \tag{2.3.3}$$

Therefore, either $\alpha = -1$ or $S(V, X) = (A + B)g(V, X)$.

Thus we can state the following:

Theorem 2.3.1. *Every almost $C(\alpha)$ manifold satisfying $B \cdot S = 0$ is an Einstein manifold provided $\alpha \neq -1$.*

From definition of almost $C(\alpha)$ manifold and Theorem (2.3.1), we can state

Theorem 2.3.2. *Every Sasakian manifold $C(1)$ and co-Kaehler manifold $C(0)$ satisfying $B \cdot S = 0$ is an Einstein manifold.*

2.4 Almost $C(\alpha)$ manifolds satisfying $B \cdot R = 0$

Let us consider an almost $C(\alpha)$ manifold $\mathbb{M}$ with $B \cdot R = 0$.

$$(B(U, V) \cdot R)(X, Y)Z = 0,$$

$$B(U, V)R(X, Y)Z - R(B(U, V)X, Y)Z - R(X, B(U, V)Y)Z - R(X, Y)B(U, V)Z = 0. \tag{2.4.1}$$

Putting $X = V = \xi$ in (2.4.1) and using (1.2.3) and (2.2.2) we get

$$\frac{2(\alpha + 1)}{n + 2}[R(U, Y)Z - \alpha\{g(U, Z)Y - g(Y, Z)U\}] = 0. \tag{2.4.2}$$

$$\textit{Either} \quad \alpha = -1 \quad \textit{or} \quad R(U, Y)Z = \alpha[g(U, Z)Y - g(Y, Z)U]. \tag{2.4.3}$$

Taking inner product of (2.4.3) with W, we get

$$R(U, Y, Z, W) = \alpha[g(U, Z)g(Y, W) - g(Y, Z)g(U, W)]. \tag{2.4.4}$$

Let $\{e_1, e_2, \ldots, e_{2n+1}\}$ be an orthonormal basis of the tangent space at each point of the manifold, putting $Y = Z = e_i$ in (2.4.4) and taking summation over i $(1 \leq i \leq 2n+1)$ we can get

$$S(U, W) = -2n\alpha g(U, W). \tag{2.4.5}$$

Thus we are in a position to state the following:

Theorem 2.4.1. *Every almost $C(\alpha)$ manifold satisfying $B \cdot R = 0$ is an Einstein manifold provided $\alpha \neq -1$.*

Also we have

Theorem 2.4.2. *Every Sasakian manifold $C(1)$ and co-Kaehler manifold $C(0)$ satisfying $B \cdot R = 0$ is an Einstein manifold.*

2.5 Almost $C(\alpha)$ manifolds satisfying $B \cdot S = L_S Q(g, S)$

Let us consider an almost $C(\alpha)$ manifold $\mathbb{M}$ with $B \cdot S = L_S Q(g, S)$, then we have

$$(B(U, V) \cdot S)(X, Y) = L_S((U \wedge V) \cdot S)(X, Y),$$

$$S(B(U, V)X, Y) + S(X, B(U, V)Y) = L_S[(S(U \wedge V)X, Y) + S(X, (U \wedge V)Y)]. \tag{2.5.1}$$

Putting $Y = \xi$ in (2.5.1), we get

$$S(B(U, V)X, \xi) + S(X, B(U, V)\xi) = L_S[(S(U \wedge V)X, \xi) + S(X, (U \wedge V)\xi)]. \tag{2.5.2}$$

Putting $U = \xi$ in (2.5.2) and in view of (1.2.11), (1.2.12) and (2.2.2) we get

$$\left[L_S + \frac{2(\alpha + 1)}{n + 2}\right][S(V, X) - (A + B)g(V, X)] = 0. \tag{2.5.3}$$

Therefore either $L_S = -\frac{2(\alpha+1)}{n+2}$ or $S(V,X) = (A+B)g(V,X)$.

Thus we are in a position to state the following:

Theorem 2.5.1. *Every almost $C(\alpha)$ manifold satisfying $B \cdot S = L_S Q(g,S)$ is an Einstein manifold provided $L_S \neq \frac{-2(\alpha+1)}{n+2}$.*

2.6 Almost $C(\alpha)$ manifolds satisfying $B \cdot R = L_R Q(g,R)$

Let us consider an almost $C(\alpha)$ manifold with $B \cdot R = L_R Q(g,R)$, then we have

$$(B(U,V)\cdot R)(X,Y)Z = L_R((U \wedge V)\cdot R)(X,Y)Z,$$

$$\begin{aligned} &B(U,V)R(X,Y)Z - R(B(U,V)X,Y)Z - R(X,B(U,V)Y)Z \\ &-R(X,Y)B(U,V)Z = L_R[(U\wedge V)R(X,Y)Z - R((U\wedge V)X,Y)Z \\ &-R(X,(U\wedge V)Y)Z - R(X,Y)(U\wedge V)Z]. \end{aligned} \tag{2.6.1}$$

Putting $X = V = \xi$ in (2.6.1) and using (1.2.3), (1.2.4) and (2.2.2) we get

$$\left[L_R + \frac{2(\alpha+1)}{n+2}\right][R(U,Y)Z - \alpha\{g(U,Z)Y - g(Y,Z)U\}] = 0. \tag{2.6.2}$$

$$\textit{Either} \quad L_R = \frac{-2(\alpha+1)}{n+2} \quad \textit{or} \quad R(U,Y)Z = \alpha\{g(U,Z)Y - g(Y,Z)U\}. \tag{2.6.3}$$

Taking inner product of (2.6.3) with W, we get

$$R(U,Y,Z,W) = \alpha\{g(U,Z)g(Y,W) - g(Y,Z)g(U,W)\}. \tag{2.6.4}$$

Let $\{e_1, e_2, \dots, e_{2n+1}\}$ be an orthonormal basis of the tangent space at each point of the manifold, putting $Y = Z = e_i$ in (2.6.4) and taking summation over i $(1 \leq i \leq 2n+1)$

we can get

$$S(U, W) = -2n\alpha g(U, W). \tag{2.6.5}$$

Thus we are in a position to state the following

Theorem 2.6.1. *Every almost $C(\alpha)$ manifold satisfying $B \cdot R = L_R Q(g, R)$ is an Einstein manifold provided $L_R \neq \frac{-2(\alpha+1)}{n+2}$.*

2.7 Ricci soliton in semi-symmetric almost $C(\alpha)$ manifold

An almost $C(\alpha)$ manifold is said to be semi-symmetric if $R \cdot R = 0$.

$$(R(X, Y) \cdot R)(U, V)W = 0, \tag{2.7.1}$$

$$R(X, Y)R(U, V)W - R(R(X, Y)U, V)W - R(U, R(X, Y)V)W - R(U, V)R(X, Y)W = 0. \tag{2.7.2}$$

Putting $X = U = \xi$ in (2.7.2) and using (1.2.3), (1.2.4) one can get.

$$R(Y, V)W = \alpha\{g(Y, W)V - g(V, W)Y\}. \tag{2.7.3}$$

Now, taking inner product of (2.7.3) with Z, we get.

$$R(Y, V, W, Z) = \alpha\{g(Y, W)g(V, Z) - g(V, W)g(Y, Z)\}. \tag{2.7.4}$$

Let $\{e_1, e_2, \ldots, e_{2n+1}\}$ be an orthonormal basis of the tangent space at each point of the manifold, putting $V = W = e_i$ in (2.7.4) and taking summation over i $(1 \leq i \leq 2n+1)$

we can get

$$S(Y,Z) = -2n\alpha g(Y,Z). \tag{2.7.5}$$

Thus we state the following lemma:

Lemma 2.7.1. *Every semi symmetric almost $C(\alpha)$ manifold is an Einstein manifold.*

Now, by using the definition of Ricci soliton i.e., (1.4.1) we can write

$$g(\nabla_Y V, Z) + g(\nabla_Z V, Y) + 2S(Y,Z) + 2\lambda g(Y,Z) = 0. \tag{2.7.6}$$

Put $V = \xi$ in (2.7.6) and using (1.2.6), (2.7.5) we get.

$$g(-\phi Y, Z) + g(-\phi Z, Y) + 4n\alpha g(Y,Z) + 2\lambda g(Y,Z) = 0,$$

$$(2\lambda - 4n\alpha)g(Y,Z) = 0. \tag{2.7.7}$$

Taking $Y = Z = e_i$ in (2.7.7) and summing over $i = 1, 2, ...2n+1$ we get the value of λ i.e.,

$$\lambda = 2n\alpha. \tag{2.7.8}$$

Theorem 2.7.2. *A Ricci soliton in semi symmetric almost $C(\alpha)$ manifold is shrinking, steady and expanding if accordingly it as Kenmotsu, co-Kaehler and Sasakain.*

Suppose $(\mathbb{M}, g)$ is an almost $C(\alpha)$ manifold and (g, V, λ) is a Ricci soliton in $(\mathbb{M}, g)$. If V is a conformal killing vector field, then

$$L_V g = \psi g. \tag{2.7.9}$$

From (1.4.1) and (2.7.9) we have

$$S(X,Y) = -\left(\lambda + \frac{\psi}{2}\right) g(X,Y), \tag{2.7.10}$$

$$QX = -\left(\lambda + \frac{\psi}{2}\right) X. \tag{2.7.11}$$

Now Consider

$$(R(X,Y)\cdot R)(U,V)W = R(X,Y)R(U,V)W - R(R(X,Y)U,V)W - R(U,R(X,Y)V)W - R(U,V)R(X,Y)W \quad (2.7.12)$$

Contracting equation (2.7.12) over V and using (2.7.10) we get

$$\begin{aligned}(R(X,Y)\cdot R)(U,V)W &= S(R(X,Y)U,W) + S(U,R(X,Y)W),\\ &= -\left(\lambda + \frac{\psi}{2}\right)[R(X,Y,U,W) + R(X,Y,W,U)],\\ (R(X,Y)\cdot R)(U,V)W &= 0,\end{aligned}$$

i.e., $(\mathbb{M}, g)$ is semi-symmetric. Conversely suppose $(R(X,Y)\cdot R)(U,V)W = 0$.

Using (2.7.5) in (1.4.1) we get

$$\begin{aligned}(L_V g)(Y,Z) - 4n\alpha g(Y,Z) + 2\lambda g(Y,Z) &= 0,\\ (L_V g)(Y,Z) &= 2(2n\alpha - \lambda)g(Y,Z),\\ L_V g &= \psi g,\end{aligned}$$

where $\psi = 2(2n\alpha - \lambda)$, then we state

Corollary 2.7.3. *Let (g,V,λ) be a Ricci soliton in an almost $C(\alpha)$ manifold. Then $(\mathbb{M}, g)$ is semi-symmetric if and only if V is conformal killing.*

2.8 Ricci soliton in almost $C(\alpha)$ manifold satisfying $\bar{M}\cdot R = 0$

The M-Projective curvature tensor $\bar{M}$ is defined by

$$\bar{M}(U,V)Z = R(U,V)Z - \frac{1}{4n}[S(V,Z)U - S(U,Z)V + g(V,Z)QU - g(U,Z)QV]. \quad (2.8.1)$$

Using (1.2.1) in (2.8.1) we have the following

$$\bar{M}(X,Y)\xi = R(\phi X,\phi Y)\xi - \frac{\alpha}{4n}[\eta(Y)X-\eta(X)Y], \quad (2.8.2)$$

$$\bar{M}(\xi,Y)Z = -\frac{\alpha}{4n}[g(Y,Z)\xi-\eta(Z)Y], \quad (2.8.3)$$

$$\bar{M}(\xi,Y)\xi = -\frac{\alpha}{4n}[\eta(Y)\xi-Y], \quad (2.8.4)$$

$$\bar{M}(\xi,\xi)Z = 0. \quad (2.8.5)$$

We assume that $(\bar{M}(X,Y)\cdot R)(U,V)W=0$; then we have

$$\bar{M}(X,Y)R(U,V)W-R(\bar{M}(X,Y)U,V)W-R(U,\bar{M}(X,Y)V)W-R(U,V)\bar{M}(X,Y)W=0. \quad (2.8.6)$$

Put $X=U=\xi$ in (2.8.6) and using (1.2.3), (1.2.4), (2.8.3) and (2.8.4) we get.

$$R(Y,V)W=\alpha\{g(Y,W)V-g(V,W)Y\}. \quad (2.8.7)$$

Taking inner product of (2.8.7) with Z, we get.

$$R(Y,V,W,Z)=\alpha\{g(Y,W)g(V,Z)-g(V,W)g(Y,Z)\}. \quad (2.8.8)$$

Let $\{e_1,e_2,\ldots,e_{2n+1}\}$ be an orthonormal basis of the tangent space at each point of the manifold, putting $V=W=e_i$ in (2.8.8) and taking summation over i $(1\leq i\leq 2n+1)$ we can get

$$S(Y,Z)=-2n\alpha g(Y,Z). \quad (2.8.9)$$

Thus, we state the following lemma:

Lemma 2.8.1. *An almost $C(\alpha)$ manifold satisfies $\bar{M}\cdot R=0$ is an Einstein manifold.*

Put $V=\xi$ in (2.7.6) and using (1.2.6), (2.8.9) we get.

$$(2\lambda-4n\alpha)g(Y,Z)=0. \quad (2.8.10)$$

Taking $Y = Z = e_i$ in (2.8.10) and summing over $i = 1, 2, \ldots, 2n+1$ we get.

$$\lambda = 2n\alpha. \tag{2.8.11}$$

Then we state the following:

Theorem 2.8.2. *A Ricci soliton in almost $C(\alpha)$ manifold satisfying $\bar{M} \cdot R = 0$ shrinking, steady and expanding if accordingly it as Kenmotsu, co-Kaehler and Sasakian.*

2.9 Ricci soliton in almost $C(\alpha)$ manifold satisfying $R \cdot \bar{M} = 0$

We assume that $R \cdot \bar{M} = 0$; then we have

$$(R(X,Y) \cdot \bar{M})(U,V)W = 0,$$

$$R(X,Y)\bar{M}(U,V)W - \bar{M}(R(X,Y)U,V)W - \bar{M}(U,R(X,Y)V)W - \bar{M}(U,V)R(X,Y)W = 0. \tag{2.9.1}$$

Put $X = U = \xi$ in (2.9.1) and using (1.2.3), (1.2.4), (2.8.3) and (2.8.4) we get

$$\bar{M}(Y,V)W = \frac{\alpha}{4n}[g(Y,W)V - g(V,W)Y]. \tag{2.9.2}$$

Using (2.9.2) in (2.8.1) and taking inner product of with Z we get

$$\begin{aligned} g(R(\phi Y, \phi V)W, Z) &= \frac{\alpha}{4n}[g(Y,W)g(V,Z) - g(V,W)g(Y,Z)] \\ &+ \alpha[g(V,W)g(Y,Z) - g(Y,W)g(V,Z) - g(\phi V,W)g(\phi Y,Z) \\ &+ g(\phi Y,W)g(\phi V,Z)] + \frac{1}{4n}[S(V,W)g(Y,Z) - S(Y,W)g(V,Z) \\ &+ g(V,W)S(Y,Z) - g(Y,W)S(V,Z)]. \end{aligned} \tag{2.9.3}$$

Let $\{e_1, e_2, \ldots, e_{2n+1}\}$ be an orthonormal basis of the tangent space at each point of the manifold, putting $V = W = e_i$ in (2.9.3) and taking summation over i $(1 \leq i \leq 2n+1)$, also using (1.1.1) we can get

$$\begin{aligned} S(\phi Y, Z) &= -\frac{\alpha}{2} g(Y,Z) + \alpha[(2n-1)g(Y,Z) + \eta(Y)\eta(Z)] \\ &+ \frac{1}{4n}[rg(Y,Z) + (2n-1)S(Y,Z)]. \end{aligned} \tag{2.9.4}$$

Interchanging Y and Z in (2.9.4) we get

$$\begin{aligned} -S(\phi Y, Z) &= -\frac{\alpha}{2} g(Y,Z) + \alpha[(2n-1)g(Y,Z) + \eta(Y)\eta(Z)] \\ &+ \frac{1}{4n}[rg(Y,Z) + (2n-1)S(Y,Z)]. \end{aligned} \tag{2.9.5}$$

Now, adding equations (2.9.4) and (2.9.5) we get the Ricci tensor

$$S(Y,Z) = \frac{2n\alpha}{2n-1}[(3-2n)g(Y,Z) - 2\eta(Y)\eta(Z)]. \tag{2.9.6}$$

Thus we state the following lemma:

Lemma 2.9.1. *An almost $C(\alpha)$ manifold which satisfies $R \cdot \bar{M} = 0$ is an η-Einstein manifold.*

Put $V = \xi$ in (2.7.6) and using (1.2.6), (2.9.6) we get.

$$\frac{4n\alpha}{2n-1}[(3-2n)g(Y,Z) - 2\eta(Y)\eta(Z)] + 2\lambda g(Y,Z) = 0. \tag{2.9.7}$$

Taking $Y = Z = e_i$ in (2.9.7) and summing over $i = 1, 2, ...2n+1$ we get.

$$\lambda = -\frac{2n\alpha}{4n^2-1}[4n - 4n^2 + 1]. \tag{2.9.8}$$

Then we state the following:

Theorem 2.9.2. *A Ricci soliton in almost $C(\alpha)$ manifold satisfying $R \cdot \bar{M} = 0$ is shrinking, steady and expanding if accordingly it as Kenmotsu, co-Kaehler and Sasakian.*

2.10 Ricci soliton in almost $C(\alpha)$ manifold satisfying $R \cdot R = L_1 Q(g, R)$

An almost $C(\alpha)$ manifold is said to be pseudo symmetric if

$$(R(X,Y) \cdot R)(U,V)W = L_1[((X \wedge Y) \cdot R)(U,V)W], \tag{2.10.1}$$

where L_1 is smooth function on M.

$$\begin{aligned} &R(X,Y)R(U,V)W - R(R(X,Y)U,V)W - R(U,R(X,Y)V)W \\ &-R(U,V)R(X,Y)W = L_1[(X \wedge Y)R(U,V)W - R((X \wedge Y)U,V)W \\ &-R(U,(X \wedge Y)V)W - R(U,V)(X \wedge Y)W]. \end{aligned} \tag{2.10.2}$$

Put $X = U = \xi$ in (2.10.2) and using (1.2.3), (1.2.4) we get

$$[L_1 + \alpha][R(Y,V)W - \alpha\{g(Y,W)V - g(V,W)Y\}] = 0. \tag{2.10.3}$$

Therefore

$$L_1 = -\alpha \quad or \quad R(Y,V)W = \alpha\{g(Y,W)V - g(V,W)Y\}. \tag{2.10.4}$$

Taking inner product of (2.10.4) with Z, we get

$$R(Y,V,W,Z) = \alpha\{g(Y,W)g(V,Z) - g(V,W)g(Y,Z)\}. \tag{2.10.5}$$

Let $\{e_1, e_2, \ldots, e_{2n+1}\}$ be an orthonormal basis of the tangent space at each point of the manifold, putting $V = W = e_i$ in (2.10.5) and taking summation over i $(1 \leq i \leq 2n+1)$ we can get

$$S(Y,Z) = -2n\alpha g(Y,Z). \tag{2.10.6}$$

Thus we state the following lemma:

Lemma 2.10.1. *An almost $C(\alpha)$ manifold which satisfies $R \cdot R = L_1 Q(g, R)$ is either an Einstein manifold or $L_1 = -\alpha$.*

Put $V = \xi$ in (2.7.6) and using (1.2.6), (2.10.6), we get

$$(2\lambda - 4n\alpha)g(Y, Z) = 0. \tag{2.10.7}$$

Taking $Y = Z = e_i$ in (2.10.7) and summing over $i = 1, 2, \ldots, 2n + 1$ we get.

$$\lambda = 2n\alpha. \tag{2.10.8}$$

Theorem 2.10.2. *A Ricci soliton in almost $C(\alpha)$ manifold satisfying $R \cdot R = L_1 Q(g, R)$ is shrinking, steady and expanding if accordingly it as Kenmotsu, co-Kaehler and Sasakian.*

2.11 Ricci soliton in almost $C(\alpha)$ manifold satisfying $\bar{M} \cdot R = L_2 Q(g, R)$

We assume that $\bar{M} \cdot R = L_2 Q(g, R)$; then we have

$$(\bar{M}(X, Y) \cdot R)(U, V)W = L_2[((X \wedge Y) \cdot R)(U, V)W], \tag{2.11.1}$$

$$\begin{aligned} &\bar{M}(X, Y)R(U, V)W - R(\bar{M}(X, Y)U, V)W - R(U, \bar{M}(X, Y)V)W \\ &-R(U, V)\bar{M}(X, Y)W = L_2[(X \wedge Y)R(U, V)W - R((X \wedge Y)U, V)W \\ &-R(U, (X \wedge Y)V)W - R(U, V)(X \wedge Y)W]. \end{aligned} \tag{2.11.2}$$

Put $X = U = \xi$ in (2.11.2) and using (2.8.3), (2.8.4), (1.2.3) and (1.2.4) we get

$$\left[L_2 + \frac{\alpha}{4n}\right][R(Y, V)W - \alpha\{g(Y, W)V - g(V, W)Y\}] = 0. \tag{2.11.3}$$

Therefore

$$L_2 = -\frac{\alpha}{4n} \quad or \quad R(Y,V)W = \alpha\{g(Y,W)V - g(V,W)Y\}. \tag{2.11.4}$$

Taking inner product of (2.11.4) with Z, we get.

$$R(Y,V,W,Z) = \alpha\{g(Y,W)g(V,Z) - g(V,W)g(Y,Z)\}. \tag{2.11.5}$$

Let $\{e_1, e_2, \ldots, e_{2n+1}\}$ be an orthonormal basis of the tangent space at each point of the manifold, putting $V = W = e_i$ in (2.11.5) and taking summation over i $(1 \leq i \leq 2n+1)$ we can get

$$S(Y,Z) = -2n\alpha g(Y,Z). \tag{2.11.6}$$

Thus we state the following lemma:

Lemma 2.11.1. *An almost $C(\alpha)$ manifold which satisfies $\bar{M} \cdot R = L_2 Q(g,R)$ is either an Einstein manifold or $L_2 = -\frac{\alpha}{4n}$.*

Put $V = \xi$ in (2.7.6) and using (1.2.6), (2.11.6) we get

$$(2\lambda - 4n\alpha)g(Y,Z) = 0. \tag{2.11.7}$$

Taking $Y = Z = e_i$ in (2.11.7) and summing over $i = 1, 2, \ldots, 2n+1$ we get.

$$\lambda = 2n\alpha. \tag{2.11.8}$$

Theorem 2.11.2. *A Ricci soliton in almost $C(\alpha)$ manifold satisfying $\bar{M} \cdot R = L_2 Q(g,R)$ is shrinking, steady and expanding if accordingly it as Kenmotsu, co-Kahler and Sasakian.*

2.12 Ricci soliton in almost $C(\alpha)$ manifold satisfying $R\cdot\bar{M}=L_3Q(g,\bar{M})$

We assume that $R\cdot\bar{M}=L_3Q(g,\bar{M})$; then we have

$$(R(X,Y)\cdot\bar{M})(U,V)W \;=\; L_3[((X\wedge Y)\cdot\bar{M})(U,V)W], \tag{2.12.1}$$

$$\begin{aligned}&R(X,Y)\bar{M}(U,V)W-\bar{M}(R(X,Y)U,V)W-\bar{M}(U,R(X,Y)V)W\\&-\bar{M}(U,V)R(X,Y)W=L_3[(X\wedge Y)\bar{M}(U,V)W-\bar{M}((X\wedge Y)U,V)W\\&-\bar{M}(U,(X\wedge Y)V)W-\bar{M}(U,V)(X\wedge Y)W].\end{aligned} \tag{2.12.2}$$

Put $X=U=\xi$ in (2.12.2) and using (2.8.3), (2.8.4), (1.2.3) and (1.2.4), we get

$$[L_3+\alpha][\bar{M}(Y,V)W-\frac{\alpha}{4n}\{g(Y,W)V-g(V,W)Y\}]=0. \tag{2.12.3}$$

Therefore

$$L_3=-\alpha \quad \text{or} \quad \bar{M}(Y,V)W=\frac{\alpha}{4n}\{g(Y,W)V-g(V,W)Y\}, \tag{2.12.4}$$

Using (2.12.4) in (2.8.1) and taking inner product of with Z we get

$$\begin{aligned}g(R(\phi Y,\phi V)W,Z) &= \frac{\alpha}{4n}[g(Y,W)g(V,Z)-g(V,W)g(Y,Z)]\\&+ \alpha[g(V,W)g(Y,Z)-g(Y,W)g(V,Z)-g(\phi V,W)g(\phi Y,Z)\\&+ g(\phi Y,W)g(\phi V,Z)]+\frac{1}{4n}[S(V,W)g(Y,Z)-S(Y,W)g(V,Z)\\&+ g(V,W)S(Y,Z)-g(Y,W)S(V,Z)].\end{aligned} \tag{2.12.5}$$

Let $\{e_1,e_2,\ldots,e_{2n+1}\}$ be an orthonormal basis of the tangent space at each point of the manifold, putting $V=W=e_i$ in (2.12.5) and taking summation over i $(1\le i\le 2n+1)$, also using (1.1.1) we can get

$$S(\phi Y, Z) = \frac{\alpha}{2}g(Y,Z) - \alpha[(2n-1)g(Y,Z) + \eta(Y)\eta(Z)] + \frac{1}{4n}[rg(Y,Z) + (2n-1)S(Y,Z)]. \tag{2.12.6}$$

Interchanging Y and Z in (2.12.6), we get

$$-S(\phi Y, Z) = \frac{\alpha}{2}g(Y,Z) - \alpha[(2n-1)g(Y,Z) + \eta(Y)\eta(Z)] + \frac{1}{4n}[rg(Y,Z) + (2n-1)S(Y,Z)]. \tag{2.12.7}$$

Now, adding equations (2.12.6) and (2.12.7) we get the Ricci tensor

$$S(Y,Z) = \frac{2n\alpha}{2n-1}[(3-2n)g(Y,Z) - 2\eta(Y)\eta(Z)]. \tag{2.12.8}$$

Thus we state the following lemma:

Lemma 2.12.1. *An almost $C(\alpha)$ manifold satisfies $R \cdot \bar{M} = L_3 Q(g, \bar{M})$ is either an η-Einstein manifold or $L_3 = -\alpha$.*

Put $V = \xi$ in (2.7.6) and using (1.2.6), (2.12.8) we get

$$\frac{4n\alpha}{2n-1}[(3-2n)g(Y,Z) + 2\eta(Y)\eta(Z)] + 2\lambda g(Y,Z) = 0. \tag{2.12.9}$$

Taking $Y = Z = e_i$ in (2.12.9) and summing over i=1, 2,. . . ,n we get

$$\lambda = -\frac{2n\alpha}{4n^2-1}[4n - 4n^2 + 1]. \tag{2.12.10}$$

Then we state the following:

Theorem 2.12.2. *A Ricci soliton in almost $C(\alpha)$ manifold satisfying $R \cdot \bar{M} = L_3 Q(g, \bar{M})$ is shrinking, steady and expanding if accordingly it as Kenmotsu, co-Kaehler and Sasakian.*

2.13 Conformal Ricci soliton in almost $C(\alpha)$ manifolds

The conformal Ricci flow equation on $\mathbb{M}$ is defined by the equation,

$$\frac{\partial g}{\partial t} + 2\left(Ric(g) + \frac{g}{n}\right) = -\rho g, \tag{2.13.1}$$

where $R(g) = -1$, ρ is a scalar non-dynamical field (time dependent scalar field) and n is the dimension of manifold.

