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A STUDY ON RICCI SOLITONS IN KAHLER MANIFOLDS

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by

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Dedicated To

My Beloved Parents,
Family Members,
My Guide and All My Teachers.

DECLARATION

I hereby declare that the thesis entitled **A Study on Ricci Solitons in Kahler Manifolds**, submitted to the Faculty of Science and Technology, Kuvempu University for the award of the degree of Doctor of Philosophy in Mathematics is the result of the research work carried out by me in the Department of Mathematics, Kuvempu University under the guidance of **Prof.C.S. Bagewadi**, Retd. Professor (Emeritus Fellow-UGC), Department of P.G. Studies and Research in Mathematics, Kuvempu University, Jnana Sahyadri, Shankaraghatta.

I further declare that this thesis or part thereof has not been previously formed the basis of the award of any degree, associateship etc., of any other University or Institution.

Place: Jnana Sahyadri

Date: 05/02/2017.

Praveena M.M.

CERTIFICATE

This is to certify that the thesis entitled **A Study on Ricci Solitons in Kahler Manifolds**, submitted to the Faculty of Science and Technology, Kuvempu University for the award of the degree of Doctor of Philosophy in Mathematics by **Praveena M.M.** is the result of bonafide research work carried out by him under my guidance in the Department of P.G. Studies and Research in Mathematics, Kuvempu University, Jnana Sahyadri, Shankaraghatta.

This thesis or part thereof has not been previously formed the basis of the award of any degree, associateship etc., of any other University or Institution.

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CONTENTS

Preface

Geometry is a part of mathematics which focuses on the study of size, shape, relative configuration and spatial properties. Differential geometry is an old mathematical discipline and well studied after the foundation of calculus laid by Newton and Leibnitz. Gauss laid down the foundation of differential geometry of surface in three dimensional Euclidean space in early nineteenth century. In fact the theory of plane and space curves and of surfaces in the three dimensional Euclidean space formed the basis for the initial development. Riemann (1854) extended the concept to more than three dimensions and the object is known as abstract surface i.e., manifold and resembles Euclidean space of certain dimension called the dimension of the manifold. Thus a line and a circle are one dimension manifolds, a plane and the surface of a ball are two dimensional manifolds. However for smooth manifolds one has to put additional structure such as differentiability and analyticity. The multilinear algebra is used to study the concepts of curvature and various other geometric properties of curves and spaces. At the end of nineteenth century Levi-Civita and Ricci developed the concept of parallel translation in the classical language of tensors.

In 1868, after the publication of Riemann work, a number of mathematicians like Beltrami (1868), Lipschitz (1869) enriched the subject by introducing the idea of Christoffel symbols, covariant differentiation and Gauss equations et. al. The manifold equipped with Riemannian metric is known as Riemannian manifold. Schouten and Dantzing (1930) introduced the concept of complex structure and a Hermitian metric on a differentiable manifold and called it as a Hermitian manifold. In 1933, K$\ddot{a}$hler presented the idea of a K$\ddot{a}$hlerian structure on a complex manifold. Ehresmann (1950) defined an almost complex structure on an even dimensional differentiable manifold.

If J is a linear endomorphism of $T_p(M)$ defined by $JU = iU$. $\forall U \in T_p(M)$, then $J^2 = -I$, where I is the identity transformation. This linear transformation is called the almost complex structure attached to M.

Here we define the torsion tensor filed T of type $(0, 2)$ of an almost complex structure J by

$$T(U, V) = [JU, JV] - J[U, JV] - J[JU, V] - [U, V],$$

for any vector fields U and V on M.

Let M be an almost complex manifold with almost complex structure J. Then J is a complex structure if and only if J has no torsion. Let M be a complex manifold with complex structure J. A Hermition metric on M is a Riemannian metric g such that $g(JU, JV) = g(U, V)$ for any vector fields U and V on M.

A complex manifold with a Hermition metric is called a Hermition manifold. The fundamental 2-form Ω of M is defined by $\Omega(U, V) = g(U, JV)$.

A Hermition metric g on a complex manifold M is called a K$\ddot{a}$hlerian metric if the fundamental 2-form Ω is closed i.e., $d\Omega = 0$. A complex manifold M with a K$\ddot{a}$hlerian metric is called a Kählerian manifold.

The more differential aspects of Kähler spaces were studied by Bochner (1947-50), Calabi and Spencer (1951), Hodge (1951), Goldberg (1960), Tachibana and Yano (1965), Mishra (1969), Kon, Blair, Chen, Bagewadi, Shahid, Deszcz, De, Shaikh et. al.

The Ricci flow is a powerful technique in understanding the geometry and topology of Riemannian manifolds. Intuitively, the idea is to set up a PDE that evolves a metric according to its Ricci curvature. The resulting equation has much in common with the heat equation, which tends to flow a given function to ever nicer functions. By analogy, the Ricci flow evolves an initial metric into improved metrics. Hamilton [36] introduced the Ricci flow and his ideas gave rise to Perelman's [43] proof of the Poincare conjecture in three dimensional topology. Cao [10] observed that the Kähler condition is preserved under Hamilton's Ricci flow, and to achieve this Cao wrote out a scalar parabolic equation satisfied by the Kähler potential of the Kähler-Ricci flow. The Kähler-Ricci flow has become a major tool in Kähler geometry, where as in the field of geometric evolution equations, the singularity modes which arise are usually ancient solutions, where the solutions exist all the way back to time minus infinity. Among such 'long-existing' solutions are the selfsimilar solutions, which in Ricci flow are called Ricci solitons. Thus Ricci solitons are generalizations of Einstein manifolds and they are also called as quasi Einstein manifolds by theoretical physicists. The detail study on Ricci soliton in Kähler manifold was carried out by many authors like Dong, Cao, Hamilton, Zau et. al.

The thesis entitled "A study on Ricci solitons in Kähler manifolds" has been partitioned into six chapters.

The first chapter is all about basic concepts, it includes the definitions and preliminaries which are used in later chapters. The first section is related to Kähler, semisymmetric and pseudo symmetric manifolds, Einstein and some curvature tensors. The Ricci solitons are included in next section. The section three includes real, complex and generalized complex space forms. Fourth section consists of definitions of umblicity and geodesity of submanifolds. Lastly we define quaternion Kähler manifold.