The notion of conformal Ricci soliton is given by

$$(L_V g)(X,Y) + 2S(X,Y) = \left[2\lambda - \left(\rho + \frac{2}{n}\right)\right] g(X,Y). \tag{2.13.2}$$

Now by using the definition of Lie derivative we can find the value of $L_\xi g$ that is given by

$$(L_\xi g)(X,Y) = g(-\phi X, Y) + g(X, -\phi Y) = 0 \tag{2.13.3}$$

By virtue of (2.13.3) in (2.13.2) we get

$$S(X,Y) = \sigma g(X,Y), \tag{2.13.4}$$

where $\sigma = \frac{1}{2}\left[2\lambda - \left(\rho + \frac{2}{n}\right)\right]$. If we put $X = Y = e_i$ in (2.13.4) where $\{e_i\}$ is an orthonormal basis, and summing over i, we get $S = \sigma n$. But for conformal Ricci flow $R(g) = -1$, which yields the value of λ

$$\lambda = \frac{\rho}{2}. \tag{2.13.5}$$

We can consequently state the following:

Theorem 2.13.1. *An almost $C(\alpha)$ manifolds admitting conformal Ricci soliton is an Einstein manifold and the scalar λ of the conformal Ricci soliton is equal to $\frac{\rho}{2}$.*

2.14 Conformal Ricci soliton in almost $C(\alpha)$ manifold satisfying $R \cdot W_2 = 0$

The W_2-curvature tensor is given by [42]

$$W_2(U,V)Z = R(U,V)Z + \frac{1}{n-1}[g(U,Z)QV - g(V,Z)QU]. \tag{2.14.1}$$

By virtue of (1.2.1), (2.13.4) and (2.14.1) we can get the following

$$W_2(\xi,V)Z = \left(-\alpha - \frac{\sigma}{n-1}\right)[g(V,Z)\xi - \eta(Z)V], \tag{2.14.2}$$

$$W_2(\xi,V)\xi = \left(-\alpha - \frac{\sigma}{n-1}\right)[\eta(V)\xi - V]. \tag{2.14.3}$$

Let us consider $(R(X,Y)\cdot W_2)(U,V)Z = 0$

$$R(X,Y)W_2(U,V)Z - W_2(R(X,Y)U,V)Z - W_2(U,R(X,Y)V)Z - W_2(U,V)R(X,Y)Z = 0. \tag{2.14.4}$$

Put $X = U = \xi$ in (2.14.4) and using (1.2.3), (1.2.4) and (2.14.2) we get

$$W_2(Y,V)Z = \left(-\alpha - \frac{\sigma}{n-1}\right)[g(V,Z)Y - g(Y,Z)V]. \tag{2.14.5}$$

Taking inner product of (2.14.5) with T we can write

$$W_2(Y,V,Z,T) = \left(-\alpha - \frac{\sigma}{n-1}\right)[g(V,Z)g(Y,T) - g(Y,Z)g(V,T)]. \tag{2.14.6}$$

Using (1.2.1) and (2.14.1) in (2.14.6) and contracting over Y and T we can get

$$\begin{aligned} S(\phi V,Z) &= \left[\alpha(n-2) + \frac{r}{n-1} + \left(-\alpha - \frac{\sigma}{n-1}\right)(n-1)\right]g(V,Z) \\ &+ \alpha\eta(V)\eta(Z) - \frac{1}{n-1}S(V,Z). \end{aligned} \tag{2.14.7}$$

Interchanging V and Z in (2.14.7)

$$\begin{aligned} -S(\phi V,Z) &= \left[\alpha(n-2) + \frac{r}{n-1} + \left(-\alpha - \frac{\sigma}{n-1}\right)(n-1)\right]g(V,Z) \\ &+ \alpha\eta(V)\eta(Z) - \frac{1}{n-1}S(V,Z). \end{aligned} \tag{2.14.8}$$

Now add (2.14.7) and (2.14.8)

$$S(V,Z)=\left(-\alpha+\frac{r}{n-1}-\sigma\right)(n-1)g(V,Z)+\alpha(n-1)\eta(V)\eta(Z). \tag{2.14.9}$$

We state the following:

Theorem 2.14.1. *An almost $C(\alpha)$ manifold satisfying the condition $R\cdot W_2=0$ is an η-Einstein manifold.*

For conformal Ricci flow $R(g)=-1$, using (2.13.4) in (2.14.9) and on contraction over V and Z we get the value of λ and it is given by

$$\lambda=-\alpha\frac{(n-1)}{n}+\frac{\rho}{2}. \tag{2.14.10}$$

Thus we state the following:

Theorem 2.14.2. *Ricci soliton in almost $C(\alpha)$ manifolds satisfying the condition $R\cdot W_2=0$ admitting conformal Ricci soliton is*

1. *shrinking if $\rho<\frac{2\alpha(n-1)}{n}$.*
2. *steady if $\rho=\frac{2\alpha(n-1)}{n}$.*
3. *expanding if $\rho>\frac{2\alpha(n-1)}{n}$.*

2.15 Conformal Ricci soliton in almost $C(\alpha)$ manifold satisfying $W_2\cdot R=0$.

Let us consider $(W_2(X,Y)\cdot R)(U,V)Z=0$

$$W_2(X,Y)R(U,V)Z-R(W_2(X,Y)U,V)Z-R(U,W_2(X,Y)V)Z-R(U,V)W_2(X,Y)Z=0. \tag{2.15.1}$$

Put $X = U = \xi$ in (2.15.1) and using (2.14.2) and (2.14.3) we get

$$\left(-\alpha - \frac{\sigma}{n-1}\right)[R(Y,V)Z + \alpha(g(V,Z)Y - g(Y,Z)V)] = 0. \tag{2.15.2}$$

Since $\left(-\alpha - \frac{\sigma}{n-1}\right) \neq 0$ and taking inner product of (2.15.2) with T we can write

$$R(Y,V,Z,T) = \alpha[g(Y,Z)g(V,T) - g(V,Z)g(Y,T)]. \tag{2.15.3}$$

Putting $Y = T = e_i$ in (2.15.3), where $\{e_i\}$ is an orthonormal basis and taking summation $i = 1, 2, \ldots, n$ we get

$$S(V,Z) = -\alpha(n-1)g(V,Z). \tag{2.15.4}$$

We state the following:

Theorem 2.15.1. *An almost $C(\alpha)$ manifold satisfying the condition $W_2 \cdot R = 0$ is an Einstein manifold.*

For conformal Ricci flow $R(g) = -1$, using (2.13.4) in (2.15.4) and on contraction over V and Z we get the value of λ and it is given by

$$\lambda = \frac{1}{2}\left(\rho + \frac{2}{n}\right) - \alpha(n-1). \tag{2.15.5}$$

We state the following theorem:

Theorem 2.15.2. *Ricci soliton in almost $C(\alpha)$ manifolds satisfying the condition $W_2 \cdot R = 0$ admitting conformal Ricci soliton is*

1. *shrinking if $\rho < 2\alpha(n-1) - \frac{2}{n}$.*

2. *steady if $\rho = 2\alpha(n-1) - \frac{2}{n}$.*

3. *expanding if $\rho > 2\alpha(n-1) - \frac{2}{n}$.*

2.16 Conformal Ricci soliton in almost $C(\alpha)$ manifold satisfying $R \cdot W_2 = L_{W_2} Q(g, W_2)$

Let us consider $R \cdot W_2 = L_{W_2} Q(g, W_2)$

$$\begin{aligned} & R(X,Y)W_2(U,V)Z - W_2(R(X,Y)U,V)Z - W_2(U,R(X,Y)V)Z \\ & -W_2(U,V)R(X,Y)Z = L_{W_2}[(X\wedge Y)W_2(U,V)Z - W_2((X\wedge Y)U,V)Z \\ & -W_2(U,(X\wedge Y)V)Z - W_2(U,V)R(X\wedge Y)Z. \end{aligned} \tag{2.16.1}$$

Put $X = U = \xi$ in (2.16.1) using (1.2.3), (1.2.4), (2.14.2) and (2.14.3) also by using the definition of endomorphism we get

$$(L_{W_2} - \alpha)[W_2(Y,V)Z - \left(-\alpha - \frac{\sigma}{n-1}\right)(g(V,Z)Y - g(Y,Z)V)]. \tag{2.16.2}$$

Since $L_{W_2} \neq \alpha$, taking inner product of (2.16.2) with T one can get

$$W_2(Y,V,Z,T) = \left(-\alpha - \frac{\sigma}{n-1}\right)[g(V,Z)g(Y,T) - g(Y,Z)g(V,T)]. \tag{2.16.3}$$

Using (1.2.1) and (2.14.1) in (2.16.3) and contracting over Y and T we can get

$$\begin{aligned} S(\phi V,Z) &= \left[\alpha(n-2) + \frac{r}{n-1} + \left(-\alpha - \frac{\sigma}{n-1}\right)(n-1)\right] g(V,Z) + \alpha\eta(V)\eta(Z) \\ &- \frac{1}{n-1}S(V,Z). \end{aligned} \tag{2.16.4}$$

Interchanging V and Z in (2.16.4)

$$\begin{aligned} -S(\phi V,Z) &= \left[\alpha(n-2) + \frac{r}{n-1} + \left(-\alpha - \frac{\sigma}{n-1}\right)(n-1)\right] g(V,Z) + \alpha\eta(V)\eta(Z) \\ &- \frac{1}{n-1}S(V,Z). \end{aligned} \tag{2.16.5}$$

Now add (2.16.4) and (2.16.5)

$$S(V,Z) = \left(-\alpha + \frac{r}{n-1} - \sigma\right)(n-1)g(V,Z) + \alpha(n-1)\eta(V)\eta(Z). \tag{2.16.6}$$

We state the following:

Theorem 2.16.1. *An almost $C(\alpha)$ manifold satisfying the condition $R \cdot W_2 = L_{W_2} Q(g, W_2)$ is an η-Einstein manifold provided $L_{W_2} \neq \alpha$.*

For conformal Ricci flow $R(g) = -1$, using (2.13.4) in (2.16.6) and on contraction over V and Z we get the value of λ and it is given by

$$\lambda = -\alpha \frac{(n-1)}{n} + \frac{\rho}{2}. \tag{2.16.7}$$

Remark 2.16.1. We can state the result similar to Theorem 2.14.2.

2.17 Conformal Ricci soliton in almost $C(\alpha)$ manifolds satisfying the condition $W_2 \cdot R = L_R Q(g, R)$

Let us consider $W_2 \cdot R = L_R Q(g, R)$

$$\begin{aligned} &W_2(X,Y)R(U,V)Z - R(W_2(X,Y)U,V)Z - R(U,W_2(X,Y)V)Z \\ &-R(U,V)W_2(X,Y)Z = L_R[(X \wedge Y)R(U,V)Z - R((X \wedge Y)U,V)Z \\ &-R(U,(X \wedge Y)V)Z - R(U,V)R(X \wedge Y)Z. \end{aligned} \tag{2.17.1}$$

Put $X = U = \xi$ in (2.17.1) and using (2.14.2) and (2.14.3) also by using the definition of endomorphism we can write

$$\left[L_R - \left(-\alpha - \frac{\sigma}{n-1}\right)\right] [R(Y,V)Z + \alpha(g(V,Z)Y - g(Y,Z)V)] = 0. \tag{2.17.2}$$

Since $L_R \neq \left(-\alpha - \frac{\sigma}{n-1}\right)$ and taking inner product of (2.17.2) with T we can write

$$R(Y,V,Z,T) = \alpha[g(Y,Z)g(V,T) - g(V,Z)g(Y,T)]. \tag{2.17.3}$$

Putting $Y = T = e_i$ in (2.17.3), where $\{e_i\}$ is an orthonormal basis and taking summation $i = 1, 2, \ldots, n$ we get

$$S(V, Z) = -\alpha(n-1)g(V, Z). \tag{2.17.4}$$

We state the following:

Theorem 2.17.1. *An almost $C(\alpha)$ manifold satisfying the condition $W_2 \cdot R = L_R Q(g, R)$ is an Einstein manifold provided $L_R \neq \left(-\alpha - \frac{\sigma}{n-1}\right)$.*

For conformal Ricci flow $R(g) = -1$, using (2.13.4) in (2.17.4) and on contraction over V and Z we get the value of λ and it is given by

$$\lambda = \frac{1}{2}\left(\rho + \frac{2}{n}\right) - \alpha(n-1) \tag{2.17.5}$$

Remark 2.17.1. We can state the result similar to Theorem 2.15.2.

2.18 Conformal Ricci soliton in almost $C(\alpha)$ manifold satisfying $W_2 \cdot S = 0$

Let us consider $(W_2(X, Y) \cdot S)(U, V) = 0$

$$S(W_2(X, Y)U, V) + S(U, W_2(X, Y)V) = 0. \tag{2.18.1}$$

Putting $V = \xi$ in (2.18.1), using (2.14.1) we get

$$S(W_2(X, Y)U, \xi) + S(U, R(\phi X, \phi Y)\xi) + [\eta(Y)S(X, U) - \eta(X)S(Y, U)] = 0. \tag{2.18.2}$$

Again putting $X = \xi$ in (2.18.2) and using (2.14.2) we get

$$\left(-\alpha - \frac{\sigma}{n-1}\right)[S(Y, U) + \sigma g(Y, U) - 2\sigma\eta(Y)\eta(U)] = 0. \tag{2.18.3}$$

Since $\left(-\alpha - \frac{\sigma}{n-1}\right) \neq 0$, so equation (2.18.3) can be written as

$$S(Y,U) = -\sigma g(Y,U) + 2\sigma\eta(Y)\eta(V). \tag{2.18.4}$$

We state the following

Theorem 2.18.1. *An almost $C(\alpha)$ manifold satisfying the condition $W_2 \cdot S = 0$ is an η-Einstein manifold.*

2.19 Conformal Ricci soliton in almost $C(\alpha)$ manifold satisfying $W_2 \cdot S = L_S Q(g,S)$

Let us consider $(W_2(X,Y) \cdot S)(U,V) = L_S Q(g,S)$

$$S(W_2(X,Y)U,V) + S(U,W_2(X,Y)V) = L_S(S((X\wedge Y)U,V) + S(U,(X\wedge Y)V)). \tag{2.19.1}$$

Putting $V = X = \xi$ in (2.19.1) and using (2.14.1) and (2.14.2) we get from above

$$\left[L_S - \left(-\alpha - \frac{\sigma}{n-1}\right)\right][S(Y,U) + \sigma g(Y,U) - 2\sigma\eta(Y)\eta(U)] = 0. \tag{2.19.2}$$

Since $L_S \neq \left(-\alpha - \frac{\sigma}{n-1}\right)$, hence equation (2.19.2) can be written as

$$S(Y,U) = -\sigma g(Y,U) + 2\sigma\eta(Y)\eta(V). \tag{2.19.3}$$

We state the following:

Theorem 2.19.1. *An almost $C(\alpha)$ manifold satisfying the condition $W_2 \cdot S = L_S Q(g,S)$ is an η-Einstein manifold provided $L_S \neq \left(-\alpha - \frac{\sigma}{n-1}\right)$.*

2.20 Conclusion

The important results finding of this chapter are as follows:

- A C-Bochner flat almost $C(\alpha)$ manifold is η-Einstein manifold.

- Every almost $C(\alpha)$ manifold satisfying $B \cdot S = 0$ and $B \cdot R = 0$ is an Einstein manifold provided $\alpha \neq -1$.

- Every almost $C(\alpha)$ manifold satisfying $B \cdot S = L_S Q(g, S)$ and $B \cdot R = L_R Q(g, R)$ is an Einstein manifold.

- A Ricci soliton in semi symmetric almost $C(\alpha)$ manifold is shrinking, steady and expanding, if accordingly it is Kenmotsu, co-Kaehler and Sasakian.

- Let (g, V, λ) be a Ricci soliton in an almost $C(\alpha)$ manifold. Then (M, g) is semi-symmetric if and only if V is conformal killing.

- The Ricci soliton in almost $C(\alpha)$ manifold satisfying $\bar{M} \cdot R = 0$, $R \cdot \bar{M} = 0$, $R \cdot R = L_1 Q(g, R)$, $\bar{M} \cdot R = L_2 Q(g, R)$ and $R \cdot \bar{M} = L_3 Q(g, \bar{M})$ is shrinking, steady and expanding, if accordingly it is Kenmotsu, co-Kaehler and Sasakian.

- An almost $C(\alpha)$ manifold admitting conformal Ricci soliton is an Einstein manifold and the scalar λ of the conformal Ricci soliton is equal to $\frac{\rho}{2}$.

- Ricci soliton in almost $C(\alpha)$ manifold satisfying the condition $R \cdot W_2 = 0$ admitting conformal Ricci soliton is

  1. shrinking, if $\rho < \frac{2\alpha(n-1)}{n}$.

  2. steady, if $\rho = \frac{2\alpha(n-1)}{n}$.

3. expanding, if $\rho > \frac{2\alpha(n-1)}{n}$.

- Ricci soliton in almost $C(\alpha)$ manifold satisfying the condition $W_2 \cdot R = 0$ admitting conformal Ricci soliton is

  1. shrinking, if $\rho < 2\alpha(n-1) - \frac{2}{n}$.
  2. steady, if $\rho = 2\alpha(n-1) - \frac{2}{n}$.
  3. expanding, if $\rho > 2\alpha(n-1) - \frac{2}{n}$.

CHAPTER 3

Publications based on this Chapter

- *A Study on Ricci Soliton in $\mathcal{S}$-Manifolds,* IOSR Journal Mathematics, Volume 13, issue 1,(2017), PP 12-22.
- *Irrotational τ-Curvature Tensor in $\mathcal{S}$-Manifolds,* (Accepted in Indian Journal of Mathematics and Mathematical Sciences).

Chapter 3

On S-Manifolds

3.1 Introduction

The notion of f-structure on a $(2n+\mathsf{S})$-dimensional manifold $\mathbb{M}$, i.e., a tensor field of type $(1,1)$ on M of rank $2n$ satisfying $f^3+f=0$, was firstly introduced in 1963 by Yano [84] as a generalization of both (almost) contact (for s = 1) and (almost) complex structures (for s = 0). During the Posterior years, this notion has been furtherly developed by several authors [20], [21], [35], [36], [47], [57], [58]. The author Nagagawa in [57] [58] introduced the notion of framed f-manifold,

f-manifolds:[58] Let $\mathbb{M}$ be an n-dimensional connected differentiable manifold of class C^∞ on which there is a non-null tensor field f of type $(1,1)$ and of class C^∞ satisfying the equation

$$f^3 + f = 0, \tag{3.1.1}$$

we call such a structure as f-structure of rank r, when the rank of f is constant everywhere and is equal to r, where r is necessarily even [84]. Then the manifold is called an f-manifold when it admits an f-structure.

Later Goldberg and Yano [35], [36] and others developed and studied these manifolds with the denomination of globally framed f-manifolds. Blair [20] introduced the concept of an S-manifold equipped with an f-structure, as analogous to the Kaehler structure in almost Hermitian case and to the Sasakian structure in the almost contact case. In the present chapter we show that, S-manifold which allows semi-symmetric and pseudo-symmetric conditions is an Einstein manifold and Ricci soliton for these manifolds is shrinking, later we obtain interesting result of Ricci soliton for irrotational τ-curvature tensor in S-manifolds.

3.2 Ricci soliton in semi-symmetric S-manifolds

An S-manifold is said to be semi-symmetric if $R \cdot R = 0$.

$$(R(\xi_\alpha, Y) \cdot R)(U, V)W = 0, \tag{3.2.1}$$

$$\begin{aligned} &R(\xi_\alpha, Y)R(U, V)W - R(R(\xi_\alpha, Y)U, V)W - R(U, R(\xi_\alpha, Y)V)W \\ &-R(U, V)R(\xi_\alpha, Y)W = 0. \end{aligned} \tag{3.2.2}$$

Using (1.3.12) in (3.2.2), we get

$$\begin{aligned} &\mathsf{S}\sum_\alpha\{g(Y, R(U, V)W)\xi_\alpha - \eta_\alpha(R(U, V)W)Y - g(Y, U)R(\xi_\alpha, V)W \\ &+\eta_\alpha(U)R(Y, V)W - g(Y, V)R(U, \xi_\alpha)W + \eta_\alpha(V)R(U, Y)W \\ &-g(Y, W)R(U, V)\xi_\alpha + \eta_\alpha(W)R(U, V)Y\} = 0. \end{aligned} \tag{3.2.3}$$

By taking an inner product of (3.2.3) with ξ_α then we get

$$\begin{aligned} &\sum_\alpha\{\mathsf{S}R(U, V, W, Y) - \eta_\alpha(R(U, V)W)\eta_\alpha(Y) - g(Y, U)\eta_\alpha(R(\xi_\alpha, V)W) \\ &+\eta_\alpha(U)\eta_\alpha(R(Y, V)W) - g(Y, V)\eta_\alpha(R(U, \xi_\alpha)W) + \eta_\alpha(V)\eta_\alpha(R(U, Y)W) \\ &-g(Y, W)\eta_\alpha(R(U, V)\xi_\alpha) + \eta_\alpha(W)\eta_\alpha(R(U, V)Y)\} = 0. \end{aligned} \tag{3.2.4}$$

By using (1.3.11), (1.3.13) in (3.2.4) we have

$$\mathsf{S}R(U,V,W,Y)+\mathsf{S}^2g(Y,V)g(U,W)-\mathsf{S}^2g(Y,U)g(V,W)=0. \tag{3.2.5}$$

Taking $U=Y=e_i$ in (3.2.5) and summing over $i=1,2,\ldots,2n+\mathsf{S}$ we get

$$S(V,W)=\mathsf{S}(2n+\mathsf{S}-1)g(V,W). \tag{3.2.6}$$

Thus we state the following:

Theorem 3.2.1. *Semi symmetric S-manifold is an Einstein manifold.*

If V is collinear with ξ, then Ricci soliton along ξ is given by

$$(L_\xi g)(X,Y)+2S(X,Y)+2\lambda g(X,Y)=0.$$

Definition 3.2.1. Let $(f,\xi_1,\xi_2,\ldots,\xi_s,\eta_1,\eta_2,\ldots,\eta_{\mathsf{s}},g)$ is the contact S-frame manifold, if V is in the linear span (combination) of $\xi_1,\xi_2,\ldots,\xi_{\mathsf{s}}$ then $V=c_1\xi_1+c_2\xi_2+\ldots+c_s\xi_s$ and the Ricci soliton is a triple (g,ξ_α,λ) with g is a Riemannian metric, $\xi_\alpha,(\alpha=1,2,....\mathsf{s})$ is a vector field and λ is a real scalar such that

$$\left(\sum_{i=1}^{\mathsf{s}}c_iL_{\xi_i}g\right)(X,Y)+2S(X,Y)+2\lambda g(X,Y)=0. \tag{3.2.7}$$

Equation (3.2.7) can be written

$$c_ig(\nabla_X\xi_\alpha,Y)+c_ig(\nabla_Y\xi_\alpha,X)+2S(X,Y)+2\lambda g(X,Y)=0. \tag{3.2.8}$$

Using (1.3.5) in (3.2.8) we get

$$c_ig(-fX,Y)+c_ig(-fY,X)+2S(X,Y)+2\lambda g(X,Y)=0. \tag{3.2.9}$$

From (3.2.6) and (3.2.9) we have

$$(\mathsf{S}(2n+\mathsf{S}-1)+\lambda)g(X,Y)=0. \tag{3.2.10}$$

Taking $X = Y = e_i$ in (3.2.10) and summing over $i = 1, 2, \ldots, 2n + \mathsf{S}$, we get the value of λ given by

$$\lambda = -\mathsf{S}(2n + \mathsf{S} - 1)(< 0).$$

Thus we state the following:

Theorem 3.2.2. *Ricci soliton in semi-symmetric* S*-manifold is shrinking.*

Corollary 3.2.3. *Ricci soliton in semi-symmetric* S*-manifold is steady if* $\mathsf{S} = 0$ *(Kaehler manifold) and is shrinking if* $\mathsf{S} = 1$ *(Sasakian manifold).*

3.3 Ricci soliton in S-manifolds satisfying $R \cdot C = 0$

The concircular curvature tensor C is given by

$$C(U, V)Z = R(U, V)Z - \frac{r}{2n(2n+1)}\{g(V, Z)U - g(U, Z)V\}, \tag{3.3.1}$$

Using (1.3.11), (1.3.12) and (1.3.13) in (3.3.1) we get

$$C(U, V)\xi_\alpha = \left[\mathsf{S} - \frac{r}{2n(2n+1)}\right] \sum_\alpha \{U\eta_\alpha(V) - \eta_\alpha(U)V\}, \tag{3.3.2}$$

$$C(\xi_\alpha, Y)Z = \left[\mathsf{S} - \frac{r}{2n(2n+1)}\right] \sum_\alpha \{g(V, Z)\xi_\alpha - V\eta_\alpha(Z)\}, \tag{3.3.3}$$

$$\eta_\alpha(C(X, Y)Z) = \left[\mathsf{S} - \frac{r}{2n(2n+1)}\right] \sum_\alpha \{g(V, Z)\eta_\alpha(U) - g(U, Z)\eta_\alpha(V)\}. \tag{3.3.4}$$