Chapter-2 is devoted to the study of generalized complex space forms. Introduction is the first section of this chapter. In the second section we study Bochner semisymmetric generalized complex space form and show that it is an Einstein manifold. Further it is shown that Ricci soliton is shrinking, steady and expanding accordingly when scalar curvature is positive, zero and negative respectively. In sections three, four, five and six we consider $R \cdot C^{*}$=0, $C^{*} \cdot R = 0$, $C^{*} \cdot S = 0$ and Einstein semisymmetric respectively, on generalized complex space forms and show that they are Einstein manifolds and correspondingly get the values of λ to determine Ricci solitons of these. Section seven includes, H-projective curvature tensor on generalized complex space form. In the next sections we study pseudo-projective curvature tensor, Bochner Ricci-generalized pseudosymmetric, $D \cdot W_2 = L_1 Q(g, W_2)$ and $divD = 0$ on generalized complex space forms. Finally, the last section includes conclusion.

Chapter-3 deals with symmetric properties of Kähler manifolds. First section is devoted to introduction part. In the second section we give the definitions and preliminaries. Section three includes almost pseudo symmetric Kähler Manifold and show that manifold is Ricci flat. In the fourth section we study almost pseudo Bochner symmetric Kähler manifold and show that it is an Einstein manifold and after it is applied to Ricci soliton. It is shrinking, steady and expanding depending upon $r < 0, r = 0$ and $r > 0$. In the fifth and sixth sections, we study almost pseudo Ricci-symmetric and almost pseudo Bochner Ricci-symmetric Kähler manifolds. We consider almost pseudo Bochner symmetric generalized complex space form, almost pseudo Ricci-symmetric generalized complex space form and Bochner flat almost pseudo Ricci-symmetric generalized complex space form in sections seven, eight and nine respectively. Tenth section includes almost pseudo symmetric Kähler manifolds admitting a special type of semi-symmetric non-metric connection $\tilde{\nabla}$. In section eleven we consider projective flat almost pseudo symmetric Kähler manifolds admitting a special type of semi-symmetric non-metric connection $\tilde{\nabla}$. Section twelve is devoted to almost pseudo symmetric Kähler manifolds admitting a special type of semi-symmetric non-metric connection $\tilde{\nabla}$ with parallel projective curvature tensor. In section thirteen we study almost pseudo projective symmetric Kähler manifolds admitting a special type of semi-symmetric non-metric connection $\tilde{\nabla}$. Section fourteen includes almost pseudo symmetric with recurrent Kähler manifolds admitting a special type of semi-symmetric non-metric connection $\tilde{\nabla}$. Finally, the chapter ends with conclusion.

Chapter-4 is devoted to the study of Eisenhart problem to Ricci solitons in Kähler manifolds. Introduction is the first section of this chapter. In the next section we study parallel second order covariant tensor and Ricci soliton in a non-flat real, complex and generalized complex space form. Finally, last section of this chapter is conclusion of all the above space forms.

Chapter-5 deals with submanifolds in real and complex space forms. First and second sections of this chapter consist of introduction and basic concepts. Third section includes parallel and semi-parallel submanifolds in a non-flat real space form. In the fourth section we study recurrent submanifolds in a non-flat real space form. In the fifth section we consider parallel and semi-parallel submanifolds in a non-flat complex space form. The last section of this chapter is conclusion.

Chapter-6 is devoted to the study of Ricci solitons in quaternion space forms. The first section is devoted to the introduction part. The second section is concerned with the basic concepts. In the third section, we consider parallel second order covariant tensor and Ricci soliton in a non-flat quaternion space form. Fourth section is devoted to semisymmetric quaternion space form and we show that space form is an Einstein manifold and after using this it is seen that Ricci soliton is shrinking. In the sections five and six we study the semisymmetric conditions $R \cdot B = 0$ and $B \cdot R = 0$ of quaternion space form. Seventh section includes hypersurface of a quaternion space form. Finally, the last section is the conclusion of above concepts.

Finally, the thesis ends with a list of bibliography and publications.

Chapter 1

Preliminaries

This chapter is introductory and consists of basic concepts, which are used in the later chapters.

1.1 K$\ddot{a}$hler manifold

Definition 1.1.1. A Kähler manifold is a complex n-dimensional manifold M, with a complex structure J and a positive-definite metric g which satisfies the following conditions

$$J^2 = -I, \quad g(JU, JV) = g(U, V) \ \ and \ \ \nabla J = 0,$$

where ∇ means covariant derivation according to the Levi-Civita connection.

Example 1. *Every Riemannian metric on Riemann surface is Kähler.*

Example 2. *The unit complex ball B^n admits a Kähler metric.*

The formulae

$$R(U, V) = R(JU, JV), \tag{1.1.1}$$

$$S(U, V) = S(JU, JV), \tag{1.1.2}$$

$$S(JU, V) + S(U, JV) = 0, \tag{1.1.3}$$

are well known for a Kähler manifold.

Using the second Binachi identity, we infer [53]

$$(divR)(U,V)W = (\nabla_W S)(U,V) - (\nabla_U S)(W,V) = (\nabla_{JV} S)(JU,W). \tag{1.1.4}$$

The scalar curvature $r = S(e_i, e_i)$ and

$$(\nabla_V S)(e_i, e_i) = \nabla_V r = dr(V). \tag{1.1.5}$$

Let Q be the Ricci operator defined by

$$g(QV,W) = S(V,W), \tag{1.1.6}$$

$$(\nabla_U S)(V,W) = g((\nabla_U Q)(V),W). \tag{1.1.7}$$

Taking $U = W = e_i$ and taking summation over i in the above equation we get [53]

$$\begin{aligned}
(\nabla_{e_i} S)(V,e_i) &= g((\nabla_{e_i} Q)(V),e_i). \\
\Rightarrow (divQ)(V) &= tr(U \to (\nabla_U Q)(V)), \\
&= \sum g((\nabla_{e_i} Q)(V),e_i),
\end{aligned}$$

$$(divQ)(V) = \frac{1}{2}dr(V), \tag{1.1.8}$$

$$(\nabla_{e_i} S)(V,e_i) = \frac{1}{2}dr(V). \tag{1.1.9}$$

A Riemannian manifold (M,g) is called locally symmetric if its curvature tensor R is parallel [11] i.e., $\nabla R = 0$, where ∇ denotes the Levi-Civita connection. The notion of semisymmetric manifold is a generalization of locally symmetric manifold and is defined by [51]

$$(R(U,V) \cdot R)(X,Y,W) = 0, \quad X,Y,U,V,W \in \chi(M).$$

For a $(0,k)$-tensor field $\tilde{T}$ on M, $k \geq 1$, we define the tensors $R \cdot \tilde{T}$, $Q(g,\tilde{T})$ and $Q(S,\tilde{T})$ by [31]

$$\begin{aligned}
(R \cdot \tilde{T})(U_1, \dots U_k, U, V) &= -\tilde{T}(R(U,V)U_1, U_2, \dots U_k) - \dots - \tilde{T}(U_1, U_2, \dots U_{k-1}, R(U,V)U_k),\\
Q(g,\tilde{T})(U_1, \dots U_k, U, V) &= -\tilde{T}((U \wedge_g V)U_1, U_2, \dots U_k) - \dots - \tilde{T}(U_1, U_2, \dots U_{k-1}, (U \wedge_g V)U_k),\\
Q(S,\tilde{T})(U_1, \dots U_k, U, V) &= -\tilde{T}((U \wedge_S V)U_1, U_2, \dots U_k) - \dots - \tilde{T}(U_1, U_2, \dots U_{k-1}, (U \wedge_S V)U_k),
\end{aligned}$$

where $(U \wedge_g V)$ and $(U \wedge_S V)$ are the endomorphism given by

$$(U \wedge_g V)Z = g(V,Z)U - g(U,Z)V, \quad (U \wedge_S V)Z = S(V,Z)U - S(U,Z)V. \quad (1.1.10)$$

The notion of pseudosymmetric manifold (in the sense of Deszcz [31]) is a generalization of semisymmetric manifold and is defined by

$$R \cdot R = L_R Q(g,R),$$

and holds on the set $U_R = \{p \in M \mid R - \frac{r}{n(n-1)}G \neq 0 \text{ at } p\}$, where G is the $(0,4)$-tensor defined by $G(V_1,V_2,V_3,V_4)$=$g((V_1 \wedge V_2)V_3, V_4)$ and L_R is some function on U_R.

A Riemannian manifold is said to be Ricci generalized pseudosymmetric (in the sense of Deszcz[31]) if

$$R \cdot R = L_R Q(S,R),$$

holds on the set $U_R = \{p \in M : Q(S,R) \neq 0 \text{ at } p\}$ and L_R is some function on U_R.

Definition 1.1.2. The Einstein tensor denoted by E is defined by

$$E(U,V) = S(U,V) - \frac{r}{n}g(U,V), \quad (1.1.11)$$

where S is a Ricci tensor and r is the scalar curvature.

Given a complex n-dimensional Kähler manifold M, the Bochner curvature tensor D and H-projective curvature tensor $\bar{P}$ are given by [61]

$$\begin{aligned}
D(U,V,W,X) &= R(U,V,W,X) - \frac{1}{2n+4}[g(V,W)S(U,X) - S(U,W)g(V,X) \\
&+ g(JV,W)S(JU,X) - S(JU,W)g(JV,X) + S(V,W)g(U,X) \\
&- g(U,W)S(V,X) + S(JV,W)g(JU,X) - g(JU,W)S(JV,X) \\
&- 2S(V,JU)g(JW,X) - 2S(JW,X)g(JU,V)] \\
&+ \frac{r}{(2n+2)(2n+4)}[g(V,W)g(U,X) - g(U,W)g(V,X) \qquad (1.1.12)\\
&+ g(JV,W)g(JU,X) - g(JU,W)g(JV,X) - 2g(JU,V)g(JW,X)].
\end{aligned}$$

$$\begin{aligned}
\bar{P}(U,V)W &= R(U,V)W - \frac{2}{n+2}[S(V,W)U - S(U,W)V - S(JV,W)JU \\
&+ S(JU,W)JV + S(JU,V)JW - S(JV,U)JW]. \qquad (1.1.13)
\end{aligned}$$

In an n-dimensional Riemannian manifold M, the τ-curvature tensor [56] is given by

$$\begin{aligned}
\tau(U,V)W &= x_0R(U,V)W + x_1S(V,W)U + x_2S(U,W)V + x_3S(U,V)W \\
&+ x_4g(V,W)QU + x_5g(U,W)QV + x_6g(U,V)QW \\
&+ x_7r(g(V,W)U - g(U,W)V), \qquad (1.1.14)
\end{aligned}$$

where R, S, Q and r are the curvature tensor, the Ricci tensor, the Ricci operator and the scalar curvature respectively.

If $x_1 = -x_2 = x_4 = -x_5$, $x_3 = x_6 = 0$ and $x_7 = 2x_2$ then it reduces to B-curvature tensor B [47] i.e.,

$$\begin{aligned}
B(U,V)W = x_0R(U,V)W &+ x_1[S(V,W)U - S(U,W)V + g(V,W)QU - g(U,W)QV] \\
&+ 2x_2r[g(V,W)U - g(U,W)V]. \qquad (1.1.15)
\end{aligned}$$

If $x_0 = a$, $x_1 = -x_2 = x_4 = -x_5 = b$, $x_3 = x_6 = 0$ and $x_7 = -\frac{1}{n}(\frac{x_0}{n-1} + 2x_1)$ then it is reduced to quasi-conformal curvature tensor C^* [63] i.e.,

$$\begin{aligned} C^*(U,V)W = aR(U,V)W &+ b[S(V,W)U - S(U,W)V + g(V,W)QU - g(U,W)QV] \\ &+ \frac{r}{n}[\frac{a}{n-1} + 2b][g(V,W)U - g(U,W)V]. \end{aligned} \tag{1.1.16}$$

If $x_0 = a$, $x_1 = -x_2 = b$, $x_3 = x_6 = x_4 = -x_5 = 0$ and $x_7 = -\frac{1}{n}(\frac{a}{n-1} + b)$ then it is reduced to pseudo-projective curvature tensor P^* [45] i.e.,

$$\begin{aligned} P^*(U,V)W &= aR(U,V)W + b[S(V,W)U - S(U,W)V] \\ &- \frac{r}{n}[\frac{a}{n-1} + b][g(V,W)U - g(U,W)V]. \end{aligned} \tag{1.1.17}$$

If $x_0 = 1$, $x_4 = -x_5 = -\frac{1}{n-1}$ and $x_1 = x_2 = x_3 = x_6 = x_7 = 0$ then it is reduced to W_2-curvature tensor W_2 [44] i.e.,

$$W_2(U,V)Z = R(U,V)Z + \frac{1}{n-1}[g(U,Z)QV - g(V,Z)QU]. \tag{1.1.18}$$

If $x_0 = 1$, $x_1 = -x_2 = x_4 = -x_5 = -\frac{1}{n-2}$, $x_3 = x_6 = 0$ and $x_7 = \frac{1}{(n-1)(n-2)}$ then it is reduced to Weyl-conformal curvature tensor C [33] i.e.,

$$\begin{aligned} C(U,V)W = R(U,V)W &- \frac{1}{n-2}[S(V,W)U - S(U,W)V + g(V,W)QU - g(U,W)QV] \\ &+ \frac{1}{(n-1)(n-2)}[g(V,W)U - g(U,W)V]. \end{aligned} \tag{1.1.19}$$