Let us assume that the condition $R((\xi_\alpha, Y) \cdot C)(U, V)W = 0$ holds on $\mathbb{M}$, then

$$R(\xi_\alpha, Y)C(U, V)W - C(R(\xi_\alpha, Y)U, V)W - C(U, R(\xi_\alpha, Y)V)W - C(U, V)R(\xi_\alpha, Y)W = 0. \tag{3.3.5}$$

Using (1.3.12) in (3.3.5), we get

$$\mathsf{S}\sum_{\alpha}\{g(Y,C(U,V)W)\xi_\alpha-\eta_\alpha(C(U,V)W)Y-g(Y,U)C(\xi_\alpha,V)W$$
$$+\eta_\alpha(U)C(Y,V)W-g(Y,V)C(U,\xi_\alpha)W+\eta_\alpha(V)C(U,Y)W$$
$$-g(Y,W)C(U,V)\xi_\alpha+\eta_\alpha(W)C(U,V)Y\}=0. \tag{3.3.6}$$

By taking an inner product of (3.3.6) with ξ_α then we get

$$\sum_{\alpha}\{\mathsf{S}C(U,V,W,Y)-\eta_\alpha(C(U,V)W)\eta_\alpha(Y)-g(Y,U)\eta_\alpha(C(\xi_\alpha,V)W)$$
$$+\eta_\alpha(U)\eta_\alpha(C(Y,V)W)-g(Y,V)\eta_\alpha(C(U,\xi_\alpha)W)+\eta_\alpha(V)\eta_\alpha(C(U,Y)W)$$
$$-g(Y,W)\eta_\alpha(C(U,V)\xi_\alpha)+\eta_\alpha(W)\eta_\alpha(C(U,V)Y)\}=0. \tag{3.3.7}$$

By using (3.3.2), (3.3.4) in (3.3.7) we have

$$C(U,V,W,Y)=\left[\mathsf{S}-\frac{r}{2n(2n+1)}\right]\{g(Y,U)g(V,W)-g(Y,V)g(U,W)\}. \tag{3.3.8}$$

Taking $U=Y=e_i$ in (3.3.8) and summing over $i=1,2,\ldots,2n+\mathsf{S}$ and using (3.3.1) we get

$$S(V,W)=\mathsf{S}(2n+\mathsf{S}-1)g(V,W). \tag{3.3.9}$$

Thus we state the following:

Theorem 3.3.1. *S-manifold satisfying the condition $R\cdot C=0$ is an Einstein manifold.*

Substituting (3.3.9) in (3.2.9) we have

$$(\mathsf{S}(2n+\mathsf{S}-1)+\lambda)g(X,Y)=0. \tag{3.3.10}$$

Taking $X=Y=e_i$ in (3.3.10) and summing over $i=1,2,\ldots,2n+\mathsf{S}$, we get the value of λ given by

$$\lambda = -\mathsf{S}(2n + \mathsf{S} - 1)(< 0).$$

Thus we state the following;

Theorem 3.3.2. *Ricci soliton in* S*-manifold satisfying the condition* $R \cdot C = 0$ *is shrinking.*

Corollary 3.3.3. *Ricci soliton in* S*-manifold satisfying* $R \cdot C = 0$ *is steady if* $\mathsf{S} = 0$ *(Kaehler manifold) and is shrinking if* $\mathsf{S} = 1$ *(Sasakian manifold).*

3.4 Ricci soliton in S-manifolds satisfying $C \cdot R = 0$

Let us assume that the condition $C((\xi_\alpha, Y) \cdot R)(U, V)W = 0$ holds on $\mathbb{M}$, then

$$C(\xi_\alpha, Y)R(U, V)W - R(C(\xi_\alpha, Y)U, V)W - R(U, C(\xi_\alpha, Y)V)W - R(U, V)C(\xi_\alpha, Y)W = 0. \tag{3.4.1}$$

Using (3.3.3) in (3.4.1), we get

$$\begin{aligned} \left[\mathsf{S} - \frac{r}{2n(2n+1)}\right] \sum_\alpha \{ & g(Y, R(U, V)W)\xi_\alpha - \eta_\alpha(R(U, V)W)Y \\ & -g(Y, U)R(\xi_\alpha, V)W + \eta_\alpha(U)R(Y, V)W - g(Y, V)R(U, \xi_\alpha)W \\ & +\eta_\alpha(V)R(U, Y)W - g(Y, W)R(U, V)\xi_\alpha + \eta_\alpha(W)R(U, V)Y\} = 0. \end{aligned} \tag{3.4.2}$$

Since $\left[\mathsf{S} - \frac{r}{2n(2n+1)}\right] \neq 0$, by taking an inner product of (3.4.2) with ξ_α then we get

$$\begin{aligned} & \sum_\alpha \{\mathsf{S}R(U, V, W, Y) - \eta_\alpha(R(U, V)W)\eta_\alpha(Y) - g(Y, U)\eta_\alpha(R(\xi_\alpha, V)W) \\ & +\eta_\alpha(U)\eta_\alpha(R(Y, V)W) - g(Y, V)\eta_\alpha(R(U, \xi_\alpha)W) + \eta_\alpha(V)\eta_\alpha(R(U, Y)W) \\ & -g(Y, W)\eta_\alpha(R(U, V)\xi_\alpha) + \eta_\alpha(W)\eta_\alpha(R(U, V)Y)\} = 0. \end{aligned} \tag{3.4.3}$$

By using (3.3.2), (3.3.4) in (3.4.3) we have

$$R(U,V,W,Y) = \mathsf{S}\{g(Y,U)g(V,W) - g(Y,V)g(U,W)\}. \tag{3.4.4}$$

Taking $U = Y = e_i$ in (3.4.4) and summing over $i = 1, 2, \ldots, 2n + \mathsf{S}$ we get

$$S(V,W) = \mathsf{S}(2n + \mathsf{S} - 1)g(V,W). \tag{3.4.5}$$

Thus we state the following:

Theorem 3.4.1. *S-manifold satisfying the condition $C \cdot R = 0$ is an Einstein manifold.*

Substituting (3.4.5) in (3.2.9) we have

$$(\mathsf{S}(2n + \mathsf{S} - 1) + \lambda)g(X,Y) = 0. \tag{3.4.6}$$

Taking $X = Y = e_i$ in (3.4.6) and summing over $i = 1, 2, \ldots, 2n + \mathsf{S}$, we get the value of λ given by

$$\lambda = -\mathsf{S}(2n + \mathsf{S} - 1)(< 0).$$

Thus we state the following;

Theorem 3.4.2. *Ricci soliton in S-manifold satisfying the condition $C \cdot R = 0$ is shrinking.*

Corollary 3.4.3. *Ricci soliton in S-manifold satisfying $C \cdot R = 0$ is steady if $\mathsf{S} = 0$ (Kaehler manifold) and is shrinking if $\mathsf{S} = 1$ (Sasakian manifold).*

3.5 Ricci soliton in S-manifolds satisfying $C \cdot C = 0$

Let us assume that the condition $C((\xi_\alpha, Y) \cdot C)(U, V)W = 0$ holds on $\mathbb{M}$, then

$$C(\xi_\alpha, Y)C(U, V)W - C(C(\xi_\alpha, Y)U, V)W - C(U, C(\xi_\alpha, Y)V)W - C(U, V)C(\xi_\alpha, Y)W = 0. \tag{3.5.1}$$

Using (3.3.3) in (3.5.1), we get

$$\begin{aligned}\left[\mathsf{S} - \frac{r}{2n(2n+1)}\right] \sum_\alpha \{g(Y, C(U, V)W)\xi_\alpha - \eta_\alpha(C(U, V)W)Y \\ -g(Y, U)C(\xi_\alpha, V)W + \eta_\alpha(U)C(Y, V)W - g(Y, V)C(U, \xi_\alpha)W \\ +\eta_\alpha(V)C(U, Y)W - g(Y, W)C(U, V)\xi_\alpha + \eta_\alpha(W)C(U, V)Y\} = 0.\end{aligned} \tag{3.5.2}$$

Since $\left[\mathsf{S} - \frac{r}{2n(2n+1)}\right] \neq 0$, by taking an inner product of (3.5.2) with ξ_α then we get

$$\begin{aligned}&\sum_\alpha \{\mathsf{S}C(U, V, W, Y) - \eta_\alpha(C(U, V)W)\eta_\alpha(Y) - g(Y, U)\eta_\alpha(C(\xi_\alpha, V)W) \\ &+\eta_\alpha(U)\eta_\alpha(C(Y, V)W) - g(Y, V)\eta_\alpha(C(U, \xi_\alpha)W) + \eta_\alpha(V)\eta_\alpha(C(U, Y)W) \\ &-g(Y, W)\eta_\alpha(C(U, V)\xi_\alpha) + \eta_\alpha(W)\eta_\alpha(C(U, V)Y)\} = 0.\end{aligned} \tag{3.5.3}$$

By using (3.3.2), (3.3.4) in (3.5.3) we have

$$C(U, V, W, Y) = \left[\mathsf{S} - \frac{r}{2n(2n+1)}\right] \{g(Y, U)g(V, W) - g(Y, V)g(U, W)\}. \tag{3.5.4}$$

Taking $U = Y = e_i$ in (3.5.4) and summing over $i = 1, 2, \ldots, 2n + \mathsf{S}$ and using (3.3.1) we get

$$S(V, W) = \mathsf{S}(2n + \mathsf{S} - 1)g(V, W). \tag{3.5.5}$$

Thus we state the following:

Theorem 3.5.1. *S-manifold satisfying the condition $C \cdot C = 0$ is an Einstein manifold.*

Substituting (3.5.5) in (3.2.9) we have

$$(\mathsf{S}(2n+\mathsf{S}-1)+\lambda)g(X,Y)=0. \tag{3.5.6}$$

Taking $X=Y=e_i$ in (3.5.6) and summing over $i=1,2,\ldots,2n+\mathsf{S}$, we get the value of λ given by

$$\lambda=-\mathsf{S}(2n+\mathsf{S}-1)(<0).$$

Thus we state the following:

Theorem 3.5.2. *Ricci soliton in* S*-manifold satisfying the condition* $C\cdot C=0$ *is shrinking.*

Corollary 3.5.3. *Ricci soliton in* S*-manifold satisfying* $C\cdot C=0$ *is steady if* $\mathsf{S}=0$ *(Kaehler manifold) and is shrinking if* $\mathsf{S}=1$ *(Sasakian manifold).*

3.6 Ricci soliton in pseudo-symmetric S-manifolds

An S-manifold is said to be Pseudo-symmetric if $R\cdot R=L_5Q(g,R)$.

$$(R(\xi_\alpha,Y)\cdot R)(U,V)W=L_5[((\xi_\alpha\wedge Y)\cdot R)(U,V)W], \tag{3.6.1}$$

$$\begin{aligned}&R(\xi_\alpha,Y)R(U,V)W-R(R(\xi_\alpha,Y)U,V)W-R(U,R(\xi_\alpha,Y)V)W\\&-R(U,V)R(\xi_\alpha,Y)W=L_5[(\xi_\alpha\wedge Y)R(U,V)W-R((\xi_\alpha\wedge Y)U,V)W\\&-R(U,(\xi_\alpha\wedge Y)V)W-R(U,V)(\xi_\alpha\wedge Y)W].\end{aligned} \tag{3.6.2}$$

Using (1.3.12) L.H.S of (3.6.2) becomes

$$\mathsf{S}\sum_{\alpha}\{g(Y,R(U,V)W)\xi_\alpha - \eta_\alpha(R(U,V)W)Y - g(Y,U)R(\xi_\alpha,V)W$$
$$+\eta_\alpha(U)R(Y,V)W - g(Y,V)R(U,\xi_\alpha)W + \eta_\alpha(V)R(U,Y)W$$
$$-g(Y,W)R(U,V)\xi_\alpha + \eta_\alpha(W)R(U,V)Y\}. \qquad (3.6.3)$$

By taking an inner product of above equation with ξ_α then we get

$$\mathsf{S}\sum_{\alpha}\{\mathsf{S}R(U,V,W,Y) - \eta_\alpha(R(U,V)W)\eta_\alpha(Y) - g(Y,U)\eta_\alpha(R(\xi_\alpha,V)W)$$
$$+\eta_\alpha(U)\eta_\alpha(R(Y,V)W) - g(Y,V)\eta_\alpha(R(U,\xi_\alpha)W) + \eta_\alpha(V)\eta_\alpha(R(U,Y)W)$$
$$-g(Y,W)\eta_\alpha(R(U,V)\xi_\alpha) + \eta_\alpha(W)\eta_\alpha(R(U,V)Y)\}. \qquad (3.6.4)$$

By using (1.3.11), (1.3.13) in (3.6.4) we have

$$\mathsf{S}\{\mathsf{S}R(U,V,W,Y) + \mathsf{S}^2 g(Y,V)g(U,W) - \mathsf{S}^2 g(Y,U)g(V,W)\}. \qquad (3.6.5)$$

Again using (1.3.12), R.H.S of (3.6.2) becomes

$$L_5[\sum_{\alpha}\{g(Y,R(U,V)W)\xi_\alpha - \eta_\alpha(R(U,V)W)Y - g(Y,U)R(\xi_\alpha,V)W$$
$$+\eta_\alpha(U)R(Y,V)W - g(Y,V)R(U,\xi_\alpha)W + \eta_\alpha(V)R(U,Y)W$$
$$-g(Y,W)R(U,V)\xi_\alpha + \eta_\alpha(W)R(U,V)Y\}]. \qquad (3.6.6)$$

By taking an inner product of above equation with ξ_α then we get

$$L_5[\sum_{\alpha}\{\mathsf{S}R(U,V,W,Y) - \eta_\alpha(R(U,V)W)\eta_\alpha(Y) - g(Y,U)\eta_\alpha(R(\xi_\alpha,V)W)$$
$$+\eta_\alpha(U)\eta_\alpha(R(Y,V)W) - g(Y,V)\eta_\alpha(R(U,\xi_\alpha)W) + \eta_\alpha(V)\eta_\alpha(R(U,Y)W)$$
$$-g(Y,W)\eta_\alpha(R(U,V)\xi_\alpha) + \eta_\alpha(W)\eta_\alpha(R(U,V)Y)\}]. \qquad (3.6.7)$$

By using (1.3.11), (1.3.13) in (3.6.7) we have

$$L_5[\mathsf{S}R(U,V,W,Y) + \mathsf{S}^2 g(Y,V)g(U,W) - \mathsf{S}^2 g(Y,U)g(V,W)]. \qquad (3.6.8)$$

Combining equations (3.6.5) and (3.6.8) we get

$$[L_5 - \mathsf{S}][\mathsf{S}R(U,V,W,Y) + \mathsf{S}^2 g(Y,V)g(U,W) - \mathsf{S}^2 g(Y,U)g(V,W)] = 0. \tag{3.6.9}$$

Therefore, either $L_5 = \mathsf{S}$ or

$$R(U,V,W,Y) = \mathsf{S}\{g(Y,U)g(V,W) - g(Y,V)g(U,W)\}. \tag{3.6.10}$$

Taking $U = Y = e_i$ in (3.6.10) and summing over $i = 1, 2, \ldots, 2n + \mathsf{S}$ we get

$$S(V,W) = \mathsf{S}(2n + \mathsf{S} - 1)g(V,W). \tag{3.6.11}$$

Thus we state the following;

Theorem 3.6.1. *Pseudo-symmetric S-manifold is an Einstein manifold provided* $L_5 \neq \mathsf{S}$.

Substituting (3.6.11) in (3.2.9) we have

$$(\mathsf{S}(2n + \mathsf{S} - 1) + \lambda)g(X,Y) = 0. \tag{3.6.12}$$

Taking $X = Y = e_i$ in (3.6.12) and summing over $i = 1, 2, \ldots, 2n + \mathsf{S}$, we get the value of λ given by

$$\lambda = -\mathsf{S}(2n + \mathsf{S} - 1)(< 0).$$

Thus we state the following:

Theorem 3.6.2. *Ricci soliton in pseudo-symmetric S-manifold is shrinking.*

Corollary 3.6.3. *Ricci soliton in pseudo-symmetric S-manifold is steady if* $\mathsf{S} = 0$ *(Kaehler manifold) and is shrinking if* $\mathsf{S} = 1$ *(Sasakian manifold).*

3.7 Ricci soliton in S-manifolds satisfying $R \cdot C = L_6 Q(g, C)$

Let us assume that the condition $R((\xi_\alpha, Y) \cdot C)(U, V)W = L_6[(\xi_\alpha \wedge Y) \cdot C](U, V)W$ holds on $\mathbb{M}$, then

$$R(\xi_\alpha, Y)C(U, V)W - C(R(\xi_\alpha, Y)U, V)W - C(U, R(\xi_\alpha, Y)V)W$$
$$-C(U, V)R(\xi_\alpha, Y)W = L_6[(\xi_\alpha \wedge Y)C(U, V)W - C((\xi_\alpha \wedge Y)U, V)W$$
$$-C(U, (\xi_\alpha \wedge Y)V)W - C(U, V)(\xi_\alpha \wedge Y)W]. \tag{3.7.1}$$

Using (1.3.12) L.H.S of (3.7.1) is

$$\mathsf{S}\sum_\alpha\{g(Y, C(U, V)W)\xi_\alpha - \eta_\alpha(C(U, V)W)Y - g(Y, U)C(\xi_\alpha, V)W$$
$$+\eta_\alpha(U)C(Y, V)W - g(Y, V)C(U, \xi_\alpha)W + \eta_\alpha(V)C(U, Y)W$$
$$-g(Y, W)C(U, V)\xi_\alpha + \eta_\alpha(W)C(U, V)Y\}. \tag{3.7.2}$$

By taking an inner product of above equation with ξ_α then we get

$$\mathsf{S}\sum_\alpha\{\mathsf{S}C(U, V, W, Y) - \eta_\alpha(C(U, V)W)\eta_\alpha(Y) - g(Y, U)\eta_\alpha(C(\xi_\alpha, V)W)$$
$$+\eta_\alpha(U)\eta_\alpha(C(Y, V)W) - g(Y, V)\eta_\alpha(C(U, \xi_\alpha)W) + \eta_\alpha(V)\eta_\alpha(C(U, Y)W)$$
$$-g(Y, W)\eta_\alpha(C(U, V)\xi_\alpha) + \eta_\alpha(W)\eta_\alpha(C(U, V)Y)\}. \tag{3.7.3}$$

By using (3.3.2), (3.3.4) in (3.7.3) we have

$$\mathsf{S}^2\left\{C(U, V, W, Y) - \left[\mathsf{S} - \frac{r}{2n(2n+1)}\right][g(Y, U)g(V, W) - g(Y, V)g(U, W)]\right\}. \tag{3.7.4}$$

Again using (1.3.12), R.H.S of (3.7.1) becomes

$$L_6 \sum_{\alpha} \{g(Y, C(U,V)W)\xi_\alpha - \eta_\alpha(C(U,V)W)Y - g(Y,U)C(\xi_\alpha, V)W$$
$$+\eta_\alpha(U)C(Y,V)W - g(Y,V)C(U,\xi_\alpha)W + \eta_\alpha(V)C(U,Y)W$$
$$-g(Y,W)C(U,V)\xi_\alpha + \eta_\alpha(W)C(U,V)Y\}. \tag{3.7.5}$$

By taking an inner product of above equation with ξ_α then we get

$$L_6 \sum_{\alpha} \{\mathsf{S}C(U,V,W,Y) - \eta_\alpha(C(U,V)W)\eta_\alpha(Y) - g(Y,U)\eta_\alpha(C(\xi_\alpha, V)W)$$
$$+\eta_\alpha(U)\eta_\alpha(C(Y,V)W) - g(Y,V)\eta_\alpha(C(U,\xi_\alpha)W) + \eta_\alpha(V)\eta_\alpha(C(U,Y)W)$$
$$-g(Y,W)\eta_\alpha(C(U,V)\xi_\alpha) + \eta_\alpha(W)\eta_\alpha(C(U,V)Y)\}. \tag{3.7.6}$$

By using (3.3.2), (3.3.4) in (3.7.6) we have

$$\mathsf{S}L_6 \left\{ C(U,V,W,Y) - \left[\mathsf{S} - \frac{r}{2n(2n+1)}\right] [g(Y,U)g(V,W) - g(Y,V)g(U,W)] \right\}. \tag{3.7.7}$$

Combining equations (3.7.4) and (3.7.7) we get

$$[\mathsf{S}L_6 - \mathsf{S}^2] \left\{ C(U,V,W,Y) - \left[\mathsf{S} - \frac{r}{2n(2n+1)}\right] [g(Y,U)g(V,W) - g(Y,V)g(U,W)] \right\} = 0. \tag{3.7.8}$$

Therefore, either $L_6 = \mathsf{S}$ or

$$C(U,V,W,Y) = \left[\mathsf{S} - \frac{r}{2n(2n+1)}\right] [g(Y,U)g(V,W) - g(Y,V)g(U,W)]. \tag{3.7.9}$$

Taking $U = Y = e_i$ in (3.7.9) and summing over $i = 1, 2, \ldots, 2n + \mathsf{S}$ we get

$$S(V,W) = \mathsf{S}(2n + \mathsf{S} - 1)g(V,W). \tag{3.7.10}$$

Thus we state the following:

Theorem 3.7.1. *An* S*-manifold satisfying the condition* $R\cdot C = L_6 Q(g,C)$ *is an Einstein manifold provided* $L_6 \neq \mathsf{S}$*.*

Substituting (3.7.10) in (3.2.9) we have

$$(\mathsf{S}(2n+\mathsf{S}-1)+\lambda)g(X,Y) = 0. \tag{3.7.11}$$

Taking $X = Y = e_i$ in (3.7.11) and summing over $i = 1, 2, \ldots, 2n+\mathsf{S}$, we get the value of λ given by

$$\lambda = -\mathsf{S}(2n+\mathsf{S}-1)(<0).$$

Thus we state the following:

Theorem 3.7.2. *Ricci soliton in* S*-manifold satisfying the condition* $R\cdot C = L_6 Q(g,C)$ *is shrinking.*

Corollary 3.7.3. *Ricci soliton in* S*-manifold satisfying* $R\cdot C = L_6 Q(g,C)$ *is steady if* $\mathsf{S}=0$ *(Kaehler manifold) and is shrinking if* $\mathsf{S}=1$ *(Sasakian manifold).*

3.8 Ricci soliton in S-manifolds satisfying $C\cdot R = L_7 Q(g,R)$

Let us assume that the condition $C((\xi_\alpha, Y)\cdot R)(U,V)W = L_7[(\xi_\alpha \wedge Y)\cdot R](U,V)W$ holds on $\mathbb{M}$, then

$$\begin{aligned}
&C(\xi_\alpha, Y)R(U,V)W - R(C(\xi_\alpha, Y)U,V)W - R(U,C(\xi_\alpha, Y)V)W \\
&-R(U,V)C(\xi_\alpha, Y)W = L_7[(\xi_\alpha \wedge Y)R(U,V)W - R((\xi_\alpha \wedge Y)U,V)W \\
&-R(U,(\xi_\alpha \wedge Y)V)W - R(U,V)(\xi_\alpha \wedge Y)W].
\end{aligned} \tag{3.8.1}$$

Using (3.4.2), (3.4.3), (3.6.6) and (3.6.7) in (3.8.1) we get

$$\left\{\mathsf{S}L_7 - \left[\mathsf{S} - \frac{r}{2n(2n+1)}\right]\right\}\{R(U,V,W,Y) - \mathsf{S}[g(Y,U)g(V,W) - g(Y,V)g(U,W)]\} = 0. \tag{3.8.2}$$

Therefore, either $L_7 = \mathsf{S} - \frac{r}{2n(2n+1)}$ or

$$R(U,V,W,Y) = \mathsf{S}\{g(Y,U)g(V,W) - g(Y,V)g(U,W)\}. \tag{3.8.3}$$

Taking $U = Y = e_i$ in (3.8.3) and summing over $i = 1, 2, \ldots, 2n + \mathsf{S}$ we get

$$S(V,W) = \mathsf{S}(2n + \mathsf{S} - 1)g(V,W). \tag{3.8.4}$$

Thus we state the following:

Theorem 3.8.1. *S-manifold satisfying the condition $C \cdot R = L_7 Q(g,R)$ is an Einstein manifold provided $L_7 \neq \mathsf{S} - \frac{r}{2n(2n+1)}$.*

Substituting (3.8.4) in (3.2.9) we have

$$(\mathsf{S}(2n + \mathsf{S} - 1) + \lambda)g(X,Y) = 0. \tag{3.8.5}$$

Taking $X = Y = e_i$ in (3.8.5) and summing over $i = 1, 2, \ldots, 2n + \mathsf{S}$, we get the value of λ given by

$$\lambda = -\mathsf{S}(2n + \mathsf{S} - 1)(< 0).$$

Thus we state the following:

Theorem 3.8.2. *Ricci soliton in S-manifold satisfying the condition $C \cdot R = L_7 Q(g,R)$ is shrinking.*

Corollary 3.8.3. *Ricci soliton in S-manifold satisfying $C \cdot R = L_7 Q(g,R)$ is steady if $\mathsf{S} = 0$ (Kaehler manifold) and is shrinking if $\mathsf{S} = 1$ (Sasakian manifold).*

3.9 Ricci soliton in S-manifolds satisfying $C \cdot C = L_8 Q(g, C)$

Let us assume that the condition $C((\xi_\alpha, Y) \cdot C)(U, V)W = L_8[(\xi_\alpha \wedge Y) \cdot C](U, V)W$ holds on $\mathbb{M}$, then

$$C(\xi_\alpha, Y)C(U, V)W - C(C(\xi_\alpha, Y)U, V)W - C(U, C(\xi_\alpha, Y)V)W$$
$$-C(U, V)C(\xi_\alpha, Y)W = L_8[(\xi_\alpha \wedge Y)C(U, V)W - C((\xi_\alpha \wedge Y)U, V)W$$
$$-C(U, (\xi_\alpha \wedge Y)V)W - C(U, V)(\xi_\alpha \wedge Y)W]. \tag{3.9.1}$$

Using (3.5.2), (3.5.3), (3.7.5) and (3.7.6) in (3.9.1) we get either $L_8 = s - \frac{r}{2n(2n+1)}$ or

$$C(U, V, W, Y) - \left[\mathsf{S} - \frac{r}{2n(2n+1)}\right][g(Y, U)g(V, W) - g(Y, V)g(U, W)] = 0. \tag{3.9.2}$$

Taking $U = Y = e_i$ in (3.9.2) and summing over $i = 1, 2, \ldots, 2n + \mathsf{S}$, using (3.3.1) we get

$$S(V, W) = \mathsf{S}(2n + \mathsf{S} - 1)g(V, W). \tag{3.9.3}$$

Thus we state the following:

Theorem 3.9.1. *S-manifold satisfying the condition $C \cdot C = L_8 Q(g, C)$ is an Einstein manifold provided $L_8 \neq S - \frac{r}{2n(2n+1)}$.*

Substituting (3.9.3) in (3.2.9) we have

$$(\mathsf{S}(2n + \mathsf{S} - 1) + \lambda)g(X, Y) = 0 \tag{3.9.4}$$

Taking $X = Y = e_i$ in (3.9.4) and summing over $i = 1, 2, \ldots, 2n + \mathsf{S}$, we get the value of λ given by

$$\lambda = -\mathsf{S}(2n + \mathsf{S} - 1)(< 0).$$

Thus we state the following:

Theorem 3.9.2. *Ricci soliton in* S*-manifold satisfying the condition* $C \cdot C = L_8 Q(g, C)$ *is shrinking.*

Corollary 3.9.3. *Ricci soliton in* S*-manifold satisfying* $C \cdot C = L_8 Q(g, C)$ *is steady if* $\mathsf{S} = 0$ *(Kaehler manifold) and is shrinking if* $\mathsf{S} = 1$ *(Sasakian manifold).*

3.10 Irrotational τ-curvature tensor in S-manifolds

In a $(2n+1)$-dimensional Riemannian manifold $\mathbb{M}$, the τ-curvature tensor [83] is given by

$$\begin{aligned} \tau(X,Y)Z &= a_0 R(X,Y)Z + a_1 S(Y,Z)X + a_2 S(X,Z)Y + a_3 S(X,Y)Z \\ &+ a_4 g(Y,Z)QX + a_5 g(X,Z)QY + a_6 g(X,Y)QZ \\ &+ a_7 r[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{3.10.1}$$

where R, S, Q and r are the curvature tensor, the Ricci tensor, the Ricci operator and the scalar curvature, respectively.