If $x_0 = 1$, $x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = 0$ and $x_7 = -\frac{1}{n(n-1)}$ then it is reduced to concircular curvature tensor $\tilde{C}$ [64] i.e.,

$$\tilde{C}(U,V)W = R(U,V)W - \frac{r}{n(n-1)}[g(V,W)U - g(U,W)V]. \tag{1.1.20}$$

If $x_0 = 1$, $x_1 = -x_2 = x_4 = -x_5 = -\frac{1}{n-2}$ and $x_3 = x_6 = x_7 = 0$ then it is reduced to conharmonic curvature tensor L^* [38] i.e.,

$$\begin{aligned} L^*(U,V)W = R(U,V)W &- \frac{1}{n-2}[S(V,W)U - S(U,W)V \\ &+ g(V,W)QU - g(U,W)QV]. \end{aligned} \tag{1.1.21}$$

1.2 Ricci Solitons

Let $\phi_t : M \longrightarrow M$, $t \in R$ be a family of diffeomorphisms and $(\phi_t : t \in R)$ is a one parameter family of abelian group called flow. It generates a vector field Y_p given by

$$Y_p f = \frac{df(\phi_t(p))}{dt}, \quad f \in C^\infty(M).$$

If X is a vector field then $L_Y X = lim_{t\to 0}\frac{\phi_t^* X - X}{t}$ is known as Lie derivative of X with respect to Y. Ricci solitons move under the Ricci flow under $\phi_t : M \longrightarrow M$ of the initial metric i.e., they are stationary points of the Ricci flow in space of metrics. If g_0 is a metric on the codomain then $g(t) = \phi_t^* g_0$ is the pullback of g_0, is a metric on the domain. Hence if g_0 is a solution of the Ricci flow on the codomain subject to condition $L_V g_{_0} + 2Ricg_{_0} + 2\lambda g_{_0} = 0$ on the codomain then $g(t)$ is the solution of the Ricci flow on the domain subject to the condition $L_V g + 2Ricg + 2\lambda g = 0$ on the domain under suitable conditions [54]. Here g_0 and $g(t)$ are metrics which satisfy Ricci flow.

Thus the equation in general

$$L_V g + 2S + 2\lambda g = 0. \tag{1.2.1}$$

is called Ricci soliton, where S is Ricci tensor of M, L_V denotes the Lie derivative operator along the vector field V and λ a real scalar. It is said to be expanding, shrinking and steady according as $\lambda > 0$, $\lambda < 0$ and $\lambda = 0$ respectively.

Example 3. *Hamilton Cigar Soliton: Let* $M = R^2$ *and* $\phi_t : R^2 \longrightarrow R^2$ *be defined by* $\phi_t(u, w) = (e^{-2t}u, e^{-2t}w)$ *forms a family of one parameter group of diffeomorphisms. The vector field* Y *generated by* $\{\phi_t\}$ *is* $Y = -2\left(u\frac{\partial}{\partial u} + w\frac{\partial}{\partial w}\right)$. *The metric* g_0 *is obtained as*

$$g_0 = \frac{du^2 + dw^2}{1 + u^2 + w^2},$$

$$\tilde{g}(t) = \phi_t^*(g_0) = \frac{du^2 + dw^2}{e^{4t} + u^2 + w^2},$$

$$Ricg_0 = \frac{2}{1 + u^2 + w^2} g_0,$$

$$L_Y g_0 = \frac{4}{1 + u^2 + w^2} g_0.$$

Using (1.2.1), we have $\lambda = 0$. *Hence, this Ricci soliton is steady and is called cigar soliton as it is asymptotic to a flat cylinder at infinity.*

1.3 Space Forms

Definition 1.3.1. A Riemannian manifold with constant sectional curvature k is called a real space form and its [62] curvature tensor satisfies the equation

$$R(U, V)W = k\{g(V, W)U - g(U, W)V\}. \tag{1.3.1}$$

Definition 1.3.2. A Kähler manifold with constant holomorphic sectional curvature k is called a complex space form and its [62] curvature tensor is given by

$$\begin{aligned} R(U, V)W &= \frac{k}{4}[g(V, W)U - g(U, W)V + g(U, JW)JV - g(V, JW)JU \\ &+ 2g(U, JV)JW]. \end{aligned} \tag{1.3.2}$$

Definition 1.3.3. An almost Hermition manifold M is called a generalized complex space form $M(f_1, f_2)$ if its [55] Riemannian curvature tensor R satisfies,

$$\begin{aligned} R(U,V)W &= f_1\{g(V,W)U - g(U,W)V\} + f_2\{g(U,JW)JV \\ &- g(V,JW)JU + 2g(U,JV)JW\}, \end{aligned} \tag{1.3.3}$$

for all $U, V, W \in TM$, where f_1 and f_2 are smooth functions on M. Then equation (1.3.3) we get the following;

$$S(V,W) = \{(n-1)f_1 + 3f_2\}g(V,W), \tag{1.3.4}$$

$$QV = [(n-1)f_1 + 3f_2]V, \tag{1.3.5}$$

$$r = n[(n-1)f_1 + 3f_2], \tag{1.3.6}$$

where S is the Ricci tensor, Q is the Ricci operator and r is scalar curvature of the space form $M(f_1, f_2)$.

1.4 Submanifolds

Definition 1.4.1. The manifold M is called a submanifold of a manifold $\widetilde{M}$, if two conditions are satisfied.

1. The set M is a subset of $\widetilde{M}$.

2. The identity map i from M into $\widetilde{M}$ is an imbedding of M into $\widetilde{M}$.

Example 4. *S^2 is a $2-$dimensional differentiable submanifold of R^3.*

Assume that $\phi : M \longrightarrow \widetilde{M}$ is an immersion of an n-dimensional Riemannian manifold M into $\widetilde{M}$. Denote by ∇ and $\widetilde{\nabla}$ the Levi-Civita connections on M and $\widetilde{M}$, respectively.

The Gauss and Weingarten formulae are given by [50]

$$\widetilde{\nabla}_V W = \nabla_V W + \sigma(V,W), \tag{1.4.1}$$

$$\widetilde{\nabla}_V N = -A_N V + \nabla^{\perp}_V N, \tag{1.4.2}$$

for all vector V, W tangent to M and normal vector field N on M, where ∇ is the Riemannian connection on M determined by the induced metric g, σ is a symmetric covariant tensor of order 2, $\nabla^{\perp}$ is the normal connection on $T^{\perp}M$ of M and A_N is the shape operator which is related to σ by $g(\sigma(V,W),N) = g(A_N V, W)$. The Gauss equation is given by

$$\widetilde{R}(U,V)W = R(U,V)W - A_{\sigma(V,W)}U + A_{\sigma(U,W)}V, \tag{1.4.3}$$

where V, W are vector fields tangent to M. The first covariant derivative of the second fundamental form σ is given by

$$(\widetilde{\nabla}_U \sigma)(V,W) = \nabla^{\perp}_U \sigma(V,W) - \sigma(\nabla_U V, W) - \sigma(V, \nabla_U W), \tag{1.4.4}$$

where $\widetilde{\nabla}$ is called the Wander-Bortolotti connection of M [21].

We denote by $\nabla^q T^*$ the covariant differential of the q^{th} order, $q \geq 1$, of a $(0,k)$-tensor field T^*, $k \geq 1$, defined on a Riemannain manifold (M,g) with the Levi-Civita connection. According to [46, 50], the tensor T^* is said to be recurrent, if the following condition holds on M

$$(\nabla T^*)(U_1, \ldots U_k; U)T^*(V_1, \ldots, V_k) = (\nabla T^*)(V_1, \ldots V_k; U)T^*(U_1, \ldots U_k), \tag{1.4.5}$$

where $U, U_1, V_1, \ldots U_k, V_k \in TM$. From (1.4.5) it follows that at a point $p \in M$ if the tensor T^* is non-zero then there exists a unique 1-form B, defined on a neighborhood X

of p, such that

$$\nabla T^{*} = T^{*} \otimes B, \quad B = d(log \parallel T^{*} \parallel), \tag{1.4.6}$$

holds on U, where $\parallel T^{*} \parallel$ denotes the norm of T^{*}, $\parallel T^{*} \parallel^{2} = g(T^{*}, T^{*})$.

The mean curvature vector field is defined by

$$H = \frac{tr.\sigma}{n}. \tag{1.4.7}$$

Definition 1.4.2. A submanifold M is said to be totally umbilical if we have

$$\sigma(U, V) = Hg(U, V). \tag{1.4.8}$$

Definition 1.4.3. A submanifold M is said to be totally geodesic if $\sigma(U, V) = 0$ for each $U, V \in TM$ and is minimal if $H = 0$ on M.

1.5 Quaternion K$\ddot{a}$hler manifold

Let $\bar{M}$ be an n $(n = 4m, m \geq 1)$-dimensional Riemannian manifold with the Riemannian metric g. $\bar{M}$ is called a quaternion Kählher manifold if there exists a 3-dimensional vector bundle μ consisting of tensors of type $(1, 1)$ with local basis of almost Hermitian structure J, K and L such that [37, 61]

(a)

$$J^{2} = K^{2} = L^{2} = -I, \tag{1.5.1}$$

$$JK = -KJ = L, \quad KL = -LK = J, \quad LJ = -JL = K, \tag{1.5.2}$$

$$g(JU, JV) = g(KU, KV) = g(LU, LV) = g(U, V), \tag{1.5.3}$$

where I denoting the identity tensor of type $(1, 1)$ in $\bar{M}$.

(b) If ϕ is a cross-section of the bundle μ, then $\nabla_{U}\phi$ is also a cross-section of the bundle

μ, U being an arbitrary vector field on $\bar{M}$ and ∇ the Riemannian connection on $\bar{M}$.

The condition **(b)** is equivalent to the following condition;

(c) There exist the local 1-forms p, q and r such that

$$\nabla_U J = r(U)K - q(U)L, \quad \nabla_U K = -r(U)J + p(U)L, \;\; \nabla_U L = q(U)J - p(U)K.$$

Example 5. *For any simple Lie group G, there is a unique Wolf space $\frac{G}{H}$ obtained as a quotient of G by a subgroup $H = H_0 \cdot SU(2)$. Here, $SU(2)$ is the subgroup associated with the highest root of G, and H_0 is its centralizer in G. The Wolf spaces with positive Ricci curvature are compact and simply connected. If G is $Sp(n+1)$, the corresponding Wolf space is the quaternionic projective space KP^n.*

The formulae [61]

$$R(U,V) = R(JU,JV) = R(KU,KV) = R(LU,LV), \tag{1.5.4}$$

$$S(U,V) = S(JU,JV) = S(KU,KV) = S(LU,LV), \tag{1.5.5}$$

$$S(U,JV) + S(JU,V) = 0, S(U,KV) + S(KU,V) = 0 \;\; \textit{and}$$

$$S(U,LV) + S(LU,V) = 0, \tag{1.5.6}$$

are well known for a quaternion Kähler manifold.

Definition 1. *A non-flat quaternion Kähler manifold $\bar{M}$ is said to be*

1. *quasi-Einstein manifold if its Ricci tensor S is non zero and satisfies the condition*

$$S(U,V) = ag(U,V) + bE(U)E(V). \tag{1.5.7}$$

2. *generalized quasi-Einstein manifold if*

$$S(U,V) = ag(U,V) + bE(U)E(V) + cF(U)F(V).$$

3. a mixed generalized quasi-Einstein manifold if

$$S(U,V) = ag(U,V) + bE(U)E(V) + cF(U)F(V) + d[E(U)F(V) + F(U)E(V)],$$

where a, b, c *and* d *are non zero scalars,* E *and* F *are two non zero 1-forms such that*

$$g(X,\rho) = E(X), g(X,\beta) = F(X), \tag{1.5.8}$$

for all vector fields X*.*

Chapter 2

On Generalized Complex Space Forms

2.1 Introduction

A K$\ddot{a}$hler manifold with constant holomorphic-sectional curvature is a complex space form and its curvature tensor form is given in equation (1.3.2). In 1989 the author Olszak [42] proved the existence of generalized complex space form. Nature of a generalized Sasakian space form with some conditions in accordance to some curvature tensors have been studied by De [27], Srivastava [60], Nagaraja [41] and Bagewadi et. al., In the context of generalized complex space forms, the authors Bharathi and Bagewadi [8] have studied some curvature tensors on generalized complex space forms. On the basis of the above we extend our study and obtain interesting results.