In particular, the τ-curvature tensor is reduced to be [83]

1. The quasi-conformal curvature tensor $\tilde{C}$ if

   $a_1 = -a_2 = a_4 = -a_5; a_3 = a_6 = 0; a_7 = -\frac{1}{2n+1}\left(\frac{a_0}{2n} + 2a_1\right);$

2. the conformal curvature tensor V if

   $a_0 = 1; a_1 = -a_2 = a_4 = -a_5 = -\frac{1}{2n-1}; a_3 = a_6 = 0; a_7 = \frac{1}{2n(2n-1)};$

3. The conharmonic curvature tensor L if

   $a_0 = 1; a_1 = -a_2 = a_4 = -a_5 = -\frac{1}{2n-1}; a_3 = a_6 = 0; a_7 = 0;$

4. The concircular curvature tensor C if

   $a_0 = 1; a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 0; a_7 = \frac{-1}{2n(2n+1)};$

5. The pseudo-projective curvature tensor P if

   $a_1 = -a_2; a_3 = a_4 = a_5 = a_6 = 0; a_7 = -\frac{1}{2n+1}\left(\frac{a_0}{2n} + a_1\right);$

6. The projective curvature tensor P_* if

   $a_0 = 1; a_1 = -a_2 = -\frac{1}{2n}; a_3 = a_4 = a_5 = a_6 = 0 = a_7 = 0;$

7. The M-projective curvature tensor if

   $a_0 = 1; a_1 = -a_2 = a_4 = -a_5 = -\frac{1}{4n}; a_3 = a_6 = a_7 = 0;$

8. The W_0-curvature tensor if

   $a_0 = 1; a_1 = -a_5 = -\frac{1}{2n}; a_2 = a_3 = a_4 = a_6 = a_7 = 0;$

9. The W_0^*-curvature tensor if

   $a_0 = 1; a_1 = -a_5 = \frac{1}{2n}; a_2 = a_3 = a_4 = a_6 = a_7 = 0;$

10. The W_1-curvature tensor if

    $a_0 = 1; a_1 = -a_2 = \frac{1}{2n}; a_3 = a_4 = a_5 = a_6 = a_7 = 0;$

11. The W_1^*-curvature tensor if

    $a_0 = 1; a_1 = -a_2 = -\frac{1}{2n}; a_3 = a_4 = a_5 = a_6 = a_7 = 0;$

12. The W_2-curvature tensor if

    $a_0 = 1; a_4 = -a_5 = -\frac{1}{2n}; a_1 = a_2 = a_3 = a_6 = a_7 = 0;$

13. The W_3-curvature tensor if

    $a_0 = 1; a_2 = -a_4 = -\frac{1}{2n}; a_1 = a_3 = a_5 = a_6 = a_7 = 0;$

14. The W_4-curvature tensor if

$$a_0 = 1; a_5 = -a_6 = \frac{1}{2n}; a_1 = a_2 = a_3 = a_4 = a_7 = 0;$$

15. The W_5-curvature tensor if

$$a_0 = 1; a_2 = -a_5 = -\frac{1}{2n}; a_1 = a_3 = a_4 = a_6 = a_7 = 0;$$

16. The W_6-curvature tensor if

$$a_0 = 1; a_1 = -a_6 = -\frac{1}{2n}; a_2 = a_3 = a_4 = a_5 = a_7 = 0;$$

17. The W_7-curvature tensor if

$$a_0 = 1; a_1 = -a_4 = -\frac{1}{2n}; a_2 = a_3 = a_5 = a_6 = a_7 = 0;$$

18. The W_8-curvature tensor if

$$a_0 = 1; a_1 = -a_3 = -\frac{1}{2n}; a_2 = a_4 = a_5 = a_6 = a_7 = 0;$$

19. The W_9-curvature tensor if

$$a_0 = 1; a_3 = -a_4 = \frac{1}{2n}; a_1 = a_2 = a_5 = a_6 = a_7 = 0;$$

Put $Z = \xi$ in (3.10.1) and using (1.3.9), (1.3.10) and (1.3.11) we get,

$$\tau(X,Y)\xi = k_1\eta(Y)X + k_2\eta(X)Y + k_3 g(X,Y)\xi + k_4\eta(X)\eta(Y)\xi, \tag{3.10.2}$$

where

$$k_1 = a_0\mathsf{S} + a_1 b_3 + a_4 b_3 + a_7 r, \qquad k_2 = -a_0 s + a_2 b_3 + a_5 b_3 - a_7 r,$$

$$k_3 = a_3 b_1 + a_6 b_3, \qquad k_4 = a_3 b_2$$

Definition 3.10.1. The rotation (curl) of τ-curvature tensor on a Riemannian manifold is given by

$$Rot\tau = (\nabla_U\tau)(X,Y,Z) + (\nabla_X\tau)(U,Y,Z) + (\nabla_Y\tau)(U,X,Z) - (\nabla_Z\tau)(X,Y,U). \tag{3.10.3}$$

By virtue of second Bianchi identity

$$(\nabla_U \tau)(X,Y,Z) + (\nabla_X \tau)(U,Y,Z) + (\nabla_Y \tau)(U,X,Z) = 0. \tag{3.10.4}$$

Using (3.10.4), equation (3.10.3) reduces to

$$curl \tau = -(\nabla_Z \tau)(X,Y,U). \tag{3.10.5}$$

If the τ-curvature tensor is irrotational then $curl\tau = 0$ and by (3.10.5) we have

$$(\nabla_Z \tau)(X,Y)U = 0. \tag{3.10.6}$$

Which implies,

$$\nabla_Z\{\tau(X,Y)U\} = \tau(\nabla_Z X, Y)U + \tau(X, \nabla_Z Y)U + \tau(X,Y)\nabla_Z U. \tag{3.10.7}$$

Put $U = \xi_\alpha$ in the above equation, we have

$$\nabla_Z\{\tau(X,Y)\xi_\alpha\} = \tau(\nabla_Z X, Y)\xi_\alpha + \tau(X, \nabla_Z Y)\xi_\alpha + \tau(X,Y)\nabla_Z \xi_\alpha. \tag{3.10.8}$$

Theorem 3.10.1. *If the τ-curvature tensor in S-manifold is irrotational, then the manifold is η-Einstein.*

Proof. Using equation (3.10.2) in (3.10.8) we get

$$\begin{aligned} -\tau(X,Y)fZ &= k_1(\nabla_Z\eta)(Y)X + k_2(\nabla_Z\eta)(X)Y + k_3 g(X,Y)(-fX) \\ &+k_4\{(\nabla_Z\eta)(X)\eta(Y)\xi_\alpha + (\nabla_Z\eta)(Y)\eta(X)\xi_\alpha - \eta(X)\eta(Y)\xi_\alpha\}. \end{aligned} \tag{3.10.9}$$

By virtue of (1.3.5) in (3.10.9) we have

$$\begin{aligned} -\tau(X,Y)fZ &= -k_1 g(fZ,Y)X - k_2 g(fZ,X)Y - k_3 g(X,Y)fZ \\ &+k_4\{-g(fZ,X)\eta(Y)\xi_\alpha - g(fZ,Y)\eta(X)\xi_\alpha - \eta(X)\eta(Y)fZ\}. \end{aligned} \tag{3.10.10}$$

Replace Z by fZ in (3.10.10) and using (1.3.1) we have

$$\begin{aligned} \tau(X,Y)Z &= k_1 g(Y,Z)X + k_2 g(X,Z)Y + k_3 g(X,Y)Z \\ &+ k_4\{g(X,Z)\eta(Y)\xi_\alpha + g(Y,Z)\eta(X)\xi_\alpha + \eta(X)\eta(Y)Z\} \end{aligned} \tag{3.10.11}$$

Using (3.10.1) and (3.10.11) we can write

$$\begin{aligned} a_0 R(X,Y,Z,W) &= k_1 g(Y,Z)g(X,W) + k_2 g(X,Z)g(Y,W) + k_3 g(X,Y)g(Z,W) \\ &+ k_4\{g(X,Z)\eta(Y)\eta(W) + g(Y,Z)\eta(X)\eta(W) + \eta(X)\eta(Y)g(Z,W)\} \\ &- a_1 S(Y,Z)g(X,W) - a_2 S(X,Z)g(Y,W) - a_3 S(X,Y)g(Z,W) \\ &- a_4 g(Y,Z)g(QX,W) - a_5 g(X,Z)g(QY,W) - a_6 g(X,Y)g(QZ,W) \\ &+ a_7 r[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \end{aligned} \tag{3.10.12}$$

Let $\{e_i\}, i = 1, 2, \ldots, (2n + \mathsf{S})$ be an orthonormal basis of the tangent space. Then summing for $1 \leq i \leq (2n + \mathsf{S})$ of the relation (3.10.12) with $X = W = e_i$ yields the Ricci tensor S is given by

$$S(Y,Z) = \nu g(X,Y) + \omega \eta_\alpha(X)\eta_\alpha(Y). \tag{3.10.13}$$

where,

$$\begin{aligned} \nu &= \frac{(2n+\mathsf{S})k_1 + k_2 + k_3 + k_4 - (2n+\mathsf{S}-1)ra_7 - ra_4}{a_0 + (2n+\mathsf{S})a_1 + a_2 + a_3 + a_5 + a_6}, \\ \omega &= \frac{2k_4}{a_0 + (2n+\mathsf{S})a_1 + a_2 + a_3 + a_5 + a_6}. \end{aligned}$$

From (3.10.13), $\mathbb{M}$ is an η-Einstein manifold. □

The above theorem (3.10.1) is shown in tabular form for different curvatures which can be obtained independently for S manifold.

Curvature tensor	Manifold	Ricci tensor S
Quasi conformal	Einstein	$S=\left\{\frac{(2n+s-1)(a_0s+2a_1b_3)-a_1r}{a_0+(2n+s-1)a_1}\right\}g$
Conformal	Einstein	$S=\left(\frac{2n-1}{1-s}\right)\left\{(2n+s-1)\left(s-\frac{2b_3}{2n-1}+\frac{r}{2n(2n-1)}\right)+\frac{r}{2n(2n-1)}\right\}g$
Conharmonic	Einstein	$S=\left(\frac{2n-1}{1-s}\right)\left\{(2n+s-1)\left(s-\frac{2b_3}{2n-1}\right)+\frac{r}{2n-1}\right\}g$
Concircular	Einstein	$S=s(2n+s-1)g$
Projective	Einstein	$S=\left\{\frac{2n(2n+s-1)}{(1-s)}\right\}\left(s-\frac{b_3}{2n}\right)g$
Pseudo projective	Einstein	$S=\left\{\frac{(2n+s-1)(a_0s+a_1b_3)}{a_0+(2n+s-1)a_1}\right\}g$
M-projective	Einstein	$S=\left(\frac{4n}{2n-s+2}\right)\left\{(2n+s-1)\left(s-\frac{b_3}{2n}\right)+\frac{r}{4n}\right\}g$
W_0	Einstein	$S=\left(\frac{2n(2n+s-1)}{(1-s)}\right)\left(s-\frac{b_3}{2n}\right)g$
W_0^*	Einstein	$S=\left(\frac{2n(2n+s-1)}{(4n+s-1)}\right)\left(s+\frac{b_3}{2n}\right)g$
W_1	Einstein	$S=\left(\frac{2n(2n+s-1)}{(4n+s-1)}\right)\left(s+\frac{b_3}{2n}\right)g$
W_1^*	Einstein	$S=\left(\frac{2n(2n+s-1)}{(1-s)}\right)\left(s-\frac{b_3}{2n}\right)g$
W_2	Einstein	$S=\left(\frac{2n(2n+s-1)}{(2n+1)}\right)\left(s-\frac{b_3}{2n}\right)g$
W_3	Einstein	$S=\left(\frac{2n}{2n-1}\right)\left\{(2n+s-1)\left(s+\frac{b_3}{2n}\right)-\frac{r}{2n}\right\}g$
W_4	Einstein	$S=s(2n+s-1)g$
W_5	Einstein	$S=s(2n+s-1)g$
W_6	Einstein	$S=\left\{\frac{2n(2n+s-1)}{1-s}\right\}\left(s-\frac{b_3}{2n}\right)g$
W_7	Einstein	$S=-\left(\frac{2n}{s}\right)\left\{(2n+s)\left(s-\frac{b_3}{2n}\right)-s-\frac{r}{2n}\right\}g$
W_8	η-Einstein	$S=\left(\frac{2n}{1-s}\right)\left\{(2n+s)\left(s-\frac{b_3}{2n}\right)-s+\frac{b_1}{2n}-\frac{b_2}{2n}\right\}g$ $+\sum_\alpha\left(\frac{2b_2}{1-s}\right)\eta_\alpha\otimes\eta_\alpha$
W_9	η-Einstein	$S=\left(\frac{2n}{2n+1}\right)\left\{(2n+s)\left(s-\frac{b_3}{2n}\right)-s+\frac{b_1}{2n}+\frac{b_2}{2n}+\frac{r}{2n}\right\}g$ $+\sum_\alpha\left(\frac{2b_2}{2n+1}\right)\eta_\alpha\otimes\eta_\alpha$

From (3.10.13) and (3.2.9) we have

$$(\nu+\lambda)g(X,Y)+\omega\eta_\alpha(X)\eta_\alpha(Y)=0 \tag{3.10.14}$$

Taking $X=Y=e_i$ in (3.10.14) and summing over $i=1,2,.......2n+\mathsf{S}$, we get the value of λ

$$\lambda=-\left(\nu+\frac{\omega}{2n+\mathsf{S}}\right). \tag{3.10.15}$$

Thus we state the following theorem:

Theorem 3.10.2. *The Ricci soliton in irrotational τ-curvature tensor in S manifolds is*

1. *shrinking if $\nu, \omega > 0$.*

2. *steady if if $\nu, \omega = 0$.*

3. *expanding if if $\nu, \omega < 0$.*

3.11 Conclusion

The influential results finding of this chapter are as follows:

- An S-manifold with semi-symmetric conditions such as $R \cdot R = 0$, $R \cdot C = 0$, $C \cdot R = 0$ and $C \cdot C = 0$ is an Einstein manifold.

- An S-manifold with pseudo-symmetric conditions such as $R \cdot R = L_5 Q(g, R)$, $R \cdot C = L_6 Q(g, C)$, $C \cdot R = L_7 Q(g, R)$ and $C \cdot C = L_8 Q(g, C)$ is an Einstein manifold provided $L_5 \neq \mathsf{S}$, $L_6 \neq \mathsf{S}$, $L_7 \neq \mathsf{S} - \frac{r}{2n(2n+1)}$ and $L_8 \neq \mathsf{S} - \frac{r}{2n(2n+1)}$ respectively.

- Ricci soliton for S-manifold with above mentioned semi-symmetric and pseudo-symmetric conditions is shrinking.

- If (g, ξ_α, λ) is a Ricci soliton in semi-symmetric S-manifold and pseudo-symmetric S-manifold, then it is steady if $\mathsf{S} = 0$ (Kaehler manifold) and is shrinking if $\mathsf{S} = 1$ (Sasakian manifold).

- If the τ-curvature tensor in S-manifold is irrotational, then the manifold is η-Einstein.

- If the τ-curvature tensor is irrotational, then the Ricci soliton in S manifolds is

1. shrinking if $\nu, \omega > 0$
2. steady if if $\nu, \omega = 0$
3. expanding if if $\nu, \omega < 0$.

where,

$$\begin{aligned} \nu &= \frac{(2n+\mathsf{S})k_1 + k_2 + k_3 + k_4 - (2n+\mathsf{S}-1)ra_7 - ra_4}{a_0 + (2n+\mathsf{S})a_1 + a_2 + a_3 + a_5 + a_6}, \\ \omega &= \frac{2k_4}{a_0 + (2n+\mathsf{S})a_1 + a_2 + a_3 + a_5 + a_6}. \end{aligned}$$

CHAPTER 4

Publications based on this Chapter

- *A Study on Ricci Soliton in Ricci-Generalized Pseudo-symmetric Sasakian Manifolds,* Proceedings of the national workshop on numerical solutions to Fluid Mechanics, ISBN: 978-81-926808-5-9, (2016), 141-150.

- *Semi-symmetric Generalized Sasakian Space Forms,* (Communicated).

Chapter 4

On Sasakian Manifolds

4.1 Introduction

If a contact metric structure (ϕ, ξ, η, g) is normal, then the structure will referred as normal contact metric structure or Sasakian structure. A Sasakian structure is in same sense an analogue of a Kaehler structure on an almost Hermitian manifold, i.e., the almost complex structure J is parallel with respect to the Hermitian metric. The Blair states that, if M is a Riemannian manifold admitting a unit Killing Vector field ξ, such that equation (1.1.7) holds on $\mathbb{M}$ is a Sasakian manifold. In particular, the usual contact metric structure on an odd dimensional sphere is a Sasakian structure. In 2008, De, Jun and Gazi studied Sasakian manifolds with quasi-conformal curvature tensor [27]. In 2009, the authors obtained the results on ϕ-quasi conformally symmetric Sasakian manifolds [29]. In 2011, He and Zhu showed that a Sasakian metric which also satisfies the gradient Ricci soliton is necessarily Einstein [41].

The nature of a Riemannian manifold mostly depends on the curvature tensor R of the manifold. It is well known that the sectional curvatures of a manifold determine curvature tensor completely. A Riemannian manifold with constant sectional curvature c is known as real-space-form and its curvature tensor is given by

$$R(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y\}.$$

A Sasakian manifold with constant ϕ-sectional curvature is a Sasakian-spaceform and it has a specific form of its curvature tensor. Similar notion also holds for Kenmotsu and cosymplectic space-forms. In order to generalize such spaceforms in a common frame Alegre, Blair and Carriazo introduced the notion of generalized Sasakian-space-forms in 2004. In the context of generalized Sasakian-space-forms, Kim [51] studied locally symmetric properties of generalized Sasakian-space-forms. In [28] De and Sarkar have studied some symmetry properties of generalized Sasakian-space-forms regarding the projective curvature tensor. In [67] Prakasha studied some pseudosymmetric properties of generalized Sasakian-space-forms with Weyl conformal curvature tensor. In [15] Bagewadi and Ingalahalli studied some results of C-Bochner curvature tensor and τ-curvature tensor of a generalized Sasakian space forms. In this chapter we study the Ricci soliton for Ricci-generalized pseudo-symmetric Sasakian manifold with different curvature tensors. also we close attention to generalized Sasakian space forms satisfying certain curvature conditions on quasi conformal curvature tensor.

4.2 Ricci-generalized pseudo-symmetric Sasakian manifold

If the tensors $R \cdot R$ and $Q(S, R)$ are linearly dependent then $\mathbb{M}$ is called Ricci-generalized pseudo-symmetric. This is equivalent to;

$$R \cdot R = L_1 Q(S, R) \qquad (4.2.1)$$

holding on the set $U_1 = \{X \in M; Q(S,R) \neq 0 \text{ at X}\}$, where L_1 is some function on U_1. The tensors $R \cdot R$, $Q(S,R)$ and $X \wedge_S Y$ are defined by;

$$\begin{aligned}(R \cdot R)(U,V,W;X,Y) &= R(X,Y)R(U,V)W - R(R(X,Y)U,V)W \\ &- R(U,R(X,Y)V)W - R(U,V)R(X,Y)W, \qquad (4.2.2)\\ Q(S,R)(U,V,W;X,Y) &= (X \wedge_S Y)R(U,V)W - R((X \wedge_S Y)U,V)W \\ &- R(U,(X \wedge_S Y)V)W - R(U,V)(X \wedge_S Y)W, \qquad (4.2.3)\end{aligned}$$

respectively, where $X \wedge_S Y$ is an endomorphism given by

$$(X \wedge_S Y)Z = S(Y,Z)X - S(X,Z)Y.$$

Assume that $\mathbb{M}$ is Ricci-generalized pseudo-symmetric Sasakian manifold

$$(R \cdot R)(U,V,W;\xi,Y) = L_1 Q(S,R)(U,V,W;\xi,Y). \qquad (4.2.4)$$

Now using (4.2.2), L.H.S of (4.2.4) yields

$$\begin{aligned}(R(\xi,Y) \cdot R)(U,V,W) &= R(\xi,Y)R(U,V)W - R(R(\xi,Y)U,V)W \\ &- R(U,R(\xi,Y)V)W - R(U,V)R(\xi,Y)W. \qquad (4.2.5)\end{aligned}$$

Taking inner product of (4.2.5) with ξ and by virtue of (1.1.9) and (1.1.10) we get

$$g((R(\xi,Y) \cdot R)(U,V,W),\xi) = R(U,V,W,Y) - g(Y,U)g(V,W) + g(U,W)g(V,Y). \qquad (4.2.6)$$

Again by using (4.2.3), we can write R.H.S of (4.2.4) as

$$\begin{aligned}Q(S,R)(U,V,W;\xi,Y) &= (\xi \wedge_S Y)R(U,V)W - R((\xi \wedge_S Y)U,V)W \\ &- R(U,(\xi \wedge_S Y)V)W - R(U,V)(\xi \wedge_S Y)W. \qquad (4.2.7)\end{aligned}$$

Taking inner product of (4.2.7) with ξ and by virtue of (1.1.9), (1.1.11) we get

$$\begin{aligned} g(Q(S,R)(U,V,W;\xi,Y),\xi) &= S(Y,R(U,V)W)-\eta(Y)S(\xi,R(U,V)W) \\ &- S(Y,U)\eta(R(\xi,V)W)+2n\eta(U)\eta(R(Y,V)W) \\ &- S(Y,V)\eta(R(U,\xi)W)+2n\eta(V)\eta(R(U,Y)W) \\ &+ 2n\eta(W)\eta(R(U,V)Y). \end{aligned} \tag{4.2.8}$$

Using equations (4.2.6), (4.2.8) in (4.2.4) we obtain the following

$$[2nL_1-1][R(U,V,W,Y)-g(Y,U)g(V,W)+g(U,W)g(Y,V)]=0. \tag{4.2.9}$$

Therefore either $L_1=\frac{1}{2n}$ or

$$R(U,V,W,Y)=g(Y,U)g(V,W)-g(U,W)g(Y,V). \tag{4.2.10}$$

Let $\{e_1,\dots,e_n,e_{n+1}=\phi(e_1),e_{n+2}=\phi(e_2),\dots,e_{2n}=\phi(e_n),\xi\}$ be an orthonormal basis of the tangent space at each point of the manifold. Putting $Y=U=e_i$ in (4.2.10) and taking summation over i, $(1\le i\le(2n+1))$, we get

$$\begin{aligned} &R(e_1,V,W,e_1)+\ldots+R(e_n,V,W,e_n)+R(\phi e_1,V,W,\phi e_1)+\ldots+R(\phi e_n,V,W,\phi e_n) \\ &+R(\xi,V,W,\xi)=[g(e_1,e_1)+\ldots+g(e_n,e_n)]g(V,W) \\ &+[g(\phi e_1,\phi e_1)+\ldots+g(\phi e_n,\phi e_n)]g(V,W) \\ &+g(\xi,\xi)g(V,W)-[g(e_1,W)g(e_1,V)+\ldots+g(e_n,W)g(e_n,V)] \\ &-[g(\phi e_1,W)g(\phi e_1,V)+\ldots+g(\phi e_n,W)g(\phi e_n,V)]-g(\xi,W)g(\xi,V) \end{aligned} \tag{4.2.11}$$

Set,

$$V=\sum_{i=1}^{n}v_ie_i+\sum_{j=1}^{n}v_j\phi e_j+v_{2n+1}\xi, \qquad W=\sum_{i=1}^{n}w_ie_i+\sum_{j=1}^{n}w_j\phi e_j+w_{2n+1}\xi \tag{4.2.12}$$

Now using $g(R(\phi X, \phi, Y)\phi Z, \phi W) = g(R(X, Y)Z, W)$ and by virtue of (1.1.2) and (4.2.12) in (4.2.11) we have

$$S(V, W) = 2ng(V, W). \tag{4.2.13}$$

We can state the following:

Theorem 4.2.1. *A $(2n + 1)$-dimensional Ricci-generalized pseudo-symmetric Sasakian manifold is Einstein provided $L_1 \neq \frac{1}{2n}$.*

Now from the Ricci soliton equation (1.4.1) we have

$$g(\nabla_X V, Y) + g(\nabla_Y V, X) + 2S(X, Y) + 2\lambda g(X, Y) = 0. \tag{4.2.14}$$

Substituting $V = \xi$ in (4.2.14) and by virtue of (1.1.5) we have

$$S(X, Y) = -2\lambda g(X, Y). \tag{4.2.15}$$

Compare equations (4.2.13) and (4.2.15)

$$(\lambda + 2n)g(V, W) = 0. \tag{4.2.16}$$

Taking $V = W = e_i$ in (4.2.16), summing over $i = 1, 2, \ldots, 2n + 1$ we get

$$\lambda = -2n.$$

Thus we have the following:

Corollary 4.2.2. *A Ricci soliton in Ricci-generalized pseudo-symmetric Sasakian manifold is shrinking.*

4.3 Pseudo-projective Ricci-generalized pseudo -symmetric Sasakian manifold

The pseudo-projective curvature tensor P is defined by

$$\begin{aligned}\bar{P}(U,V)Z &= aR(U,V)Z + b[S(V,Z)U - S(U,Z)V] \\ &- \frac{r}{2n+1}\left(\frac{a}{2n}+b\right)[g(V,Z)U - g(U,Z)V],\end{aligned} \tag{4.3.1}$$

where $a, b \neq 0$ are constants. Taking $Z = \xi$ in (4.3.1), using (1.1.7) and (1.1.11) we get

$$\bar{P}(U,V)\xi = \gamma[\eta(V)U - \eta(U)V]. \tag{4.3.2}$$

Similarly using (1.1.10) and (1.1.11) in (4.3.1) we get,

$$\eta(\bar{P}(U,V)Z) = \gamma[g(V,Z)\eta(U) - g(U,Z)\eta(V)]. \tag{4.3.3}$$

where $\gamma = (a+2nb) - \frac{r}{2n+1}\left(\frac{a}{2n}+b\right)$ Assume that $\mathbb{M}$ is pseudo-projective Ricci-generalized pseudo symmetric Sasakian manifold, then

$$(R \cdot \bar{P})(U,V,W;\xi,Y) = L_2 Q(S,\bar{P})(U,V,W;\xi,Y) \tag{4.3.4}$$

holds on $\mathbb{M}$.

L.H.S. of equation (4.3.4) takes the form

$$\begin{aligned}(R(\xi,Y)\cdot\bar{P})(U,V,W) &= R(\xi,Y)\bar{P}(U,V)W - \bar{P}(R(\xi,Y)U,V)W \\ &- \bar{P}(U,R(\xi,Y)V)W - \bar{P}(U,V)R(\xi,Y)W.\end{aligned} \tag{4.3.5}$$

Taking inner product of (4.3.5) with ξ and by virtue of (1.1.9), (1.1.10), (4.3.2) and (4.3.3) we get

$$\begin{aligned}g((R(\xi,Y)\cdot\bar{P})(U,V,W),\xi) &= \bar{P}(U,V,W,Y) \\ &- \gamma[g(Y,U)g(V,W) + g(U,W)g(V,Y)].\end{aligned} \tag{4.3.6}$$

R.H.S. of equation (4.3.4) takes the form

$$\begin{aligned} Q(S,\bar{P})(U,V,W;\xi,Y) &= (\xi\wedge_S Y)\bar{P}(U,V)W - \bar{P}((\xi\wedge_S Y)U,V)W \\ &- \bar{P}(U,(\xi\wedge_S Y)V)W - \bar{P}(U,V)(\xi\wedge_S Y)W. \qquad (4.3.7) \end{aligned}$$

Taking inner product of (4.3.7) with ξ and by virtue of (1.1.9), (1.1.11), (4.3.2) and (4.3.3) we get

$$\begin{aligned} g(Q(S,\bar{P})(U,V,W;\xi,Y),\xi) &= S(Y,\bar{P}(U,V)W) - \eta(Y)S(\xi,\bar{P}(U,V)W) \\ &- S(Y,U)\eta(\bar{P}(\xi,V)W) + 2n\eta(U)\eta(\bar{P}(Y,V)W) \\ &- S(Y,V)\eta(\bar{P}(U,\xi)W) + 2n\eta(V)\eta(\bar{P}(U,Y)W) \\ &+ 2n\eta(W)\eta(\bar{P}(U,V)Y). \qquad (4.3.8) \end{aligned}$$

Using the equations (4.3.6), (4.3.8) in (4.3.4) we obtain

$$[2nL_2-1][\bar{P}(U,V,W,Y)-\gamma[g(Y,U)g(V,W)+g(U,W)g(Y,V)]=0. \qquad (4.3.9)$$

Therefore, either $L_2=\frac{1}{2n}$ or

$$\bar{P}(U,V,W,Y)=\gamma[g(Y,U)g(V,W)-g(U,W)g(Y,V)] \qquad (4.3.10)$$

Let $\{e_1,\ldots,e_n,e_{n+1}=\phi(e_1),e_{n+2}=\phi(e_2),\ldots,e_{2n}=\phi(e_n),\xi\}$ be an orthonormal basis of the tangent space at each point of the manifold. Putting $Y=U=e_i$ in (4.3.10) and using (4.3.1), taking summation over i,$(1\le i\le(2n+1))$, we get

$$\begin{aligned} &\bar{P}(e_1,V,W,e_1)+\ldots+\bar{P}(e_n,V,W,e_n)+\bar{P}(\phi e_1,V,W,\phi e_1)+\ldots+\bar{P}(\phi e_n,V,W,\phi e_n) \\ &+\bar{P}(\xi,V,W,\xi)=\gamma\{[g(e_1,e_1)+\ldots+g(e_n,e_n)]g(V,W) \\ &+[g(\phi e_1,\phi e_1)+\ldots+g(\phi e_n,\phi e_n)]g(V,W)+g(\xi,\xi)g(V,W) \\ &-[g(e_1,W)g(e_1,V)+....+g(e_n,W)g(e_n,V)] \\ &-[g(\phi e_1,W)g(\phi e_1,V)+....+g(\phi e_n,W)g(\phi e_n,V)]-g(\xi,W)g(\xi,V)\}. \qquad (4.3.11) \end{aligned}$$

Now using $g(R(\phi X, \phi, Y)\phi Z, \phi W) = g(R(X,Y)Z,W)$ and by virtue of (1.1.2) and (4.2.12) in (4.3.11) we have

$$S(V,W) = 2ng(V,W). \tag{4.3.12}$$

We can state the following:

Theorem 4.3.1. *A* $(2n+1)$ *dimensional pseudo-projective Ricci-generalized pseudo symmetric Sasakian manifold is Einstein provided* $L_2 \neq \frac{1}{2n}$.

Comparing (4.2.15) and (4.3.12) we get

$$(\lambda + 2n)g(V,W) = 0. \tag{4.3.13}$$

Taking $V = W = e_i$ in (4.3.13), summing over $i = 1, 2, \ldots, 2n+1$ we get

$$\lambda = -2n$$

Thus we have the following:

Corollary 4.3.2. *A Ricci soliton in pseudo-projective Ricci-generalized pseudo-symmetric Sasakian manifold is shrinking.*

4.4 Quasi-conformal Ricci-generalized pseudo -symmetric Sasakian manifold

The quasi-conformal curvature tensor $\tilde{C}$ is given by

$$\begin{aligned} \tilde{C}(U,V)Z &= aR(U,V)Z + b[S(V,Z)U - S(U,Z)V + g(V,Z)QU - g(U,Z)QV] \\ &- \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right)[g(V,Z)U - g(U,Z)V], \end{aligned} \tag{4.4.1}$$

where $a, b \neq 0$ are constants. Taking $Z = \xi$ in (4.4.1), using (1.1.7) and (1.1.11) we get

$$\tilde{C}(U,V)\xi = \gamma^{'}[\eta(V)U - \eta(U)V]. \tag{4.4.2}$$

Similarly using (1.1.10) and (1.1.11) in (4.4.1) we get,

$$\eta(\tilde{C}(U,V)Z) = \gamma^{'}[g(V,Z)\eta(U) - g(U,Z)\eta(V)], \tag{4.4.3}$$

where $\gamma^{'} = (a+4nb) - \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right)$. Assume that $\mathbb{M}$ is quasi-conformal Ricci-generalized pseudo symmetric Sasakian manifold, then

$$(R \cdot \tilde{C})(U,V,W;\xi,Y) = L_3 Q(S,\tilde{C})(U,V,W;\xi,Y) \tag{4.4.4}$$

holds on $\mathbb{M}$.

L.H.S of (4.4.4) takes the form

$$\begin{aligned}(R(\xi,Y)\cdot\tilde{C})(U,V,W) &= R(\xi,Y)\tilde{C}(U,V)W - \tilde{C}(R(\xi,Y)U,V)W \\ &- \tilde{C}(U,R(\xi,Y)V)W - \tilde{C}(U,V)R(\xi,Y)W. \end{aligned} \tag{4.4.5}$$

Taking inner product of (4.4.5) with ξ and by virtue of (1.1.9), (1.1.10), (4.4.2) and (4.4.3) we get

$$\begin{aligned} g((R(\xi,Y)\cdot\tilde{C})(U,V,W),\xi) &= \tilde{C}(U,V,W,Y) \\ &- \gamma^{'}[g(Y,U)g(V,W) + g(U,W)g(V,Y)]. \end{aligned} \tag{4.4.6}$$

R.H.S of (4.4.4) takes the form

$$\begin{aligned} Q(S,\tilde{C})(U,V,W;\xi,Y) &= (\xi\wedge_S Y)\tilde{C}(U,V)W - \tilde{C}((\xi\wedge_S Y)U,V)W \\ &- \tilde{C}(U,(\xi\wedge_S Y)V)W - \tilde{C}(U,V)(\xi\wedge_S Y)W. \end{aligned} \tag{4.4.7}$$

Taking inner product of (4.4.7) with ξ and by virtue of (1.1.9), (1.1.11) we get

$$\begin{aligned} g(Q(S,\tilde{C})(U,V,W;\xi,Y),\xi) &= S(Y,\tilde{C}(U,V)W) - \eta(Y)S(\xi,\tilde{C}(U,V)W) \\ &- S(Y,U)\eta(\tilde{C}(\xi,V)W) + 2n\eta(U)\eta(\tilde{C}(Y,V)W) \\ &- S(Y,V)\eta(\tilde{C}(U,\xi)W) + 2n\eta(V)\eta(\tilde{C}(U,Y)W) \\ &+ 2n\eta(W)\eta(\tilde{C}(U,V)Y). \end{aligned} \tag{4.4.8}$$

Using equations (4.4.6) and (4.4.8) in (4.4.4) we get

$$[2nL_3 - 1][\tilde{C}(U,V,W,Y) - \gamma'[g(Y,U)g(V,W) + g(U,W)g(Y,V)] = 0. \tag{4.4.9}$$

Therefore, either $L_3 = \frac{1}{2n}$ or

$$\tilde{C}(U,V,W,Y) = \gamma'[g(Y,U)g(V,W) - g(U,W)g(Y,V)]. \tag{4.4.10}$$

Let $\{e_1,\ldots,e_n,e_{n+1} = \phi(e_1),e_{n+2} = \phi(e_2),\ldots,e_{2n} = \phi(e_n),\xi\}$ be an orthonormal basis of the tangent space at each point of the manifold. Putting $Y = U = e_i$ in (4.4.10) and using (4.4.1), taking summation over i,$(1 \leq i \leq (2n+1))$, we get

$$\begin{aligned} &\tilde{C}(e_1,V,W,e_1) + \ldots\ldots + \tilde{C}(e_n,V,W,e_n) + \tilde{C}(\phi e_1,V,W,\phi e_1) + \ldots + \tilde{C}(\phi e_n,V,W,\phi e_n) \\ &+\tilde{C}(\xi,V,W,\xi) = \gamma'\{[g(e_1,e_1) + \ldots. + g(e_n,e_n)]g(V,W) \\ &+[g(\phi e_1,\phi e_1) + \ldots. + g(\phi e_n,\phi e_n)]g(V,W) + g(\xi,\xi)g(V,W) \\ &-[g(e_1,W)g(e_1,V) + \ldots. + g(e_n,W)g(e_n,V)] \\ &-[g(\phi e_1,W)g(\phi e_1,V) + \ldots. + g(\phi e_n,W)g(\phi e_n,V)] - g(\xi,W)g(\xi,V)\}. \end{aligned} \tag{4.4.11}$$

Now using $g(R(\phi X,\phi,Y)\phi Z,\phi W) = g(R(X,Y)Z,W)$ and by virtue of (1.1.2) and (4.2.12) in (4.4.11) we have

$$S(V,W) = \left[\frac{2n(a+4nb) - br}{a+(2n-1)b}\right]g(V,W). \tag{4.4.12}$$

We can state the following:

Theorem 4.4.1. *A* $(2n+1)$ *dimensional quasi-conformal Ricci-generalized pseudo symmetric Sasakian manifold is Einstein provided* $L_3 \neq \frac{1}{2n}$.

Comparing (4.2.15) and (4.4.12) we get

$$\left(\lambda + \left[\frac{2n(a+4nb) - br}{a+(2n-1)b}\right]\right) g(V,W) = 0 \tag{4.4.13}$$

Taking $V = W = e_i$ in (4.4.13), summing over $i = 1, 2, \ldots, 2n+1$ we get

$$\lambda = -\left[\tfrac{2n(a+4nb)-br}{a+(2n-1)b}\right].$$

Thus we have the following:

Corollary 4.4.2. *A Ricci soliton in quasi-conformal Ricci-generalized pseudo-symmetric Sasakian manifold is shrinking.*

4.5 Concircular Ricci-generalized pseudo-symmetric Sasakian manifold

The concircular curvature tensor C is defined by equation (3.3.1), taking $Z = \xi$ in (3.3.1) and using (1.1.7), (1.1.10) and (1.1.11) we get

$$C(U,V)\xi = \left[1 - \frac{r}{2n(2n+1)}\right][\eta(V)U - \eta(U)V], \tag{4.5.1}$$

$$\eta(C(U,V)Z) = \left[1 - \frac{r}{2n(2n+1)}\right][g(V,Z)\eta(U) - g(U,Z)\eta(V)]. \tag{4.5.2}$$

Assume that $\mathbb{M}$ is concircular Ricci-generalized pseudo symmetric Sasakian manifold, then

$$(R \cdot C)(U,V,W;\xi,Y) = L_4 Q(S,C)(U,V,W;\xi,Y), \tag{4.5.3}$$

holds on $\mathbb{M}$.

L.H.S of (4.5.3) takes the form

$$\begin{aligned}(R(\xi,Y)\cdot C)(U,V,W) &= R(\xi,Y)C(U,V)W - C(R(\xi,Y)U,V)W \\ &- C(U,R(\xi,Y)V)W - C(U,V)R(\xi,Y)W. \qquad (4.5.4)\end{aligned}$$

Taking inner product of (4.5.4) with ξ and by virtue of (1.1.9), (1.1.10), (4.5.1) and (4.5.2) we get

$$\begin{aligned}&g((R(\xi,Y)\cdot C)(U,V,W),\xi) = C(U,V,W,Y) \\ &-\left[1-\frac{r}{2n(2n+1)}\right][g(Y,U)g(V,W)+g(U,W)g(V,Y)]. \qquad (4.5.5)\end{aligned}$$

R.H.S of (4.5.3) takes the form

$$\begin{aligned}&Q(S,C)(U,V,W;\xi,Y) = (\xi\wedge_S Y)C(U,V)W - C((\xi\wedge_S Y)U,V)W - C(U,(\xi\wedge_S Y)V)W \\ &-C(U,V)(\xi\wedge_S Y)W. \qquad (4.5.6)\end{aligned}$$

Taking inner product of (4.5.6) with ξ and by virtue of (1.1.9) and (1.1.11) we get:

$$\begin{aligned}g(Q(S,C)(U,V,W;\xi,Y),\xi) &= S(Y,C(U,V)W) - \eta(Y)S(\xi,C(U,V)W) \\ &- S(Y,U)\eta(C(\xi,V)W) + 2n\eta(U)\eta(C(Y,V)W) \\ &- S(Y,V)\eta(C(U,\xi)W) + 2n\eta(V)\eta(C(U,Y)W) \\ &+ 2n\eta(W)\eta(C(U,V)Y). \qquad (4.5.7)\end{aligned}$$

Using (4.5.5) and (4.5.7) in (4.5.3) we obtain

Either $L_4 = \frac{1}{2n}$ or

$$C(U,V,W,Y) = \left[1-\frac{r}{2n(2n+1)}\right][g(Y,U)g(V,W)-g(U,W)g(Y,V)]. \qquad (4.5.8)$$

Let $\{e_1, ..., e_n, e_{n+1} = \phi(e_1), e_{n+2} = \phi(e_2), ..., e_{2n} = \phi(e_n), \xi\}$ be an orthonormal basis of the tangent space at each point of the manifold. Putting $Y = U = e_i$ in (4.5.8) and using (3.3.1), taking summation over i,$(1 \leq i \leq (2n+1))$, we get

$$
\begin{aligned}
& C(e_1, V, W, e_1) + ... + C(e_n, V, W, e_n) + C(\phi e_1, V, W, \phi e_1) + ... + C(\phi e_n, V, W, \phi e_n) \\
& + C(\xi, V, W, \xi) = \left[1 - \frac{r}{2n(2n+1)}\right] \{[g(e_1, e_1) + ... + g(e_n, e_n)]g(V, W) \\
& + [g(\phi e_1, \phi e_1) + ... + g(\phi e_n, \phi e_n)]g(V, W) + g(\xi, \xi)g(V, W) \\
& - [g(e_1, W)g(e_1, V) + ... + g(e_n, W)g(e_n, V)] \\
& - [g(\phi e_1, W)g(\phi e_1, V) + ... + g(\phi e_n, W)g(\phi e_n, V)] - g(\xi, W)g(\xi, V)\}. \qquad (4.5.9)
\end{aligned}
$$

Now using $g(R(\phi X, \phi, Y)\phi Z, \phi W) = g(R(X, Y)Z, W)$ and by virtue of (1.1.2) and (4.2.12) in (4.5.9) we have

$$
S(V, W) = 2ng(V, W). \qquad (4.5.10)
$$

We can state the following:

Theorem 4.5.1. *A* $(2n+1)$ *dimensional concircular Ricci-generalized pseudo symmetric Sasakian manifold is Einstein provided* $L_4 \neq \frac{1}{2n}$.

Comparing (4.2.15) and (4.5.10) we get

$$
(\lambda + 2n)g(V, W) = 0. \qquad (4.5.11)
$$

Taking $V = W = e_i$ in (4.5.11), summing over $i = 1, 2, \ldots, 2n+1$ we get

$$
\lambda = -2n.
$$

Thus we have the following:

Corollary 4.5.2. *A Ricci soliton in concircular Ricci-generalized pseudo symmetric Sasakian manifold is shrinking.*

4.6 Semi-symmetric generalized Sasakian space forms

Let $\mathbb{M}(f_1, f_2, f_3)$ be an $(2n+1)$ dimensional semi symmetric generalized Sasakian space forms

$$(R \cdot R)(U, V, W, Z; X, Y) = 0. \tag{4.6.1}$$

Then from (1.5.1) we have

$$R(R(X,Y)U,V,W,Z) + R(U,R(X,Y)V,W,Z) + R(U,V,R(X,Y)W,Z)$$
$$+R(U,V,W,R(X,Y)Z) = 0. \tag{4.6.2}$$

In view of (1.1.16), for $X = U = \xi$, (4.6.2) yields

$$(f_1 - f_3)[R(Y,V,W,Z) + (f_1 - f_3)g(Y,W)g(V,Z) - (f_1 - f_3)g(Y,Z)g(V,W)] = 0. \tag{4.6.3}$$

Since $(f_1 - f_3) \neq 0$, we have

$$R(Y,V,W,Z) = (f_1 - f_3)g(Y,Z)g(V,W) - (f_1 - f_3)g(Y,W)g(V,Z). \tag{4.6.4}$$

Let $\{e_1, e_2, \ldots, e_{2n+1}\}$ be an orthonormal basis of the tangent space at each point of the manifold. Putting $Y = Z = e_i$ in (4.6.4) and taking summation over i,$(1 \leq i \leq (2n+1))$ we get

$$S(V,W) = 2n(f_1 - f_3)g(V,W). \tag{4.6.5}$$

Therefore, $\mathbb{M}(f_1, f_2, f_3)$ is an Einstein manifold. Hence we state the following:

Theorem 4.6.1. *Let $\mathbb{M}(f_1, f_2, f_3)$ be an $(2n+1)$ dimensional generalized Sasakian space forms. If $\mathbb{M}(f_1, f_2, f_3)$ is semi symmetric then $\mathbb{M}(f_1, f_2, f_3)$ is an Einstein manifold.*

Corollary 4.6.2. *If (g, V, λ) is Ricci soliton in semi-symmetric generalized Sasakian space forms, whereas V is conformal killing vector field, then the Ricci soliton is shrinking if $f_1 < f_3$, steady if $f_1 = f_3$ and expanding if $f_1 > f_3$.*

Proof. From Theorem (4.6.1) and by using Ricci soliton equation (1.4.1) we have

$$(\mathcal{L}_V g)(V,W) + 4n(f_1 - f_3)g(V,W) + 2\lambda g(V,W) = 0. \tag{4.6.6}$$

Let $\{e_1, e_2, \ldots, e_{2n+1}\}$ be an orthonormal basis of the tangent space at each point of the manifold. Putting $V = W = e_i$ in (4.6.6) and taking summation over i,$(1 \leq i \leq (2n+1))$ we get,

$$(\mathcal{L}_V g)(e_i, e_i) + 4n(2n+1)(f_1 - f_3) + 2(2n+1)\lambda = 0.$$

Since $[e_i, e_j] = 0$, for all $1 \leq i, j \leq (2n+1)$, then we get,

$$\lambda = -2n(f_1 - f_3).$$

□

4.7 Pseudo-symmetric generalized Sasakian space forms

Let $\mathbb{M}(f_1, f_2, f_3)$ be an $(2n+1)$ dimensional generalized Sasakian space forms

$$(R \cdot R)(U,V,W,Z;X,Y) = L_R Q(g,R)(U,V,W,Z;X,Y). \tag{4.7.1}$$

Then from (1.5.1) and (1.5.2) we have

$$\begin{aligned}
&-R(R(X,Y)U,V,W,Z) - R(U,R(X,Y)V,W,Z) - R(U,V,R(X,Y)W,Z)\\
&-R(U,V,W,R(X,Y)Z) = L_R[R((X \wedge Y)U,V,W,Z) + R(U,(X \wedge Y)V,W,Z)\\
&+R(U,V,(X \wedge Y)W,Z) + R(U,V,W,(X \wedge Y)Z)].
\end{aligned} \tag{4.7.2}$$

In view of (1.1.16), for $X = U = \xi$, (4.7.2) yields

$$(f_1 - f_3)[R(Y,V,W,Z) + (f_1 - f_3)g(Y,W)g(V,Z) - (f_1 - f_3)g(Y,Z)g(V,W)]$$
$$= -L_R[R(Y,V,W,Z) + (f_1 - f_3)g(Y,W)g(V,Z) - (f_1 - f_3)g(Y,Z)g(V,W)],$$

$$[L_R + (f_1 - f_3)][R(Y,V,W,Z) + (f_1 - f_3)g(Y,W)g(V,Z) - (f_1 - f_3)g(Y,Z)g(V,W)] = 0.$$

Therefore, either $L_R = -(f_1 - f_3)$ or

$$R(Y,V,W,Z) = (f_1 - f_3)[g(Y,Z)g(V,W) - g(Y,W)g(V,Z)]. \tag{4.7.3}$$

Let $\{e_1, e_2, \ldots, e_{2n+1}\}$ be an orthonormal basis of the tangent space at each point of the manifold. Putting $Y = Z = e_i$ in (4.7.3) and taking summation over i,$(1 \leq i \leq (2n+1))$ we get

$$S(V,W) = 2n(f_1 - f_3)g(V,W). \tag{4.7.4}$$

Therefore, $\mathbb{M}(f_1, f_2, f_3)$ is an Einstein manifold. Hence we state the following:

Theorem 4.7.1. *Let $\mathbb{M}(f_1, f_2, f_3)$ be an $(2n+1)$ dimensional generalized Sasakian space forms. If $\mathbb{M}(f_1, f_2, f_3)$ is pseudo-symmetric then $\mathbb{M}(f_1, f_2, f_3)$ is an Einstein manifold provided $L_R \neq -(f_1 - f_3)$.*

Corollary 4.7.2. *If (g, V, λ) is Ricci soliton in pseudo-symmetric generalized Sasakian space forms, whereas V is conformal killing vector field, then the Ricci soliton is shrinking if $f_1 < f_3$, steady if $f_1 = f_3$ and expanding if $f_1 > f_3$.*

4.8 Quasi-conformal semi-symmetric generalized Sasakian space forms

The quasi-conformal curvature tensor is given by equation (4.4.1), by virtue of (1.1.13), (1.1.15) and (1.1.16) we obtain the following

$$\tilde{C}(U,V)\xi = D[\eta(V)U - \eta(U)V], \tag{4.8.1}$$

$$\tilde{C}(\xi,U)V = D[g(U,V)\xi - \eta(U)V], \tag{4.8.2}$$

$$\tilde{C}(\xi,U)\xi = D[\eta(U)\xi - U]. \tag{4.8.3}$$

Where $D = a(f_1 - f_3) + 2nb(f_1 - f_3) + b(2nf_1 + 3f_2 - f_3) - \frac{r}{2n+1}\left[\frac{a}{2n} + 2b\right]$.