2.2 Bochner semisymmetric generalized complex space form

Let us consider the condition $R \cdot D = 0$ in $M(f_1, f_2)$. Then for any tangent vectors U, W, X, Y and Z, the above condition leads to

$$(R(U,W) \cdot D)(X,Y,Z) = 0. \tag{2.2.1}$$

This implies

$$\begin{aligned} R(U,W)D(X,Y)Z \; &- \; D(R(U,W)X,Y)Z - D(X,R(U,W)Y)Z \\ &- \; D(X,Y)R(U,W)Z = 0. \end{aligned} \tag{2.2.2}$$

Taking inner product with vector field T we have,

$$\begin{aligned} g(R(U,W)D(X,Y)Z,T) \; &- \; g(D(R(U,W)X,Y)Z,T) - g(D(X,R(U,W)Y)Z,T) \\ &- \; g(D(X,Y)R(U,W)Z,T) = 0. \end{aligned} \tag{2.2.3}$$

Using equations (1.1.12) and (1.3.3) in (2.2.3) and contracting U and Y, further applying contraction over W and T of the simplified equation we get

$$f_2 \left\{ \frac{2n-8}{2n+4} S(X,Z) - \frac{5n+2}{(2n+4)(2n+2)} r g(X,Z) \right\} = 0. \tag{2.2.4}$$

If $f_2 \neq 0$, then

$$\frac{2n-8}{2n+4} S(X,Z) - \frac{5n+2}{(2n+4)(2n+2)} r g(X,Z) = 0. \tag{2.2.5}$$

This comes to the below form,

$$S(X,Z) = \frac{5n+2}{(2n-8)(2n+2)} r g(X,Z). \tag{2.2.6}$$

That is $M(f_1, f_2)$ is an Einstein manifold.

Hence, we have the following:

Theorem 2.2.1. *If n-dimensional generalized complex space form $M(f_1, f_2)$ satisfies $R \cdot D = 0$ then it is an Einstein manifold provided $f_2 \neq 0$.*

Using equation (2.2.6) in (1.2.1), we get

$$(L_V g)(X, Z) + 2\left[\frac{5n+2}{(2n-8)(2n+2)}\right] rg(X, Z) + 2\lambda g(X, Z) = 0, \tag{2.2.7}$$

setting $X = Z = e_i$ in (2.2.7) and then taking summation over i $(1 \leq i \leq n)$, we obtain

$$divV + \frac{5n+2}{(2n-8)(2n+2)} rn + \lambda n = 0. \tag{2.2.8}$$

If V is solenoidal then $divV = 0$. Therefore the equation (2.2.8) can be reduced to

$$\lambda = -\frac{5n+2}{(2n-8)(2n+2)} r. \tag{2.2.9}$$

Thus, we state the following:

Corollary 2.2.2. *Let (g, V, λ) be a Ricci soliton in a generalized complex space form satisfying Bochner semisymmetric condition. If V is solenoidal then it is shrinking, steady and expanding accordingly scalar curvature is positive, zero and negative respectively.*

2.3 Ricci soliton in generalized complex space form satisfying $R \cdot C^* = 0$

Let R and C^* satisfy the equation $R \cdot C^* = 0$ in $M(f_1, f_2)$. Then this equation leads to

$$(R(U, W) \cdot C^*)(X, Y, Z) = 0, \tag{2.3.1}$$

where U, W, X, Y and Z are any tangent vectors.

Equation (2.3.1) can be written as

$$R(U,W)C^*(X,Y)Z - C^*(R(U,W)X,Y)Z - C^*(X,R(U,W)Y)Z - C^*(X,Y)R(U,W)Z = 0.$$

By taking inner product T we have

$$g(R(U,W)C^*(X,Y)Z,T) - g(C^*(R(U,W)X,Y)Z,T) - g(C^*(X,R(U,W)Y)Z,T)$$
$$-g(C^*(X,Y)R(U,W)Z,T) = 0. \tag{2.3.2}$$

Applying equations (1.1.16) and (1.3.3) in (2.3.2) and setting $U = Y = e_i$, further setting $W = T = e_i$ to the simplified equation, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i $(1 \leq i \leq n)$ we obtain

$$S(X,Z) = \frac{\alpha_7 + \beta_7}{3bf_2n(n-4)(n-1)} r.g(X,Z), \tag{2.3.3}$$

where $\alpha_7 = f_1(n-1)[-n + a(2+n) + b(2n^2 + n - 4)]$ and $\beta_7 = 6f_2(n-2)[a + 2(n-1)b]$. Hence, we state the following:

Theorem 2.3.1. *If n-dimensional generalized complex space form $M(f_1, f_2)$ satisfies $R \cdot C^* = 0$ then it is an Einstein manifold.*

Applying equation (2.3.3) in (1.2.1) we get,

$$(L_V g)(X,Z) + 2\left[\frac{\alpha_7 + \beta_7}{3bf_2n(n-4)(n-1)}\right] rg(X,Z) + 2\lambda g(X,Z) = 0. \tag{2.3.4}$$

On contraction over X and Z, then we gain

$$(L_V g)(e_i,e_i) + 2\left[\frac{\alpha_7 + \beta_7}{3bf_2n(n-4)(n-1)}\right] rg(e_i,e_i) + 2\lambda g(e_i,e_i) = 0. \tag{2.3.5}$$

The equation (2.3.5) leads to,

$$divV + \frac{\alpha_7 + \beta_7}{3bf_2(n-4)(n-1)} r + \lambda n = 0. \tag{2.3.6}$$

If $divV = 0$ then V is solenoidal. Therefore the equation (2.3.6) is reduced

$$\lambda = -\frac{\alpha_7 + \beta_7}{3bf_2 n(n-4)(n-1)} r.$$

Thus, we write the following:

Corollary 2.3.2. *Let (g, V, λ) be a Ricci soliton in a generalized complex space form satisfying quasi-conformal semisymmetric condition, then V is solenoidal if and only if it is shrinking, steady and expanding depending upon the scalar curvature is positive, zero and negative respectively.*

2.4 Ricci soliton in generalized complex space form satisfying $C^* \cdot R = 0$

Consider the semisymmetric condition $(C^*(U,V) \cdot R)(X,Y,W) = 0$ in $M(f_1, f_2)$, then U, V, X, Y and W are tangent vectors, this equation can be expressed as:

$$\begin{aligned} C^*(U,V)R(X,Y)W &- R(C^*(U,V)X,Y)W - R(X,C^*(U,V)Y)W \\ &- R(X,Y)C^*(U,V)W = 0. \end{aligned} \tag{2.4.1}$$

Taking inner product with T in the above equation we get

$$\begin{aligned} &g(C^*(U,V)R(X,Y)W,T) - g(R(C^*(U,V)X,Y)W,T) - g(R(X,C^*(U,V)Y)W,T) \\ &-g(R(X,Y)C^*(U,V)W,T) = 0. \end{aligned} \tag{2.4.2}$$

Using equations (1.1.16) and (1.3.3) in (2.4.2) and putting $U = Y = e_i$, further putting $V = T = e_i$, i $(1 \leq i \leq n)$ to the simplified equation we gain

$$S(X, W) = \frac{\alpha_5}{\beta_5} r g(X, W), \tag{2.4.3}$$

where $\alpha_5 = [a+(n-2)b](n-1)f_1+[a+2b(n-1)]f_2$ and $\beta_5 = [f_1[a+b(n-2)]+4f_2b]n(n-1)$. This means, generalized complex space form is an Einstein manifold.