Let $\mathbb{M}(f_1, f_2, f_3)$ be an $(2n+1)$-dimensional quasi-conformal semi-symmetric generalized Sasakian space forms

$$(R(X,Y)\cdot\tilde{C})(U,V,W,Z) = 0. \tag{4.8.4}$$

Then from (1.5.1) we have

$$-\tilde{C}(R(X,Y)U,V,W,Z) - \tilde{C}(U,R(X,Y)V,W,Z) - \tilde{C}(U,V,R(X,Y)W,Z)$$
$$-\tilde{C}(U,V,W,R(X,Y)Z) = 0. \tag{4.8.5}$$

In view of (1.1.16) and (4.8.2) for $X = U = \xi$, (4.8.5) yields

$$(f_1 - f_3)\{\tilde{C}(Y,V,W,Z) + D[g(Y,W)g(V,Z) - g(Y,Z)g(V,W)]\} = 0.$$

Since $(f_1 - f_3) \neq 0$, we have

$$\tilde{C}(Y,V,W,Z) = D[g(Y,Z)g(V,W) - g(Y,W)g(V,Z)]. \tag{4.8.6}$$

Let $\{e_1, e_2, \ldots, e_{2n+1}\}$ be an orthonormal basis of the tangent space at each point of the manifold. Putting $Y = Z = e_i$ in (4.8.6) and taking summation over i,$(1 \leq i \leq (2n+1))$, using equation (4.4.1) we get

$$S(V,W) = D' g(V,W). \tag{4.8.7}$$

Where $D' = \frac{2n[(a+2nb)(f_1-f_3)+b(2nf_1+3f_2-f_3)]-br}{a+b(2n+1)}$

Theorem 4.8.1. *Let $\mathbb{M}(f_1,f_2,f_3)$ be an $(2n+1)$-dimensional generalized Sasakian space forms. If $\mathbb{M}(f_1,f_2,f_3)$ is quasi-conformal semi-symmetric then $\mathbb{M}(f_1,f_2,f_3)$ is an Einstein manifold.*

4.9 Quasi-conformal pseudo-symmetric generalized Sasakian space forms

Let $\mathbb{M}(f_1,f_2,f_3)$ be an $(2n+1)$-dimensional quasi-conformal pseudo-symmetric generalized Sasakian space forms

$$(R\cdot\tilde{C})(U,V,W,Z;X,Y) = L_{\tilde{C}}Q(g,\tilde{C})(U,V,W,Z;X,Y). \tag{4.9.1}$$

Then from (1.5.1) we have

$$\begin{aligned}
&-\tilde{C}(R(X,Y)U,V,W,Z) - \tilde{C}(U,R(X,Y)V,W,Z) - \tilde{C}(U,V,R(X,Y)W,Z)\\
&-\tilde{C}(U,V,W,R(X,Y)Z) = L_{\tilde{C}}[\tilde{C}((X\wedge Y)U,V,W,Z) + \tilde{C}(U,(X\wedge Y)V,W,Z)\\
&+\tilde{C}(U,V,(X\wedge Y)W,Z) + \tilde{C}(U,V,W,(X\wedge Y)Z).
\end{aligned} \tag{4.9.2}$$

In view of (1.1.6) and (4.8.2), for $X=U=\xi$, (4.9.2) yields

$$[L_{\tilde{C}} + (f_1-f_3)]\{\tilde{C}(Y,V,W,Z) + D[g(Y,W)g(V,Z) - g(Y,Z)g(V,W)]\} = 0.$$

Therefore either $L_{\tilde{C}} = -(f_1-f_3)$ or

$$\tilde{C}(Y,V,W,Z) = D[g(Y,Z)g(V,W) - g(Y,W)g(V,Z)]. \tag{4.9.3}$$

Let $\{e_1, e_2, , e_{2n+1}\}$ be an orthonormal basis of the tangent space at each point of the manifold. Putting $Y = Z = e_i$ in (4.9.3) and taking summation over i, $(1 \leq i \leq (2n+1))$, using equation (4.4.1) we get

$$S(V, W) = D' g(V, W), \tag{4.9.4}$$

where $D' = \frac{2n[(a+2nb)(f_1-f_3)+b(2nf_1+3f_2-f_3)]-br}{a+b(2n+1)}$

Theorem 4.9.1. *Let $\mathbb{M}(f_1, f_2, f_3)$ be an $(2n+1)$-dimensional generalized Sasakian space forms. If $\mathbb{M}(f_1, f_2, f_3)$ is quasi-conformal pseudo-symmetric then $\mathbb{M}(f_1, f_2, f_3)$ is an Einstein manifold provided $L_{\tilde{C}} \neq -(f_1 - f_3)$.*

4.10 Generalized Sasakian space forms satisfies the condition $\tilde{C} \cdot \tilde{C} = 0$

Let $\mathbb{M}(f_1, f_2, f_3)$ be an $(2n+1)$-dimensional generalized Sasakian space forms. Let $\tilde{C} \cdot \tilde{C}$ be a $(0, 6)$-tensor and $\tilde{C} \cdot \tilde{C} = 0$.

$$\begin{aligned}
&-\tilde{C}(\tilde{C}(X, Y)U, V, W, Z) - \tilde{C}(U, \tilde{C}(X, Y)V, W, Z) - \tilde{C}(U, V, \tilde{C}(X, Y)W, Z) \\
&-\tilde{C}(U, V, W, \tilde{C}(X, Y)Z) = 0. \qquad (4.10.1)
\end{aligned}$$

In view of (4.8.2), for $X = U = \xi$, (4.10.1) yields

$$-D[\tilde{C}(Y, V, W, Z) + D\{g(Y, W)g(V, Z) - g(Y, Z)g(V, W)\}] = 0$$

Since $D \neq 0$, where $D = a(f_1 - f_3) + 2nb(f_1 - f_3) + b(2nf_1 + 3f_2 - f_3) - \frac{r}{2n+1}\left[\frac{a}{2n} + 2b\right]$. Then we have

$$\tilde{C}(Y, V, W, Z) = D[g(Y, Z)g(V, W) - g(Y, W)g(V, Z)] \tag{4.10.2}$$

Let $\{e_1, e_2, \ldots, e_{2n+1}\}$ be an orthonormal basis of the tangent space at each point of the manifold. Putting $Y = Z = e_i$ in (4.10.2) and taking summation over i, $(1 \leq i \leq (2n+1))$, using equation (4.4.1) we get

$$S(V,W) = D^{'}g(V,W), \tag{4.10.3}$$

where $D' = \frac{2n[(a+2nb)(f_1-f_3)+b(2nf_1+3f_2-f_3)]-br}{a+b(2n+1)}$

Theorem 4.10.1. *Let* $\mathbb{M}(f_1, f_2, f_3)$ *be an* $(2n+1)$*-dimensional generalized Sasakian space forms. If* $(0,6)$*-tensor* $\tilde{C} \cdot \tilde{C} = 0$ *holds on* $\mathbb{M}(f_1, f_2, f_3)$*, then* $\mathbb{M}(f_1, f_2, f_3)$ *is an Einstein manifold.*

4.11 Conclusion

The important results finding Sasakian manifold and generalized Sasakian space forms are as follows:

- A (2n+1)-dimensional Ricci- generalized pseudo-symmetric Sasakian manifold, pseudo-projective Ricci-generalized pseudo-symmetric Sasakian manifold, quasi-conformal Ricci-generalized pseudo-symmetric Sasakian manifold, concircular Ricci-genaralized pseudo-symmetric Sasakian manifold are Einstein manifolds. And Ricci soliton for these manifolds is shrinking.

- Let $\mathbb{M}(f_1, f_2, f_3)$ be an $(2n+1)$-dimensional generalized Sasakian space form. If $\mathbb{M}(f_1, f_2, f_3)$ is semi-symmetric then $\mathbb{M}(f_1, f_2, f_3)$ is an Einstein manifold.

- The triple (g, V, λ) is Ricci soliton in semi symmetric generalized Sasakian space form, iff V is conformal killing vector field.

- Let $\mathbb{M}(f_1, f_2, f_3)$ be a $(2n + 1)$-dimensional generalized Sasakian space form. If $\mathbb{M}(f_1, f_2, f_3)$ is pseudo-symmetric then $\mathbb{M}(f_1, f_2, f_3)$ is an Einstein manifold provided $L_R \neq -(f_1 - f_3)$.

- The triple (g, V, λ) is Ricci soliton in pseudo-symmetric generalized Sasakian space form, iff V is conformal killing vector field provided $L_R \neq -(f_1 - f_3)$.

- Let $\mathbb{M}(f_1, f_2, f_3)$ be an $(2n + 1)$-dimensional generalized Sasakian-space-form. If $\mathbb{M}(f_1, f_2, f_3)$ is quasi-conformal semi-symmetric then $\mathbb{M}(f_1, f_2, f_3)$ is an Einstein manifold.

- Let $\mathbb{M}(f_1, f_2, f_3)$ be an $(2n + 1)$-dimensional generalized Sasakian space form. If $\mathbb{M}(f_1, f_2, f_3)$ is quasi-conformal pseudo-symmetric then $\mathbb{M}(f_1, f_2, f_3)$ is an Einstein manifold provided $L_C \neq -(f_1 - f_3)$.

- Let $\mathbb{M}(f_1, f_2, f_3)$ be an $(2n + 1)$-dimensional generalized Sasakian space form. If $(0,\ 6)$-tensor $C \cdot C = 0$ holds on $\mathbb{M}(f_1, f_2, f_3)$, then $\mathbb{M}(f_1, f_2, f_3)$ is an Einstein manifold.

CHAPTER 5

Publications based on this Chapter

- *A Study on Ricci Soliton in Kenmotsu Manifold Admitting Semi-symmetric Metric Connection,* Proceedings of the InterNational Conference On DGAFM-2016, ISBN: 978-93-5265-439-0, 166-175.

- *Pseudo-symmetric Para-Kenmotsu Manifolds Admitting Conformal Ricci Soliton,* (Communicated).

Chapter 5

On Kenmotsu Manifolds

5.1 Introduction

In 1972 Kenmotsu introduced and studied the notion of Kenmotsu manifolds [50]. They set up one of the three classes of almost contact metric manifolds $\mathbb{M}$ whose automorphism group attains the maximum dimension [81]. For such a manifold, the sectional curvature of plane sections containing ξ is a constant, say c. (1) If $c > 0$, $\mathbb{M}$ is a homogeneous Sasakian manifold of constant ϕ-sectional curvature. (2) If $c = 0$, $\mathbb{M}$ is global Riemannian product of a line or a circle with a Kahler manifold of constant holomorphic sectional curvature. (3) If $c < 0$, $\mathbb{M}$ is a warped product space $R \times_f C^n$. Kenmotsu [50] characterized the differential geometric properties of manifolds of class (3); the structure so obtained is now known as Kenmotsu structure. A Kenmotsu structure is not Sasakian [50].

Kenmotsu showed that a locally Kenmotsu manifold is a warped product $I \times_f N$ of an interval I and a Kaehler manifold N with warping function $f(t) = se^t$, where s is a non-zero constant and Kenmotsu proved that if the manifold is locally symmetric, that is, if $\nabla R = 0$, then it has constant curvature -1 so it is locally isometric to the Hyperbolic space $H^{2m+1}(-1)$. Kenmotsu manifolds have been studied by various authors [12], [17], [45].

The author Sato [70] introduced the notion of almost para-contact manifolds. Before Sato, Takahashi [79], defined almost contact manifolds (in particular, Sasakian manifolds) equipped with an associated pseudo-Riemannian metric. Kaneyuki et al. [49] defined the notion of almost paracontact structure on pseudo-Riemannian manifold of dimension $n = (2m+1)$. Later Zamkovoy [86] showed that any almost paracontact structure admits a pseudo-Riemannian metric with signature $(n+1, n)$. Later, Adati and Matsumoto defined and studied p-Sasakian and sp-Sasakian manifolds which are regarded as a special kind of an almost contact Riemannian manifolds. Before Sato, Kenmotsu defined a class of almost contact Riemannian manifolds. In 1995, Sinha and Sai Prasad [71] have defined a class of almost para contact metric manifolds namely para-Kenmotsu (p-Kenmotsu) and special para-Kenmotsu (sp-Kenmotsu) manifolds as analogues of p-Sasakian and sp-Sasakian manifolds.

In 1924, Friedman and Schouten [34] introduced the idea of a semi-symmetric linear connection in a differentiable manifold. Then in 1932, Hayden [40] introduced a metric connection $\tilde{\nabla}$ with a non-zero torsion on a Riemannian manifold. Such a connection is called Hayden connection. A systematic study of semi-symmetric metric connection on a Riemannian manifold has been given by Yano [85] in 1970. Semi-symmetric metric connection on Riemannian manifolds is also studied by various authors such as [11], [13], [55], [66], [69], [77]. In this chapter we study semi-symmetric, pseudo-symmetric, pseudo-projective semi-symmetric Kenmotsu manifolds admitting semi-symmetric metric connection and obtain Ricci soliton for these manifolds with respect to Levi-Civita connection. Also we study Ricci soliton in para-Kenmotsu manifold admitting conformal

Ricci soliton.

5.2 Semi-symmetric metric connection on Kenmotsu manifolds

The torsion tensor $\tilde{T}$ for n-dimensional differentiable manifold with linear connection $\tilde{\nabla}$ is given by

$$\tilde{T}(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y]. \tag{5.2.1}$$

If the torsion tensor vanishes, then the connection $\tilde{\nabla}$ is symmetric, otherwise it is non-symmetric. The connection $\tilde{\nabla}$ is said to be a metric connection, if there is a Riemannian metric g in $\mathbb{M}$ such that $\tilde{\nabla} g = 0$, then the connection $\tilde{\nabla}$ is a metric connection, otherwise it is non-metric.

A linear connection is said to be a semi-symmetric connection in a Riemannian manifold if its torsion tensor $\tilde{T}$ is of the form

$$\tilde{T}(X,Y) = \varphi(Y)X - \varphi(X)Y, \tag{5.2.2}$$

where the 1-form φ is defined by $\varphi(X) = g(X,\rho)$, and ρ is vector field.

In an almost contact metric manifold, a semi-symmetric metric connection is defined by

$$\tilde{T}(X,Y) = \eta(Y)X - \eta(X)Y.$$

A relation between the semi-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ of M has been obtained by Yano [85] and it is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi, \tag{5.2.3}$$

put $Y = \xi$ in (5.2.3) and using (1.1.20) we get,

$$\tilde{\nabla}_X \xi = 2[X - \eta(X)\xi]. \tag{5.2.4}$$

Furthermore, a relation between the curvature tensor $\tilde{R}$ and R of type $(1,3)$ of the connections $\tilde{\nabla}$ and ∇ respectively is given by

$$\tilde{R}(X,Y)Z = R(X,Y)Z - \alpha(Y,Z)X + \alpha(X,Z)Y - g(Y,Z)FX + g(X,Z)FY, \tag{5.2.5}$$

where α is a tensor field of type $(0,2)$ given by

$$\alpha(Y,Z) = (\nabla_Y \eta)(Z) - \eta(Y)\eta(Z) + \tfrac{1}{2}\eta(\xi)g(Y,Z),$$

$$\alpha(Y,Z) = (\tilde{\nabla}_Y \eta)(Z) - \frac{1}{2}g(Y,Z), \tag{5.2.6}$$

and F is a tensor field of type $(1,1)$ given by $g(FY,Z) = \alpha(Y,Z)$ for any vector fields Y, Z.

From (5.2.5), it follows that

$$\tilde{S}(Y,Z) = S(Y,Z) - (n-2)\alpha(Y,Z) - ag(Y,Z), \tag{5.2.7}$$

where $\tilde{S}$ denotes the Ricci tensor with respect to $\tilde{\nabla}$, a=Trace of α. For a Kenmotsu manifold, in view of (5.2.6) and (1.1.19), we get from (5.2.5) that

$$\begin{aligned} \tilde{R}(X,Y)Z &= R(X,Y)Z - 3[g(Y,Z)X - g(X,Z)Y] + 2[\eta(Y)X - \eta(X)Y]\eta(Z) \\ &- 2[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]. \end{aligned} \tag{5.2.8}$$

So from (5.2.8) and (1.1.22) we have

$$\tilde{R}(X,Y)\xi = 2[\eta(X)Y - \eta(Y)X], \tag{5.2.9}$$

$$\tilde{R}(\xi,Y)Z = 2[\eta(Z)Y - g(Y,Z)\xi], \tag{5.2.10}$$

$$\tilde{R}(\xi,Y)\xi = 2[Y - \eta(Y)\xi]. \tag{5.2.11}$$

On contracting (5.2.8), we get

$$\tilde{S}(Y,Z) = S(Y,Z) - (3n-5)g(Y,Z) + 2(n-2)\eta(Y)\eta(Z), \tag{5.2.12}$$

where $\tilde{S}$ and S are the Ricci tensors o the connections $\tilde{\nabla}$ and ∇ respectively. So in a Kenmotsu manifold, the Ricci tensor of the semi-symmetric metric connection is symmetric. It follows from (1.1.20) and (5.2.12) that

$$\tilde{S}(\phi Y, \phi Z) = \tilde{S}(Y,Z) + 2(n-1)\eta(Y)\eta(Z), \tag{5.2.13}$$

$$\tilde{S}(Y,\xi) = -2(n-1)\eta(Y), \tag{5.2.14}$$

$$\tilde{Q}Y = -2(n-1)Y. \tag{5.2.15}$$

Again, contracting (5.2.12) over Y, Z, we get

$$\tilde{r} = r - 2(n-1),$$

where $\tilde{Q}$ is the Ricci operator, $\tilde{r}$ and r are the scalar curvatures of the connections $\tilde{\nabla}$ and ∇ respectively.

From (1.4.1) we have

$$(\tilde{L}_V g)(X,Y) + 2\tilde{S}(X,Y) + 2\tilde{\lambda} g(X,Y) = 0, \tag{5.2.16}$$

where $\tilde{L}_V$ denotes the Lie derivative operator along the conformal killing vector field V with respect to Levi-Civita connection.

$$(\tilde{L}_V g)(X,Y) = (L_V g)(X,Y) + \eta(V)g(Y,Z) - g(Y,V)\eta(Z) - g(Z,V)\eta(Y). \tag{5.2.17}$$

Using (5.2.3) and (5.2.17) in (5.2.16) we have

$$\tilde{S}(X,Y) = 2\eta(X)\eta(Y) - \frac{(2\lambda+3)}{2} g(X,Y). \tag{5.2.18}$$

From (5.2.12) and (5.2.18) we get

$$S(X,Y) = \left(\frac{6n - 2\tilde{\lambda} - 13}{2}\right) g(X,Y) + 2(3-n)\eta(X)\eta(Y). \tag{5.2.19}$$

Thus we can state following:

Theorem 5.2.1. *A Kenmotsu manifold admitting semi-symmetric metric connection is an η-Einstein manifold.*

5.3 Ricci soliton in semi-symmetric Kenmotsu manifolds with respect to semi-symmetric metric connection

Let us consider a semi-symmetric Kenmotsu manifold $\mathbb{M}(n \geq 3)$ admitting a semi-symmetric metric connection, then

$$(\tilde{R}(X,Y) \cdot \tilde{R})(U,V)Z = 0 \tag{5.3.1}$$

$$\tilde{R}(X,Y)\tilde{R}(U,V)Z - \tilde{R}(\tilde{R}(X,Y)U,V)Z - \tilde{R}(U,\tilde{R}(X,Y)V)Z$$
$$-\tilde{R}(U,V)\tilde{R}(X,Y)Z = 0. \tag{5.3.2}$$

Put $X = U = \xi$ in (5.3.2) and using (5.2.10) and (5.2.11) we get

$$\tilde{R}(Y,V)Z = 2\{g(Y,Z)V - g(V,Z)Y\}. \tag{5.3.3}$$

Taking inner product of (5.3.3) with W, we get

$$\tilde{R}(Y,V,Z,W) = 2\{g(Y,Z)g(V,W) - g(V,Z)g(Y,W)\}. \tag{5.3.4}$$

Taking $Y = W = e_i$ in (5.3.4) and summing over $i = 1, 2, \ldots, n$, we get

$$\tilde{S}(V, Z) = -2(n-1)g(Y, Z). \tag{5.3.5}$$

From (5.2.18) and (5.3.5) and contracting over V, Z we get the value of λ that is

$$\lambda = \tfrac{3n-4+4n(n-1)}{2n}.$$

Theorem 5.3.1. *Ricci soliton in semi-symmetric Kenmotsu manifold admitting semi-symmetric metric connection is expanding with respect to Levi-Civita connection.*

5.4 Ricci soliton in pseudo-projective semi-symmetric Kenmotsu manifolds with respect to semi-symmetric metric connection

In a Kenmotsu manifold $\mathbb{M}$ of dimension $n \geq 3$, the pseudo-projective curvature tensor $\tilde{P}$ with respect to semi-symmetric metric connection $\tilde{\nabla}$ is given by

$$\begin{aligned} \tilde{P}(X,Y)Z &= a\tilde{R}(X,Y)Z + b[\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y] \\ &- \frac{\tilde{r}}{n}\left(\frac{a}{n-1} + b\right)[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{5.4.1}$$

for $X, Y, Z \in \Gamma(TM)$, where $\tilde{R}$, $\tilde{S}$ are the Riemannian curvature tensor, Ricci tensor with respect to the connection $\tilde{\nabla}$ respectively.

From (5.4.1) we have,

$$\tilde{P}(\xi, Y)Z = \left[a + 2b(n-1) + \frac{\tilde{r}}{n}\left(\frac{a}{n-1} + b\right)\right][\eta(Z)Y - g(Y,Z)\xi], \tag{5.4.2}$$

$$\tilde{P}(\xi, Y)\xi = \left[a + 2b(n-1) + \frac{\tilde{r}}{n}\left(\frac{a}{n-1} + b\right)\right][Y - \eta(Y)\xi]. \tag{5.4.3}$$

Let us consider a pseudo-projective semi-symmetric Kenmotsu manifold $\mathbb{M}(n \geq 3)$ admitting a semi-symmetric metric connection, then

$$(\tilde{R}(X,Y)\cdot\tilde{P})(U,V)Z = 0, \tag{5.4.4}$$

$$\tilde{R}(X,Y)\tilde{P}(U,V)Z - \tilde{P}(\tilde{R}(X,Y)U,V)Z - \tilde{P}(U,\tilde{R}(X,Y)V)Z$$
$$-\tilde{P}(U,V)\tilde{R}(X,Y)Z = 0. \tag{5.4.5}$$

Put $X = U = \xi$ in (5.4.5)and using (5.4.2) and (5.4.3), we get

$$\tilde{P}(Y,V)Z = \left[a + 2b(n-1) + \frac{\tilde{r}}{n}\left(\frac{a}{n-1} + b\right)\right]\{g(Y,Z)V - g(V,Z)Y\}. \tag{5.4.6}$$

Taking inner product of (5.4.6) with W, we get

$$\tilde{P}(Y,V,Z,W) = \left[a + 2b(n-1) + \frac{\tilde{r}}{n}\left(\frac{a}{n-1} + b\right)\right]\{g(Y,Z)g(V,W) - g(V,Z)g(Y,W)\}. \tag{5.4.7}$$

Taking $Y = W = e_i$ in (5.4.7), using (5.4.1) and summing over $i = 1, 2, \ldots, n$ we get

$$\tilde{S}(V,Z) = -\left[\frac{(n-1)(a+2b(n-1))}{a+b(n-1)}\right]g(Y,Z). \tag{5.4.8}$$

From (5.2.18) and (5.4.8) and contracting over V, Z we get the value of λ that is

$$\lambda = \tfrac{4+n(n-1)(a+2b(n-1))-3n}{n(a+b(n-1))}.$$

Theorem 5.4.1. *Ricci soliton in pseudo-projective semi-symmetric Kenmotsu manifold admitting semi-symmetric metric connection is expanding with respect to Levi-Civita connection.*

5.5 Ricci soliton in pseudo-symmetric Kenmotsu manifolds with respect to semi-symmetric metric connection

Let us consider a pseudo-symmetric Kenmotsu manifold $\mathbb{M}(n \geq 3)$ admitting a semi-symmetric metric connection, then

$$\tilde{R}(X,Y) \cdot \tilde{R}(U,V)Z = L_{\tilde{R}}[((X \wedge Y) \cdot \tilde{R})(U,V)Z], \tag{5.5.1}$$

where L_R is smooth function on $\mathbb{M}$.

$$\begin{aligned} &\tilde{R}(X,Y)\tilde{R}(U,V)Z - \tilde{R}(\tilde{R}(X,Y)U,V)Z - \tilde{R}(U,\tilde{R}(X,Y)V)Z \\ &-\tilde{R}(U,V)\tilde{R}(X,Y)Z = L_{\tilde{R}}[(X \wedge Y)\tilde{R}(U,V)Z - \tilde{R}((X \wedge Y)U,V)Z \\ &-\tilde{R}(U,(X \wedge Y)V)Z - \tilde{R}(U,V)(X \wedge Y)Z]. \end{aligned} \tag{5.5.2}$$

Put $X = U = \xi$ in (5.5.2) and using (5.2.10) and (5.2.11) we get

$$[L_{\tilde{R}} - 1][\tilde{R}(Y,V)Z - 2\{g(Y,Z)V - g(V,Z)Y\}] = 0. \tag{5.5.3}$$

Taking inner product of (5.5.3) with W, we get either $L_{\tilde{R}} = 1$ or

$$\tilde{R}(Y,V,Z,W) = 2\{g(Y,Z)g(V,W) - g(V,Z)g(Y,W)\}. \tag{5.5.4}$$

Taking $Y = W = e_i$ in (5.5.4) and summing over $i = 1, 2, \ldots, n$ we get.

$$\tilde{S}(V,Z) = -2(n-1)g(Y,Z). \tag{5.5.5}$$

From (5.2.18) and (5.5.5) and contracting over V, Z we get the value of λ that is

$$\lambda = \tfrac{3n-4+4n(n-1)}{2n}.$$

Theorem 5.5.1. *Ricci soliton in pseudo-symmetric Kenmotsu manifold admitting semi-symmetric metric connection is expanding with respect to Levi-Civita connection.*

5.6 Ricci soliton in para-Kenmotsu manifolds satisfying $R \cdot C = L_C Q(g, C)$ admitting conformal Ricci soliton

Definition 5.6.1. An n-dimensional para-Kenmotsu manifold $\mathbb{M}$ is called concircularly pseudo-symmetric if the tensors $R \cdot C$ and $Q(g, C)$ are linearly dependent. This is equivalent to

$$R \cdot C = L_C Q(g, C) \tag{5.6.1}$$

holding on the set $U_C = \{x \in M; C \neq 0 at x\}$, where L_C is smooth function on U_C.