Hence, we state the following:

Theorem 2.4.1. *If n-dimensional generalized complex space form $M(f_1, f_2)$ satisfies $C^* \cdot R = 0$ then it is an Einstein manifold.*

Using equation (2.4.3) in (1.2.1) we get

$$(L_V g)(X, W) + 2\left[\frac{\alpha_5}{\beta_5}\right] r g(X, W) + 2\lambda g(X, W) = 0. \tag{2.4.4}$$

Letting $X = W = e_i$ and taking summation over i in the above equation, we get

$$divV + \left[\frac{\alpha_5}{\beta_5}\right] nr + \lambda n = 0. \tag{2.4.5}$$

If V is solenoidal then $divV = 0$. Therefore the last equation can be reduced to

$$\lambda = -\frac{\alpha_5}{\beta_5} r.$$

Thus, we obtain the following:

Corollary 2.4.2. *A Ricci soliton in a generalized complex space form satisfying $C^* \cdot R = 0$ is shrinking, steady and expanding if accordingly scalar curvature is positive, zero and negative respectively.*

2.5 Ricci soliton in generalized complex space form satisfying $C^* \cdot S = 0$

We assume $(C^*(X,Y) \cdot S)(Z,W) = 0$ in $M(f_1, f_2)$, this equation readily read as;

$$S(C^*(X,Y)Z,W) + S(Z, C^*(X,Y)W) = 0. \tag{2.5.1}$$

Applying equations (1.1.16) and (1.3.3) in (2.5.1) and setting $X = W = e_i$, further using equations (1.3.4), (1.3.5) and (1.3.6) to the simplified equation we obtain

$$S(Y,Z) = -\gamma g(Y,Z), \tag{2.5.2}$$

where $\gamma = (2(n-1)f_1 + 3f_2)$. By using equation (2.5.2) in (1.2.1) we get

$$(L_V g)(Y,Z) + 2[-\gamma]g(Y,Z) + 2\lambda g(Y,Z) = 0. \tag{2.5.3}$$

By substituting $Y = Z = e_i$ in (2.5.3), we obtain

$$divV - \gamma n + \lambda n = 0. \tag{2.5.4}$$

If V is solenoidal then $divV = 0$. Therefore the equation (2.5.4) can be reduced to

$$\lambda = \gamma.$$

Thus, we state the following:

Corollary 2.5.1. *Let (g, V, λ) be a Ricci soliton in generalized complex space form satisfying $C^* \cdot S = 0$. If V is solenoidal then it is expanding.*

2.6 Einstein semisymmetric generalized complex space form

Definition 2.6.1. A n-dimensional generalized complex space form is called Einstein semisymmetric if

$$(R(X,Y)\cdot E)(U,W)=0, \tag{2.6.1}$$

for any vector fields X, Y, U and W.

Now, we consider the generalized complex space form which is Einstein semisymmetric, i.e., equation (2.6.1) can be expressed as

$$E(R(X,Y)U,W)+E(U,R(X,Y)W)=0. \tag{2.6.2}$$

In view of (1.1.11) equation (2.6.2) becomes

$$\begin{aligned} S(R(X,Y)U,W) &- \frac{r}{n}g(R(X,Y)U,W) \\ +S(U,R(X,Y)W) &- \frac{r}{n}g(U,R(X,Y)W)=0. \end{aligned} \tag{2.6.3}$$

Using equation (1.3.3) in (2.6.3) and replacing $X=U=e_i$ we infer after simplification that

$$f_1\left[-nS(Y,W)+rg(Y,W)\right]=0. \tag{2.6.4}$$

If $f_1\neq 0$, then

$$S(Y,W)=\frac{r}{n}g(Y,W). \tag{2.6.5}$$

Then we can said the following:

Theorem 2.6.1. *A generalized complex space form in which Einstein semisymmetric satisfies is an Einstein manifold provided $f_1 \neq 0$.*

Using equation (2.6.5) in (1.2.1), we get

$$(L_V g)(Y, W) + 2\frac{r}{n}g(Y, W) + 2\lambda g(Y, W) = 0. \tag{2.6.6}$$

Setting $Y = W = e_i$ in (2.6.6), we get

$$divV + r + \lambda n = 0. \tag{2.6.7}$$

If V is solenoidal then $divV = 0$. Therefore the above equation (2.6.7) can be reduced to

$$\lambda = \frac{-r}{n}. \tag{2.6.8}$$

Thus, we can state the following:

Corollary 2.6.2. *Let (g, V, λ) be a Ricci soliton in a generalized complex space form satisfying Einstein semisymmetric condition. Then V is solenoidal if and only if it is shrinking, steady and expanding accordingly scalar curvature is positive, zero and negative respectively.*

2.7 H-projective curvature tensor on generalized complex space form

We consider the condition $(R(U, V) \cdot \bar{P})(X, Y, W) = 0$ in generalized complex space form. Then for any tangent vectors U, V, X, Y and W, the above implies:

$$R(U, V)\bar{P}(X, Y)W - \bar{P}(R(U, V)X, Y)W - \bar{P}(X, R(U, V)Y)W - \bar{P}(X, Y)R(U, V)W = 0.$$

Taking inner product with Z we have

$$g(R(U,V)\bar{P}(X,Y)W,Z) - g(\bar{P}(R(U,V)X,Y)W,Z) - g(\bar{P}(X,R(U,V)Y)W,Z)$$
$$-g(\bar{P}(X,Y)R(U,V)W,Z) = 0. \tag{2.7.1}$$

Using equations (1.1.13) and (1.3.3) in (2.7.1) and contraction U,Y, further applying contraction V,Z in the resulting equation we gain

$$S(X,W) = \frac{2f_1(n-2) - 4f_2}{f_1(n^2-5n+2) - 2nf_2} r g(X,W).$$

Hence, we state the following:

Theorem 2.7.1. *If n-dimensional generalized complex space form $M(f_1,f_2)$ satisfies $R \cdot \bar{P} = 0$ then it is an Einstein manifold.*

Taking covariant derivative of (1.1.13) we get

$$\begin{aligned}(\nabla_Z\bar{P})(U,V)W &= (\nabla_Z R)(U,V)W - \frac{2}{n+2}[(\nabla_Z S)(V,W)U - (\nabla_Z S)(U,W)V \\ &- (\nabla_Z S)(JY,W)JX + (\nabla_Z S)(JX,W)JY + (\nabla_Z S)(JX,V)JZ \\ &- (\nabla_Z S)(JY,U)JZ]. \end{aligned} \tag{2.7.2}$$