For a n-dimensional manifold, the concircular curvature tensor C is given by

$$C(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)}[g(Y, Z)X - g(X, Z)Y]. \tag{5.6.2}$$

Equation (5.6.1) can be written as

$$\begin{aligned} &R(X, Y)C(U, V)Z - C(R(X, Y)U, V)Z - C(U, R(X, Y)V)Z \\ &-C(U, V)R(X, Y)Z = L_C[(X \wedge Y)C(U, V) - C((X \wedge Y)U, V)Z \\ &-C(U, (X \wedge Y)V)Z - C(U, V)(X \wedge Y)Z]. \end{aligned} \tag{5.6.3}$$

Taking $X = U = \xi$ in (5.6.3) and by the definition of endomorphism i.e., $(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y$ also by virtue of (1.1.30), (5.6.2) we can obtain

$$(L_C + 1)\{C(Y, V)Z + \left(1 + \frac{r}{n(n-1)}\right)[g(V, Z)Y - g(Y, Z)V]\} = 0. \tag{5.6.4}$$

There either $L_C = -1$ or

$$C(Y,V)Z = \left(1 + \frac{r}{n(n-1)}\right)[g(Y,Z)V - g(V,Z)Y]. \tag{5.6.5}$$

Now on contraction of (5.6.5) and using (5.6.2) we get the Ricci tensor

$$S(V,Z) = -(n-1)g(V,Z). \tag{5.6.6}$$

Thus we state the following:

Lemma 5.6.1. *Concircularly pseudo-symmetric para-Kenmotsu manifold is Einstein provided $L_C \neq -1$.*

Theorem 5.6.2. *Ricci soliton in para-Kenmotsu manifold satisfying the condition $R \cdot C = L_C Q(g,C)$ which admit conformal Ricci soliton (g, V, λ, ρ) is*

1. *shrinking, if $\rho < 2(n-2)(\lambda < 0)$.*

2. *steady, if $\rho = 2(n-2)(\lambda = 0)$.*

3. *expanding, if $\rho > 2(n-2)(\lambda > 0)$.*

Proof. By the definition of conformal Ricci soliton i.e., equation (1.4.3) and Lie derivative we have

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) + 2S(X,Y) = \left[2\lambda - \left(\rho + \frac{2}{n}\right)\right] g(X,Y). \tag{5.6.7}$$

Using (1.1.28) in (5.6.7) we get

$$S(X,Y) = \left[\lambda - 1 - \frac{\rho}{2} - \frac{1}{n}\right] g(X,Y) + \eta(X)\eta(Y). \tag{5.6.8}$$

From (5.6.6) and (5.6.8) we can get the value of λ given by

$$\lambda = \frac{\rho}{2} - (n-2). \tag{5.6.9}$$

Which completes the proof. □

5.7 Ricci soliton in para-Kenmotsu manifolds satisfying $C \cdot R = L_R Q(g, R)$ admitting conformal Ricci soliton

Let us assume the pseudo-symmetric condition

$$C \cdot R = L_R Q(g, R). \tag{5.7.1}$$

Equation (5.7.1) can be written as

$$\begin{aligned} &C(X,Y)R(U,V)Z - R(C(X,Y)U,V)Z - R(U,C(X,Y)V)Z \\ &-R(U,V)C(X,Y)Z = L_R[(X \wedge Y)R(U,V) - R((X \wedge Y)U,V)Z \\ &-R(U,(X \wedge Y)V)Z - R(U,V)(X \wedge Y)Z]. \end{aligned} \tag{5.7.2}$$

Taking $X = U = \xi$ in (5.7.2) and by the definition of endomorphism i.e., $(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y$ also by virtue of (1.1.30), (5.2.3) we can obtain

$$\left[L_R + \left(1 + \frac{r}{n(n-1)}\right)\right]\{R(Y,V)Z + [g(V,Z)Y - g(Y,Z)V]\} = 0. \tag{5.7.3}$$

There either $L_R = -\left(1 + \frac{r}{n(n-1)}\right)$ or

$$R(Y,V)Z = g(Y,Z)V - g(V,Z)Y \tag{5.7.4}$$

Now on contraction of (5.7.4) we get the Ricci tensor

$$S(V,Z) = -(n-1)g(V,Z). \tag{5.7.5}$$

Thus we state the following:

Lemma 5.7.1. *Para-Kenmotsu manifold satisfying the pseudosymmetric condition $C \cdot R = L_R Q(g, R)$ is Einstein provided $L_R \neq -\left(1 + \frac{r}{n(n-1)}\right)$.*

Theorem 5.7.2. *Ricci soliton in para-Kenmotsu manifold satisfying the condition $C \cdot R = L_R Q(g, R)$ which admit conformal Ricci soliton (g, V, λ, ρ) is*

1. *shrinking, if $\rho < 2(n-2)(\lambda < 0)$.*

2. *steady, if $\rho = 2(n-2)(\lambda = 0)$.*

3. *expanding, if $\rho > 2(n-2)(\lambda > 0)$.*

5.8 Ricci soliton in para-Kenmotsu manifolds satisfying $R \cdot \bar{P} = L_{\bar{P}} Q(g, \bar{P})$ admitting conformal Ricci soliton

Definition 5.8.1. An n-dimensional para-Kenmotsu manifold $\mathbb{M}$ is called pseudo-projectively-pseudo-symmetric if the tensors $R \cdot \bar{P}$ and $Q(g, \bar{P})$ are linearly dependent. This is equivalent to

$$R \cdot \bar{P} = L_{\bar{P}} Q(g, \bar{P}). \tag{5.8.1}$$

holding on the set $U_{\bar{P}} = \{x \in M; \bar{P} \neq 0 at\ x\}$, where $L_{\bar{P}}$ is smooth function on $U_{\bar{P}}$.

For an n-dimensional manifold the pseudo-projective curvature tensor defined by

$$\begin{aligned} \bar{P}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] \\ &- \frac{r}{n}\left(\frac{a}{n-1} + b\right)[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{5.8.2}$$

Equation (5.8.1) can be written as

$$R(X,Y)\bar{P}(U,V)Z - \bar{P}(R(X,Y)U,V)Z - \bar{P}(U,R(X,Y)V)Z$$

$$-\bar{P}(U,V)R(X,Y)Z = L_{\bar{P}}[(X\wedge Y)\bar{P}(U,V) - \bar{P}((X\wedge Y)U,V)Z$$

$$-\bar{P}(U,(X\wedge Y)V)Z - \bar{P}(U,V)(X\wedge Y)Z]. \tag{5.8.3}$$

Taking $X = U = \xi$ in (5.8.3) and by the definition of endomorphism i.e., $(X\wedge Y)Z = g(Y,Z)X - g(X,Z)Y$ also by virtue of (1.1.30), (5.8.2) we can obtain

$$(L_{\bar{P}}+1)\{\bar{P}(Y,V)Z + \left[a + b(n-1) + \frac{r}{n}\left(\frac{a}{n-1} + b\right)\right][g(V,Z)Y - g(Y,Z)V]\} = 0. \tag{5.8.4}$$

There either $L_{\bar{P}} = -1$ or

$$\bar{P}(Y,V)Z = \left[a + b(n-1) + \frac{r}{n}\left(\frac{a}{n-1} + b\right)\right][g(Y,Z)V - g(V,Z)Y] \tag{5.8.5}$$

Now on contraction of (5.8.5) and using (5.8.2) we get the Ricci tensor

$$S(V,Z) = -(n-1)g(V,Z). \tag{5.8.6}$$

Thus we state the following:

Lemma 5.8.1. *Pseudo-projectively pseudo-symmetric para-Kenmotsu manifold is Einstein provided* $L_{\bar{P}} \neq -\left(a + b(n-1) + \frac{r}{n}\left(\frac{a}{n-1} + b\right)\right)$.

Theorem 5.8.2. *Ricci soliton in para-Kenmotsu manifold satisfying the condition* $R\cdot\bar{P} = L_{\bar{P}}Q(g,\bar{P})$ *which admit conformal Ricci soliton* (g,V,λ,ρ) *is*

1. *shrinking, if* $\rho < 2(n-2)(\lambda < 0)$.

2. *steady, if* $\rho = 2(n-2)(\lambda = 0)$.

3. *expanding, if* $\rho > 2(n-2)(\lambda > 0)$.

5.9 Ricci soliton in para-Kenmotsu manifolds satisfying $\bar{P} \cdot R = L_R Q(g, R)$ admitting conformal Ricci soliton

Let us assume the pseudo-symmetric condition

$$\bar{P} \cdot R = L_R Q(g, R). \tag{5.9.1}$$

Equation (5.9.1) can be written as

$$\begin{aligned} &\bar{P}(X,Y)R(U,V)Z - R(\bar{P}(X,Y)U,V)Z - R(U,\bar{P}(X,Y)V)Z \\ &-R(U,V)\bar{P}(X,Y)Z = L_R[(X \wedge Y)R(U,V) - R((X \wedge Y)U,V)Z \\ &-R(U,(X \wedge Y)V)Z - R(U,V)(X \wedge Y)Z]. \end{aligned} \tag{5.9.2}$$

Taking $X = U = \xi$ in (5.9.2) and by the definition of endomorphism i.e., $(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y$ also by virtue of (1.1.30), (5.8.2) we can obtain

$$\left[L_R + \left(a + b(n-1) + \frac{r}{n}\left(\frac{a}{n-1} + b\right)\right)\right]\{R(Y,V)Z + [g(V,Z)Y - g(Y,Z)V]\} = 0. \tag{5.9.3}$$

There either $L_R = -\left(a + b(n-1) + \frac{r}{n}\left(\frac{a}{n-1} + b\right)\right)$ or

$$R(Y,V)Z = g(Y,Z)V - g(V,Z)Y. \tag{5.9.4}$$

Now on contraction of (5.9.4) we get the Ricci tensor

$$S(V,Z) = -(n-1)g(V,Z). \tag{5.9.5}$$

Thus we state the following:

Lemma 5.9.1. *Para-Kenmotsu manifold satisfying the pseudo-symmetric condition $\bar{P} \cdot R = L_R Q(g, R)$ is Einstein provided $L_R \neq -\left(a + b(n-1) + \frac{r}{n}\left(\frac{a}{n-1} + b\right)\right)$.*

Theorem 5.9.2. *Ricci soliton in para-Kenmotsu manifold satisfying the condition $\bar{P} \cdot R = L_R Q(g, R)$ which admitting conformal Ricci soliton (g, V, λ, ρ) is*

1. *shrinking, if $\rho < 2(n-2)(\lambda < 0)$.*

2. *steady, if $\rho = 2(n-2)(\lambda = 0)$.*

3. *expanding, if $\rho > 2(n-2)(\lambda > 0)$.*

5.10 Conclusion

The influential results finding of this chapter are as follows:

- A Kenmotsu manifold admitting semi-symmetric metric connection is η-Einstein manifold.

- Ricci soliton in semi-symmetric, pseudo-symmetric, pseudo projective semi-symmetric Kenmotsu manifolds admitting semi-symmetric metric connection is expanding with respect to Levi-Civita connection.

  We have obtained the following result for conformal Ricci soliton in para-Kenmotsu manifolds:

- Ricci soliton in para-Kenmotsu manifold satisfying the pseudo-symmetric conditions $R \cdot C = L_C Q(g, C)$, $C \cdot R = L_R Q(g, R)$, $R \cdot \bar{P} = L_{\bar{P}} Q(g, \bar{P})$ and $\bar{P} \cdot R = L_R Q(g, R)$ which admit conformal Ricci soliton (g, V, λ, ρ) is

1. shrinking, if $\rho < 2(n-2)(\lambda < 0)$.

2. steady, if $\rho = 2(n-2)(\lambda = 0)$.

3. expanding, if $\rho > 2(n-2)(\lambda > 0)$.

CHAPTER 6

Publications based on this Chapter

- *A Study on $(LCS)_n$-Manifolds Admitting η-Ricci Soliton,* (Accepted in JMI International Journal Of Mathematical Sciences).
- *Ricci Soliton in Irrotational $(LCS)_n$-Manifolds,* (Communicated).

Chapter 6

On $(LCS)_n$-manifolds

6.1 Introduction

The notion of Lorentzian concircular structure manifolds (briefly $(LCS)_n$-manifolds) with an example was introduced by Shaikh [74], which generalize the notion of LP-Sasakian manifolds introduced by Mantsumoto [53] and also by Mihai and Rosca [54]. Then Shaikh and Baishya [75], [76] investigated the applications of $(LCS)_n$-manifolds to the general theory of relativity and cosmology. The $(LCS)_n$-manifolds are also studied by Atceken et al. [10], [9] and many authors. The interest in studying Ricci solitons has considerably increased and has been carried out in many contexts; on Kenmotsu manifolds, α-Sasakian manifolds, trans-Sasakian manifolds, Lorentzian α-Sasakian manifolds, $(LCS)_n$-manifolds, f-Kenmotsu manifolds respectively. Recently Blaga studied the η-Ricci soliton on Lorentzian para-Sasakian manifolds and on para-Kenmotsu manifolds [19], [18]. In [68] the authors studied the η-Ricci solitons on para-Sasakian manifolds. Recently Chandra, Hui and Shaikh [25], Hui and Chakraborty [43], [44] have also studied Ricci solitons and η-Ricci solitons in $(LCS)_n$-manifolds. However authors [25] have used Esienhart problem to study Ricci solitons In this chapter we study the η-Ricci soliton condition for semi-symmetric and pseudo-symmetric $(LCS)_n$ manifold, also irrotational $(LCS)_n$ manifolds and get the following results.

6.2 η-Ricci soliton on $(LCS)_n$-manifolds

Let $M(\phi, \xi, \eta, g)$ be an n-dimensional Lorentzian concircular structure manifold and let $(M, (g, \xi, \lambda, \mu))$ be a $(LCS)_n$ η-Ricci soliton. Then the relation (1.4.2) implies

$$(L_\xi g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) + 2\mu\eta(X)\eta(Y) = 0,$$

$$2S(X, Y) = -(L_\xi g)(X, Y) - 2\lambda g(X, Y) - 2\mu\eta(X)\eta(Y). \tag{6.2.1}$$

Here $L_\xi g$ denotes the Lie derivative of Riemannian metric g along a vector field ξ, by the definition of Lie derivative we have

$$(L_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X). \tag{6.2.2}$$

Using (1.1.35) in (6.2.2) we obtain

$$(L_\xi g)(X, Y) = 2\alpha\{g(X, Y) + \eta(X)\eta(Y)\}. \tag{6.2.3}$$

Using (6.2.1) and (6.2.3) we can write

$$S(X, Y) = (-\alpha - \lambda)g(X, Y) + (-\alpha - \mu)\eta(X)\eta(Y). \tag{6.2.4}$$

Thus we state the following theorem:

Theorem 6.2.1. *An $(LCS)_n$ η-Ricci soliton $(M, (g, \xi, \lambda, \mu))$ is an η-Einstein manifold.*

In particular, if $\mu = 0$ in (6.2.4), then it reduces to

$$S(X,Y) = (-\alpha - \lambda)g(X,Y) - \alpha\eta(X)\eta(Y). \tag{6.2.5}$$

Thus we state the following:

Corollary 6.2.2. *An $(LCS)_n$-Ricci soliton $(M,(g,\xi,\lambda))$ is an η-Einstein manifold.*

6.3 η-Ricci soliton on pseudo-projective pseudo -symmetric $(LCS)_n$-manifolds

An $(LCS)_n$-manifold M is said to be pseudo-symmetric if M satisfies the condition $R\cdot\bar{P} = L_{\bar{P}}Q(g,\bar{P})$, where $L_{\bar{P}}$ is some smooth function on M and $\bar{P}$ is the pseudo-projective curvature tensor and it is given by

$$\begin{aligned}\bar{P}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] \\ &- \frac{r}{n}\left(\frac{a}{n-1}+b\right)[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{6.3.1}$$

Using (1.1.44), (6.2.5) in (6.3.1) we get

$$\eta(\bar{P}(X,Y)Z) = \vartheta[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{6.3.2}$$

$$\bar{P}(\xi,Y)Z = \vartheta[g(Y,Z)\xi - \eta(Z)Y], \tag{6.3.3}$$

where, $\vartheta = \left[a(\alpha^2-\rho) + b(-\alpha-\lambda) - \frac{r}{n}\left(\frac{a}{n-1}+b\right)\right]$.

$$(R\cdot\bar{P})(U,V,Z;\xi,Y) = L_{\bar{P}}(Q(g,\bar{P})(U,V,Z;\xi,Y)). \tag{6.3.4}$$

L.H.S of (6.3.4) takes the form

$$\begin{aligned}(R\cdot\bar{P})(U,V,Z;\xi,Y) &= R(\xi,Y)\bar{P}(U,V)Z - \bar{P}(R(\xi,Y)U,V)Z \\ &- \bar{P}(U,R(\xi,Y)V)Z - \bar{P}(U,V)R(\xi,Y)Z. \end{aligned} \tag{6.3.5}$$

Taking inner product of (6.3.5) with ξ and by virtue of (1.1.43) and (6.3.2) we can obtain

$$\begin{aligned} g((R\cdot\bar{P})(U,V,Z;\xi,Y),\xi) &= -(\alpha^2-\rho)\bar{P}(U,V,Z,Y) \\ &+ (\alpha^2-\rho)\vartheta[g(Y,U)g(V,Z)-g(Y,V)g(U,Z)]. \end{aligned} \tag{6.3.6}$$

R.H.S of (6.3.4) takes the form

$$\begin{aligned} Q(g,\bar{P})(U,V,Z;\xi,Y) &= (\xi\wedge Y)\bar{P}(U,V)Z-\bar{P}((\xi\wedge Y)U,V)Z \\ &- \bar{P}(U,(\xi\wedge Y)V)Z-\bar{P}(U,V)R(\xi\wedge Y)Z. \end{aligned} \tag{6.3.7}$$

Taking inner product of (6.3.7) with ξ and by using the definition of endomorphism i.e., $(X\wedge Y)Z=g(Y,Z)X-g(X,Z)Y$ and (6.3.2) we can obtain

$$\begin{aligned} g(Q(g,\bar{P})(U,V,Z;\xi,Y),\xi) &= -\bar{P}(U,V,Z,Y) \\ &+ \vartheta[g(Y,U)g(V,Z)-g(Y,V)g(U,Z)]. \end{aligned} \tag{6.3.8}$$

Using equations (6.3.6) and (6.3.8) in (6.3.4) we can get

Either $L_{\bar{P}}=(\alpha^2-\rho)$ or

$$-\bar{P}(U,V,Z,Y)+\vartheta[g(Y,U)g(V,Z)-g(Y,V)g(U,Z)]=0. \tag{6.3.9}$$

Let $\{e_1,e_2,\dots,e_n\}$ be an orthonormal basis of the tangent space at each point of the manifold. Putting $U=Y=e_i$ in (6.3.9) and taking summation over $i,(1\leq i\leq n)$ and using equation (6.3.1) we get

$$S(V,Z)=\left[\frac{a(\alpha^2-\rho)+b(-\alpha-\lambda)}{a+b(n-1)}\right]g(V,Z). \tag{6.3.10}$$

Thus we can state the following:

Theorem 6.3.1. *A pseudo-projective pseudo-symmetric $(LCS)_n$ η-Ricci soliton $(M,(g,\xi,\lambda,\mu))$ is an Einstein manifold provided $L_{\bar{P}}\neq(\alpha^2-\rho)$.*

Similarly we obtain the same result for pseudo-projective semi-symmetric $(LCS)_n$-manifold and we can state the following:

Corollary 6.3.2. *An pseudo-projectively semi-symmetric $(LCS)_n$ η-Ricci soliton $(M,(g,\xi,\lambda,\mu))$ is an Einstein manifold.*

In particular, if $\mu = 0$ in (6.2.4) and comparing with (6.3.10) and contracting we get the value of λ as

$$\lambda = \frac{an(\alpha^2-\rho)}{n(b-1)} + \frac{\alpha(a+b(n-1))}{n(b-1)} - \alpha. \tag{6.3.11}$$

Thus, we can state the following:

Corollary 6.3.3. *A Ricci soliton in pseudo-projective pseudo-symmetric manifolds is given by (6.3.11)*

6.4 η-Ricci soliton on $(LCS)_n$-manifolds admitting pseudo-symmetric condition $\bar{P}\cdot R = L_R Q(g,R)$

$$(\bar{P}(\xi,Y)\cdot R)(U,V)Z = L_R[((\xi\wedge Y)\cdot R)(U,V)Z], \tag{6.4.1}$$

which implies

$$\begin{aligned}
&\bar{P}(\xi,Y)R(U,V)Z - R(\bar{P}(\xi,Y)U,V)Z - R(U,\bar{P}(\xi,Y)V)Z\\
&-R(U,V)\bar{P}(\xi,Y)Z = L_R[(\xi\wedge Y)R(U,V)Z - R((\xi\wedge Y)U,V)Z\\
&-R(U,(\xi\wedge Y)V)Z - R(U,V)(\xi\wedge Y)Z].
\end{aligned} \tag{6.4.2}$$

Taking inner product of (6.4.2) with ξ and using (6.3.3) and (1.1.44) we can get

Either $L_R = \vartheta$, or

$$R(U,V,Z,Y) = (\alpha^2-\rho)[g(Y,U)g(V,Z) - g(Y,V)g(U,Z)]. \tag{6.4.3}$$

Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of the tangent space at each point of the manifold. Putting $U = Y = e_i$ in (6.4.3) and taking summation over i,$(1 \leq i \leq n)$, we get

$$S(V, Z) = (\alpha^2 - \rho)(n-1)g(V, Z). \tag{6.4.4}$$

Thus we can state the following:

Theorem 6.4.1. *An $(LCS)_n$ η-Ricci soliton $(M, (g, \xi, \lambda, \mu))$ admitting pseudo-symmetric condition $\bar{P} \cdot R = L_R Q(g, R)$ is an Einstein manifold provided $L_R \neq \vartheta$.*

Consequently we state the following:

Corollary 6.4.2. *An $(LCS)_n$ η-Ricci soliton $(M, (g, \xi, \lambda, \mu))$ admitting semi-symmetric condition $\bar{P} \cdot R = 0$ is an Einstein manifold.*

6.5 Ricci soliton in irrotational pseudo-projective $(LCS)_n$-manifolds

Let (g, V, λ) be a Ricci soliton in an n-dimensional $(LCS)_n$-manifold M. From (1.1.35) we have

$$(L_\xi g)(X, Y) = 2\alpha[g(X, Y) - \eta(X)\eta(Y)]. \tag{6.5.1}$$

From (1.4.1) and (6.5.1) we get

$$S(X, Y) = -[(\alpha + \lambda)g(X, Y) + \alpha\eta(X)\eta(Y)]. \tag{6.5.2}$$

The above equation yields that

$$QX = -[(\alpha+\lambda)X + \alpha\eta(X)\xi], \tag{6.5.3}$$

$$S(X,\xi) = -\lambda\eta(X), \tag{6.5.4}$$

$$r = -\lambda n - \alpha(n-1). \tag{6.5.5}$$

The pseudo-projective curvature tensor $\bar{P}$ is given by (6.3.1). Put $Z = \xi$ in (6.3.1) and using (1.1.42) and (6.5.2) we get

$$\bar{P}(X,Y)\xi = \vartheta[\eta(Y)X - \eta(X)Y], \tag{6.5.6}$$

where $\vartheta = a(\alpha^2 - \rho) - \lambda b - \frac{r}{n}\left(\frac{a}{n-1} + b\right)$.

The rotation (curl) of pseudo-projective curvature tensor $\bar{P}$ on a Riemannian manifold is given by

$$Rot\bar{P} = (\nabla_U\bar{P})(X,Y,Z) + (\nabla_X\bar{P})(U,Y,Z) + (\nabla_Y\bar{P})(U,X,Z) - (\nabla_Z\bar{P})(X,Y,U). \tag{6.5.7}$$

By virtue of second Bianchi identity

$$(\nabla_U\bar{P})(X,Y,Z) + (\nabla_X\bar{P})(U,Y,Z) + (\nabla_Y\bar{P})(U,X,Z) = 0. \tag{6.5.8}$$

i.e.,

$$curl\bar{P} = -(\nabla_Z\bar{P})(X,Y,U).$$

If the pseudo-projective curvature tensor is irrotational then $curl\bar{P} = 0$ and we have $(\nabla_Z\bar{P})(X,Y,U) = 0$. Which implies

$$\nabla_Z\{\bar{P}(X,Y)U)\} = \bar{P}(\nabla_Z X,Y)U + \bar{P}(X,\nabla_Z Y)U + \bar{P}(X,Y)\nabla_Z U. \tag{6.5.9}$$

Put $U = \xi$ in (6.5.9) and by virtue of (1.1.34), (1.1.35), (1.1.36) and (6.5.6) we can get

$$\bar{P}(X,Y)Z = \vartheta\{g(Y,Z)X - g(X,Z)Y\}. \tag{6.5.10}$$

Taking inner product of (6.5.10) with W

$$\bar{P}(X,Y,Z,W) = \vartheta\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}. \tag{6.5.11}$$

On contraction of equation (6.5.11) over X and W and using (6.3.1) we get

$$S(Y,Z) = \left[\frac{(a(\alpha^2-\rho) - b\lambda)(n-1)}{a+b(n-1)}\right] g(Y,Z) \tag{6.5.12}$$

Put $Y = Z = \xi$ in (6.5.12) and using (6.5.4) we can get the value of λ is given by

$$\lambda = -(n-1)(\alpha^2 - \rho). \tag{6.5.13}$$

We have the following well known established theorem [8]

Theorem 6.5.1. *If $S : \alpha(x_1, x_2, \ldots, x_n) = c$ is a surface (abstract surface or manifold) in R^n then the gradient vector field $\nabla\alpha$ (connected only at points of S) is a non-vanishing normal vector field on the entire surface (abstract surface or manifold) S.*

Remark 6.5.1. Taking a real valued scalar function α associated with an $(LCS)_n$-manifold with $M = R^n$ and $\alpha = c$ we have, $\nabla\alpha$ as a non vanishing normal vector on the submanifold $S \subset M$ and directional derivative of α with respect to ξ, $\rho = \xi\alpha = \xi.\nabla\alpha = |\xi||\nabla\alpha|\cos(\hat{\xi}, \nabla\alpha)$

1. If ξ is tangent to S then $\xi\alpha = 0$.

2. If ξ is tangent to M but not to S then $\xi\alpha \neq 0$.

3. If the angle between ξ and $\nabla\alpha$ is acute then $0 < \cos(\hat{\xi}, \nabla\alpha) < 1$, then $\xi\alpha = k|\nabla\alpha|$, $0 < k < 1$ and $\xi\alpha > 0$.

4. If the angle between ξ and $\nabla\alpha$ is obtuse then $-1 < \cos(\hat{\xi}, \nabla\alpha) < 0$, then $\xi\alpha = k|\nabla\alpha|$, $-1 < k < 0$ and $\xi\alpha < 0$.

We can consequently state the following theorem

Theorem 6.5.2. *A Ricci soliton in irrotational pseudo-projective $(LCS)_n$-manifolds is*

1. *Shrinking, if the characteristic vector field ξ is orthogonal to $\nabla\alpha$.*

2. *Shrinking, if the angle between characteristic vector ξ and the gradient vector $\nabla\alpha$ is acute.*

3. *Shrinking, if the angle between characteristic vector ξ and the gradient vector $\nabla\alpha$ is obtuse then it is Shrinking if $\alpha^2 > k|\nabla\alpha|$, Expanding $\alpha^2 < k|\nabla\alpha|$ and steady $\alpha^2 = k|\nabla\alpha|$.*

Proof. From (6.5.11) and remark (6.5.1), items (1), (3) and (4) respectively we have

1. $\lambda = -(n-1)\alpha^2, \lambda < 0;$

2. $\lambda = -(n-1)(\alpha^2 + k|\nabla\alpha|), \lambda < 0;$

3. $\lambda = -(n-1)(\alpha^2 - k|\nabla\alpha|), \lambda < 0,$

which concludes the proof. □

6.6 Ricci soliton in irrotational quasi-conformal $(LCS)_n$-manifolds

The quasi-confirmal curvature tensor $\tilde{C}$ is given by

$$\begin{aligned}\tilde{C}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &- \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)[g(Y,Z)X - g(X,Z)Y]. \qquad (6.6.1)\end{aligned}$$

Put $Z = \xi$ in (6.6.1) and using (1.1.42), (6.5.2) we get

$$\tilde{C}(X,Y)\xi = \vartheta_2[\eta(Y)X - \eta(X)Y]. \tag{6.6.2}$$

Where $\vartheta_2 = a(\alpha^2 - \rho) - b(2\lambda + \alpha) - \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)$.

The rotation (curl) of quasi-conformal curvature tensor $\tilde{C}$ on a Riemannian manifold is given by

$$Rot\tilde{C} = (\nabla_U \tilde{C})(X,Y,Z) + (\nabla_X \tilde{C})(U,Y,Z) + (\nabla_Y \tilde{C})(U,X,Z) - (\nabla_Z \tilde{C})(X,Y,U). \tag{6.6.3}$$

By virtue of second Bianchi identity

$$(\nabla_U \tilde{C})(X,Y,Z) + (\nabla_X \tilde{C})(U,Y,Z) + (\nabla_Y \tilde{C})(U,X,Z) = 0. \tag{6.6.4}$$

i.e.,

$$curl\tilde{C} = -(\nabla_Z \bar{M})(X,Y,U).$$

If the quasi-conformal curvature tensor is irrotational then $curl\tilde{C} = 0$ and we have $(\nabla_Z \tilde{C})(X,Y,U) = 0$. Which implies

$$\nabla_Z\{\tilde{C}(X,Y)U)\} = \tilde{C}(\nabla_Z X,Y)U + \tilde{C}(X,\nabla_Z Y)U + \tilde{C}(X,Y)\nabla_Z U. \tag{6.6.5}$$

Put $U = \xi$ in (6.6.5) and by virtue of (1.1.34), (1.1.35), (1.1.36) and (6.6.2) we can get

$$\tilde{C}(X,Y)Z = \vartheta_2\{g(Y,Z)X - g(X,Z)Y\}. \tag{6.6.6}$$

Taking inner product of 6.6.6 with W

$$\tilde{C}(X,Y,Z,W) = \vartheta_2\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}. \tag{6.6.7}$$

On contraction of equation (6.6.7) over X and W and using (6.6.1) we get

$$S(Y,Z) = \left(\frac{a(n-1)(\alpha^2 - \rho) - b(n-1)(2\lambda + \alpha) - br}{a + b(n-2)}\right) g(Y,Z). \tag{6.6.8}$$

Put $Y = Z = \xi$ in (6.6.8) and using (6.5.4) and (6.5.5) we can get the value of λ is given by

$$\lambda = -(n-1)(\alpha^2 - \rho). \tag{6.6.9}$$

We can consequently state the following theorem:

Theorem 6.6.1. *A Ricci soliton in irrotational quasi-conformal $(LCS)_n$-manifolds is*

1. *Shrinking, if the characteristic vector field ξ is orthogonal to $\nabla\alpha$.*

2. *Shrinking, if the angle between characteristic vector ξ and the gradient vector $\nabla\alpha$ is acute.*

3. *Shrinking, if the angle between characteristic vector ξ and the gradient vector $\nabla\alpha$ is obtuse then it is Shrinking if $\alpha^2 > k|\nabla\alpha|$, Expanding $\alpha^2 < k|\nabla\alpha|$ and steady $\alpha^2 = k|\nabla\alpha|$.*

Proof. From (6.6.9) and remark:(6.5.1), items (1), (3) and (4) respectively we have

1. $\lambda = -(n-1)\alpha^2, \lambda < 0;$

2. $\lambda = -(n-1)(\alpha^2 + k|\nabla\alpha|), \lambda < 0;$

3. $\lambda = -(n-1)(\alpha^2 - k|\nabla\alpha|), \lambda < 0,$

which concludes the proof. □

6.7 Ricci soliton in irrotational M-Projective $(LCS)_n$-manifolds

The M-projective curvature tensor $\bar{M}$ is given by

$$\bar{M}(X,Y)Z = R(X,Y)Z - \frac{1}{2(n-1)}\left[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\right]. \tag{6.7.1}$$

Put $Z=\xi$ in (6.7.1) and using (1.1.42), (6.5.2) we get

$$\bar{M}(X,Y)\xi = \vartheta_3[\eta(Y)X - \eta(X)Y]. \tag{6.7.2}$$

Where $\vartheta_3 = (\alpha^2-\rho) + \left(\frac{2\lambda+\alpha}{2(n-1)}\right)$.

The rotation (curl) of M-projective curvature tensor $\bar{M}$ on a Riemannian manifold is given by

$$Rot\bar{M} = (\nabla_U\bar{M})(X,Y,Z) + (\nabla_X\bar{M})(U,Y,Z) + (\nabla_Y\bar{M})(U,X,Z) - (\nabla_Z\bar{M})(X,Y,U). \tag{6.7.3}$$

By virtue of second Bianchi identity

$$(\nabla_U\bar{M})(X,Y,Z) + (\nabla_X\bar{M})(U,Y,Z) + (\nabla_Y\bar{M})(U,X,Z) = 0. \tag{6.7.4}$$

i.e.,

$$curl\bar{M} = -(\nabla_Z\bar{M})(X,Y,U).$$

If the M-projective curvature tensor is irrotational then $curl\bar{M}=0$ and we have $(\nabla_Z\bar{M})(X,Y,U)=0$. Which implies

$$\nabla_Z\{\bar{M}(X,Y)U)\} = \bar{M}(\nabla_Z X,Y)U + \bar{M}(X,\nabla_Z Y)U + \bar{M}(X,Y)\nabla_Z U \tag{6.7.5}$$

Put $U = \xi$ in (6.7.5) and by virtue of (1.1.34), (1.1.35), (1.1.36) and (6.7.2) we can get

$$\bar{M}(X,Y)Z = k_3\{g(Y,Z)X - g(X,Z)Y\}. \tag{6.7.6}$$

Taking inner product of (6.7.6) with W

$$\bar{M}(X,Y,Z,W) = k_3\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}. \tag{6.7.7}$$

On contraction of equation (6.7.7) over X and W and using 6.7.1 we get

$$S(Y,Z) = \frac{2(n-1)}{n}\left[(\alpha^2-\rho)(n-1) + \frac{\lambda(n-2)}{2(n-1)}\right] g(Y,Z). \tag{6.7.8}$$

Put $Y = Z = \xi$ in (6.7.8) and using (6.5.4) and (6.5.5) we can get the value of λ is given by

$$\lambda = -(n-1)(\alpha^2-\rho). \tag{6.7.9}$$

We can consequently state the following theorem:

Theorem 6.7.1. *A Ricci soliton in irrotational M-projective $(LCS)_n$ manifolds is*

1. *Shrinking, if the characteristic vector field ξ is orthogonal to $\nabla\alpha$.*

2. *Shrinking, if the angle between characteristic vector ξ and the gradient vector $\nabla\alpha$ is acute.*

3. *Shrinking, if the angle between characteristic vector ξ and the gradient vector $\nabla\alpha$ is obtuse then it is Shrinking if $\alpha^2 > k|\nabla\alpha|$, Expanding $\alpha^2 < k|\nabla\alpha|$ and steady $\alpha^2 = k|\nabla\alpha|$.*

Proof. From (6.7.9) and remark:(6.5.1), items (1), (3) and (4) respectively we have

1. $\lambda = -(n-1)\alpha^2, \lambda < 0;$

2. $\lambda = -(n-1)(\alpha^2 + k|\nabla\alpha|), \lambda < 0$;

3. $\lambda = -(n-1)(\alpha^2 - k|\nabla\alpha|), \lambda < 0$,

which concludes the proof. □

6.8 Conclusion

The influential results finding of this chapter are as follows:

- An $(LCS)_n\eta$-Ricci soliton $(M, (g, \xi, \lambda))$ is an η-Einstein manifold.

- A pseudo-projectively pseudo-symmetric $(LCS)_n$ η-Ricci soliton $(M, (g, \xi, \lambda, \mu))$ is an Einstein manifold provided $L_{\bar{P}} \neq (\alpha^2 - \rho)$.

- A pseudo-projectively semi-symmetric $(LCS)_n$ η-Ricci soliton $(M, (g, \xi, \lambda, \mu))$ is an Einstein manifold.

- An $(LCS)_n$ η-Ricci soliton $(M, (g, \xi, \lambda, \mu))$ admitting semi-symmetric condition $\bar{P} \cdot R = L_R Q(g, R)$ is an Einstein manifold provided $L_R \neq \left[a(\alpha^2 - \rho) + b(-\alpha - \lambda) - \frac{r}{n}\left(\frac{a}{n-1} + b\right)\right]$.

- An $(LCS)_n$ η-Ricci soliton $(M, (g, \xi, \lambda, \mu))$ admitting semi-symmetric condition $\bar{P} \cdot R = 0$ is an Einstein manifold.

- A Ricci soliton in irrotational pseudo-projective, irrotational quasi-conformal, irrotational M-projective $(LCS)_n$ manifolds is

  1. Shrinking, if the characteristic vector field ξ is orthogonal to $\nabla\alpha$.

2. Shrinking, if the angle between characteristic vector ξ and the gradient vector $\nabla\alpha$ is acute.

3. Shrinking, if $\alpha^2 > k|\nabla\alpha|$, Expanding $\alpha^2 < k|\nabla\alpha|$ and steady $\alpha^2 = k|\nabla\alpha|$, if the angle between characteristic vector ξ and the gradient vector $\nabla\alpha$ is obtuse.

BIBLIOGRAPHY

Bibliography

[1] P. Alegre and A. Sarkar, *Structures on Generalized Sasakian space-forms,* Differential Geom and its application, 26, 656, (2008).

[2] P.Alegre, D. E. Blair, and A.Carriazo, *Generalized Sasakian space-forms,* Israel J. Math., 14, 157, (2004).

[3] P. Alegre and A. Sarkar, *Generalized Sasakian space-forms and conformal changes of metrics,* Results maths., 59, (2011), 485-493.

[4] Ali Akbar, *Some Results on Almost $C(\lambda)$ manifolds,* Internationl Journal of Mathematical Sciences Engineering and Applications(IJMSEA), 7(1), (2013).

[5] Ali Akbar and Avijit Sarkar, *On the Conharmonic and Concircular curvature tensors of almost $C(\lambda)$ manifolds,* International Journal of Advanced Mathematical Sciences, 1(3), (2013).

[6] Ali Akbar and A. Sarkar, *Almost $C(\lambda)$ manifolds admitting W_2 curvarture tensor,* Jornal of Rajasthan Academy of Physical Sciences, ISSN, 13, (2014).

[7] S.R. Ashoka, C.S. Bagewadi, and Gurupadavva Ingalahalli, *Curvature tensor of almost $C(\lambda)$ manifolds,* Malaya J. Mat., 2(1), (2014), 10-15.

[8] S.R. Ashoka, C.S. Bagewadi, and Gurupadavva Ingalahalli, *A geometry on Ricci solitons in* $(LCS)_n$ *manifolds,* Differential Geometry-Dynamical Systems, 16, 2014, pp. 50-62.

[9] M. Atceken and S. K. Hui, *Slant and pseudo-slant submanifolds of* $(LCS)_n$ *-manifolds,* Czechoslovak Math. J., 63 (2013), 177-190.

[10] M. Atceken, *On geometry of submanifolds of* $(LCS)_n$ *-manifolds,* Int. J. Math. and Math. Sci., 2012, doi:10.1155/2012/304-647.

[11] C. S. Bagewadi, Prakasha, D. G., Venkatesha, *Projective curvature tensor on a Kenmotsu manifold with respect to semi-symmetric metric connection,* Stud. Cercet. Stiint. Ser. Mat. Univ. Bacau., 17, (2007), 21-32.

[12] C. S. Bagewadi and Venkatesha, *Some curvature tensors on a Kenmotsu manifold,* The Tensor Society. Tensor. New Series, 68(2), 2007, pp. 140-147.

[13] C. S. Bagewadi, Prakasha, D. G., Venkatesha, *Conformally and quasi-conformally conservative curvature tensors on a trans-Sasakian manifold with respect to semi-symmetric metric connections,* Differ. Geom. Dyn. Syst., 10, (2008), 263-274.

[14] C. S. Bagewadi and G. Ingalahalli, *Ricci solitons in Lorentzian* α*-Sasakian manifolds,* Acta Mathematica, 28(1), 2012, pp. 59-68.

[15] C. S. Bagewadi and G. Ingalahalli, *A study on curvature tensor of a generalized Sasakian sapce form,* Acta Universitatis Apulensis, 38, (2014), pp. 81-93.

[16] C. S. Bagewadi and Sushilabai Adigond, *L.P. Eisenhat problem to Ricci solitons in almost $C(\alpha)$ manifolds,* Bulletin of Calcutta Mathematical Society, 108(1),(2016),01-08.

[17] T. Q. Binh, L. Tamassy, U. C. De, and M. Tarafdar, *Some remarks on almost Kenmotsu manifolds,* Mathematica Pannonica, 13(1), 2002, pp. 31-39.

[18] A. M. Blaga, *Eta-Ricci solitons on para-Kenmotsu manifolds,* Balkan Journal of Geometry and Its Applications, 20(1), 2015, pp. 1-13.

[19] A. M. Blaga, *η-Ricci Solitons on Lorentzian Para-Sasakian Manifolds,* Filomat, 30(2), (2016), 489-496.

[20] D.E. Blair, *Geometry of manifolds with structural group $U(n) \times O(s)$,* J. Differential Geom., 4, (1970), 155-167

[21] D.E. Blair, G. Ludden and K. Yano, *Differential geometric structures on principal toroidal bundles,* Trans. Amer. Math. Soc., 181, (1973), 175-184.

[22] C. Călin and M. Crasmareanu, *From the Eisenhart problem to Ricci solitons in f-Kenmotsu manifolds,* Bulletin of the Malaysian Mathematical Sciences Society, 33(3), 2010, pp. 367-368.

[23] C. Calin, M. Crasmareanu, *Eta-Ricci solitons on Hopf hypersurfaces in complex space forms,* Revue Roumaine de Mathematiques pures et appliques, 57(1), (2012), 5563.

[24] E. Cartan, *Surune classe remarquable d' esspace de Riemannian,* Bulletin de la Societe Mathematique de France, 54, (1926), 214-264.

[25] S. Chandra, S. K. Hui and A. A. Shaik, *Second order parallel tensors and Ricci solitons on* $(LCS)_n$*-manifolds,* Commun. Korean Math. Soc., 30, (2015), 123-130.

[26] J. T. Cho, M. Kimura, *Ricci solitons and real hypersurfaces in a complex space form,* Tohoku Math. J., 61(2), (2009), 205-212.

[27] U.C. De, J. B. Jun and A. K. Gazi, *Sasakian manifolds with quasi-conformal curvature tensor,* Bull. Korean Math. Soc., 45(2), (2008), pp. 313-319.

[28] U.C. De and A. Sarkar, *On the projective curvature tensor of generalized Sasakian space-forms and conformal changes of metrics,* Quaestiones Mathematicae, 33(2), 245(2010).

[29] U.C. De, C. Ozgur and A. K. Mondal, *On* ϕ*-quasi-conformally symmetric Sasakian manifolds,* Indag. Mathern., N.S., 20(2), (2009), 191-200.

[30] S. Debnath and A. Battacharya, *Second order parallel tensor in Trans-Sasakian Manifolds and connection with Ricci soliton,* Lobachevski Journal of Mathematics, Vol. 33, No. 4, (2012), 312-316.

[31] R. Deszcz, *On pseudo symmetric spaces,* Bull. Soc. Math., Belg. Ser., 44 (1992), 1-34.

[32] L.P. Eisenhart, *Symmetric tensors of the second order whose first covariant serivates are zero,* Trans. Amer. Math. Soc. 25, no-2 (1923), 297-306.

[33] A. E. Fischer, *An introduction to conformal Ricci flow,* Class. Quantum Grav.21(2004), S171-S218.

[34] A. Friedmann, J. A. Schouten, *Uber die Geometrie der halbsymmetrischen Ubertragung,* Math. Z., 21, (1924), 211-223.

[35] S.I. Goldberg and K. Yano, *On normal globally framed f-manifolds,* Tohoku Math. J. 22 (1970), 362-370.

[36] S.I. Goldberg and K. Yano, *Globally framed f-manifolds,* Illinois J. Math., 15, (1971), 456-474.

[37] R. S. Hamilton, *Three manifold with positive Ricci curvature,* J. Differential Geom., 17(2),(1982), 256-306.

[38] R. S. Hamilton, *Four - manifolds with positive curvature operator,* J.Differential Geom., 24(2),(1986),153-179.

[39] I. Hasegawa, Y. Oknyama and T. Abe, *On the p-th Sasakian manifolds,* J. Hokkaido Univ. Ed. SectII A, 37(1), (1986), 1-16.

[40] H. A. Hayden, *Subspaces of a space with torsion,* Proc. London Math. Soc., 34, (1932), 27-50.

[41] C. He and M. Zhu, *Ricci solitons on Sasakian manifolds,* ariXv:1109.4407v2[math.DG], 2011.

[42] S. K. Hui and A. Sarkar, *On the W2 curvature tensor of generalized Sasakian space-forms,* Math. Pannonica.

[43] S. K. Hui and D. Chakraborty, *Some types of Ricci solitons on $(LCS)_n$-manifolds,* J. Math. Sci. Advances and Applications, 37, (2016), 1-17.

[44] S. K. Hui and D. Chakraborty, *η-Ricci solitons on η-Einstein $(LCS)_n$-manifolds,* Acta Univ. Palac. Olom., Fac. Rer. Nat., Math., 55(2), (2016).

[45] G. Ingalahalli, C. S. Bagewadi, *A study on conservative C-Bochner curvature tensor in K-contact and Kenmotsu manifolds admitting semi-symmetric metric connection,* ISRN Geometry(2012).

[46] G. Ingalahalli and C. S. Bagewadi, *Ricci solitons in α-Sasakain manifolds,* ISRN Geometry, Article ID 421384, 14 pages, 2012.

[47] S. Ishihara, *Normal structure f satisfying $f_3 + f = 0$,* Kodai Math. Sem. Rep., 18, (1966), 36-47.

[48] D. Janssens and L. Vanhecke, *Almost contact structures and curvature tensors,* Kodai Math. J., 4, (1981), 1-27.

[49] S. Kaneyuki, F.L. Williams, *Almost paracontact and parahodge structure on manifolds,* Nagoya Math. J., 99, (1985), 173-187.

[50] K. Kenmotsu, *A class of almost contact Riemannian manifolds,* Tohoku Math. J., 24, (1972), 93-103.

[51] U. K. Kim, *Conformally flat generalized Sasakian-space-forms and locally symmetric generalized Sasakian-space-forms,* Note di Mathematica 26, (2006), 55-67.

[52] S.V. Kharitonova, *Almost $C(\lambda)$ manifolds,* Journal of Mathematical Sciences, 177(5), (2011).

[53] K. Matsumoto, *On Lorentzian almost paracontact manifolds,* Bull. of Yamagata Univ. Nat. Sci. 12, (1989), 151-156.

[54] I. Mihai and R. Rosca, *On Lorentzian para-Sasakian manifolds,* Classical Anal., World Sci. Publ., Singapore, 155-169, (1992).

[55] C. Murathan, C. Ozgur, *Riemannian manifolds with a semi-symmetric metric connection satisfying some semisymmetry conditions,* Proc. Est. Acad. Sci., 57(4), (2008), 210-216.

[56] H. G. Nagaraja and C. R. Premalatha, *Ricci solitons in Kenmotsu manifolds,* Journal of Mathematical Analysis, 3(2), 2012, pp. 18-24.

[57] H. Nakagawa, *f-structures induced on submanifolds in spaces, almost Hermitian or Kaehlerian,* Kodai Math. Sem. Rep., 18 (1966), 161-183.

[58] H. Nakagawa, *On framed f-manifolds,* Kodai Math. Sem. Rep., 18 (1966), 293-306.

[59] Nirabhra Basu and Arindam Bhattacharyya, *Conformal Ricci soliton in Kenmotsu manifold,* Global Journal of Advanced Research on Classical and Modern Geometries, 4(1), (2015), pp.15-21.

[60] K. Nomizu, *On hypersurface satisfying a certain condition on the curvature tensor,* Tohoku Math. J., 20, 45(1968).

[61] Y.A. Ogawa, *Conditions for a compact Kaehalerian space to be locally symmetric,* Natur. Sci. Report, Ochanomizu Univ., 28,21, (1977).

[62] Z. Olszak and R. Rosca, *Normal locally conformal almost cosymplectic manifolds,* publ. Math. Debrecen, 39(1991).

[63] C. Özgür, *On Kenmotsu manifolds satisfying certain pseudosymmetry conditions,* World Applied Sciences Journal, 1(2), (2006), 144-149.

[64] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications,* arxiv.org/abs/math/0211159v1.

[65] G. Perelman, *Ricci flow with surgery on three manifolds,* arxiv.org/abs/math/0303109.

[66] D. G. Prakasha, C. S. Bagewadi, Venkatesha , *Conservative conformal and quasi conformal curvature tensors on K-contact manifolds with respect to semi-symmetric metric connection,* Tamsui Oxf. J. Math. Sci., 25(1), (2009), 27-38.

[67] D.G. Prakasha, *On Generalized Sasakian-Space-Forms with Weyl-Conformal Curvature Tensor,* Lobachevskii Journal of Mathematics, 33(3), 2012.

[68] D. G. Prakasha and B. S. Hadimani, *η-Ricci solitons on para-Sasakian manifolds,* Journal of Geometry, (2016) DOI 10.1007/s00022-016-0345-z.

[69] M. Prvanovic, *On some classes of semi-symmetric metric connection in a locally decomposable Riemannian space,* Facta. Univ. (Nis), Ser. Math. Inform., 10, (1995), 105-116.

[70] I. Sato, *On a structure similar to the almost contact structure I,* Tensor N.S, 30, (1976), 219-224.

[71] B.B. Sinha, K.L. Prasad, *A class of Almost paracontact metric manifold,* Bull. Calcutta Math, Soc., 87, (1995), 307-312.

[72] R. Sharma, *Second order parallel tensor on contact manifolds II,* C.R. Math Rep. Acad. Sci. Canada XIII, No-6, 6 (1991), 259-264.

[73] R. Sharma, *Certain results on K-contact and (k, μ)-contact manifolds,* Journal of Geometry, 89(1-2), (2008), pp.138-147.

[74] A. A. Shaikh, *On Lorentzian almost paracontact manifolds with a structure of the concircular type,* Kyungpook Math. J., 43, (2003), 305-314.

[75] A. A. Shaikh and K. K. Baishya, *On concircular structure spacetimes,* J. Math. Stat., 1, (2005), 129-132.

[76] A. A. Shaikh and K. K. Baishya, *On concircular structure spacetimes II,* American J. Appl. Sci., 3(4), (2006), 1790-1794.

[77] A. A. Shaikh, S. K. Hui, *On pseudo cyclic Ricci symmetric manifolds admitting semi-symmetric metric connection,* Sci. Ser. A Math. Sci. (N.S.), 20, (2010), 73-80.

[78] Z.I. Szabo, *Structure theorems on Riemannian spaces satiesfying $R(X, Y) \cdot R = 0$,* I, The local version, J. Differential Geom., 17, 531, (1982).

[79] T. Takahashi, *Sasakian manifold with pseudo-Riemannian metric,* Thoku Math. J., 21(2), (1969), 644-653.

[80] S. Tanno, *Locally symmetric K-contact Riemannian manifolds,* Proc. Japan Acad., 43, (1967), 581.

[81] S. Tanno, *The automorphism groups of almost contact Riemannian manifolds,* Tohoku Math. J., 2, (1969), 21-38.

[82] M. M. Tripathi, *Ricci solitons in contact metric manifolds,* http://arxiv.org/abs/0801.42-22.

[83] M. M. Tripati and P. Gupta, *On τ-curvature tensor in K-contact and Sasakian manifolds,* International Electronic Journal of Geometry, 4(1), (2011), pp. 32-47.

[84] K. Yano, *On a structure defined by a tensor field f of type $(1,1)$ satisfying $f_3 + f = 0$,* Tensor N.S. 14 (1963), 99-109.

[85] K. Yano, *On semi-symmetric metric connections,* Rev. Roumaine Math. Pures Appl., 15, (1970), 1579-1586.

[86] S. Zamkovoy, *Canonical connections on paracontact manifolds,* Ann. Glob. Anal. Geom., 36(1), (2009), 37-60.

[87] Zerrin Senturk, *Pseudosymmetry in semi-Riemannian manifolds.*