Consider the equation $(R(U,V)\cdot\nabla\bar{P})(X,Y,Z,W) = 0$ in generalized complex space form. Then for any tangent vectors $U,V,\ X,Y,Z$ and W, it can be written as:

$$R(U,V)(\nabla_X\bar{P})(Y,Z)W - (\nabla_{R(U,V)X}\bar{P})(Y,Z)W - (\nabla_X\bar{P})(R(U,V)Y,Z)W$$
$$-(\nabla_X\bar{P})(Y,R(U,V)Z)W - (\nabla_X\bar{P})(Y,Z)R(U,V)W = 0.$$

Taking inner product with the vector field T we have

$$g(R(U,V)(\nabla_X\bar{P})(Y,Z)W,T) - g((\nabla_{R(U,V)X}\bar{P})(Y,Z)W,T) - g((\nabla_X\bar{P})(R(U,V)Y,Z)W,T)$$
$$-g((\nabla_X\bar{P})(Y,R(U,V)Z)W,T) - g((\nabla_X\bar{P})(Y,Z)R(U,V)W,T) = 0. \tag{2.7.3}$$

Using equations (1.1.4), (1.3.3) and (2.7.2) in (2.7.3) and substituting $U = Y = e_i$, further applying contraction over V and W in the resulting equation we get

$$\begin{aligned}\left\{\frac{f_1(-f_1(5+n) - 2f_2 n(n+1))}{n+3}\right\} dr(X) &= 0 \\ dr(X) &= 0 \;\; for\ all\ X.\end{aligned}$$

This implies r is constant.

Hence, we have the following:

Theorem 2.7.2. *If n-dimensional generalized complex space form $M(f_1, f_2)$ satisfies $R \cdot \nabla \bar{P} = 0$ then its scalar curvature is constant.*

Let the generalized complex space form satisfy $(\bar{P}(X,Y) \cdot S)(Z,W) = 0$, where X, Y, Z and W are tangent vectors this equation readily read as:

$$S(\bar{P}(X,Y)Z, W) + S(Z, \bar{P}(X,Y)W) = 0. \tag{2.7.4}$$

Applying (1.1.13) and (1.3.3) in (2.7.4) and contracting over X and W, further using (1.3.6) to the resulting equation we get

$$S(Y,Z) = \frac{(n+2)}{(3n^2-4n+4)f_1 + (6n-12)f_2 + 16} rg(Y,Z).$$

Hence, we have the following:

Theorem 2.7.3. *A n-dimensional generalized complex space form $M(f_1, f_2)$ satisfying $\bar{P} \cdot S = 0$ is an Einstein manifold.*

2.8 Pseudo-projective curvature tensor on generalized complex space form

Consider the semisymmetric condition $(R(U,V)\cdot P^*)(X,Y,W)=0$ in $M(f_1,f_2)$, which is satisfied by R and P^* and for any tangent vectors U,V,X,Y and W, this equation can be expressed as:

$$R(U,V)P^*(X,Y)W-P^*(R(U,V)X,Y)W-P^*(X,R(U,V)Y)W-P^*(X,Y)R(U,V)W=0.$$

Taking inner product with Z, then we have

$$\begin{aligned}&g(R(U,V)P^*(X,Y)W,Z)-g(P^*(R(U,V)X,Y)W,Z)-g(P^*(X,R(U,V)Y)W,Z)\\&-g(P^*(X,Y)R(U,V)W,Z)=0.\end{aligned} \tag{2.8.1}$$

Using equations (1.1.17) and (1.3.3) in (2.8.1) and putting $U=Y=e_i$, further again putting $V=Z=e_i$ in the simplified equation we obtain

$$S(X,W)=\frac{f_1(bn(n-1)(n-2))+f_2(-a+3a(n-1)+3b(n-1)^2-b(n-1))}{n(n-1)[f_1(n(n-2))+f_2(2a-4b+3bn)]}rg(X,W).$$

This means $M(f_1,f_2)$ is an Einstein manifold and we state the following:

Theorem 2.8.1. *If n-dimensional generalized complex space form $M(f_1,f_2)$ satisfies $R\cdot P^*=0$ then it is an Einstein manifold.*

Taking covariant derivative of (1.1.17) we get

$$\begin{aligned}(\nabla_Z P^*)(U,V)W &= a(\nabla_Z R)(U,V)W+b[(\nabla_Z S)(V,W)U-(\nabla_Z S)(U,W)V]\\&-\frac{dr(Z)}{n}[\frac{a}{n-1}+b][g(V,W)U-g(U,W)V].\end{aligned} \tag{2.8.2}$$

We assume $(R(U,V)\cdot\nabla P^*)(X,Y,Z,W) = 0$, where U,V,X,Y,Z and W are tangent vectors. This equation takes the following form:

$$R(U,V)(\nabla_X P^*)(Y,Z)W - (\nabla_{R(U,V)X}P^*)(Y,Z)W - (\nabla_X P^*)(R(U,V)Y,Z)W$$
$$-(\nabla_X P^*)(Y,R(U,V)Z)W - (\nabla_X P^*)(Y,Z)R(U,V)W = 0.$$

Taking inner product with the vector field T we have

$$g(R(U,V)(\nabla_X P^*)(Y,Z)W,T) - g((\nabla_{R(U,V)X}P^*)(Y,Z)W,T) - g((\nabla_X P^*)(R(U,V)Y,Z)W,T)$$
$$-g((\nabla_X P^*)(Y,R(U,V)Z)W,T) - g((\nabla_X P^*)(Y,Z)R(U,V)W,T) = 0. \tag{2.8.3}$$

Applying equations (1.1.4), (1.3.3) and (2.8.2) in (2.8.3) and letting $U = Y = e_i$, further again letting $V = W = e_i$ in the resulting equation we get

$$\begin{aligned}\left\{b((n-1)f_1^2 - 3nf_2) + a(f_1 f_2 - \frac{3}{2})\right\} dr(X) &= 0\\ dr(X) &= 0 \quad for\ all\ X.\end{aligned}$$

This implies r is constant.

Hence, we have the following:

Theorem 2.8.2. *If n-dimensional generalized complex space form $M(f_1,f_2)$ satisfies $R\cdot\nabla P^* = 0$ then its scalar curvature is constant.*

We consider the equation $(P^*(X,Y)\cdot S(Z,W)) = 0$ satisfied by S and P^* in generalized complex space form and for any tangent vectors X,Y,Z and W. It can be written as follows:

$$S(P^*(X,Y)Z,W) + S(Z,P^*(X,Y)W) = 0. \tag{2.8.4}$$

Using equations (1.1.17) and (1.3.3) in equation (2.8.4) and contracting X and W we obtain

$$S(Z,Y) = \frac{[-b(n-1)^2 f_1 - 3af_2 - 3b(n-1)f_2]n}{[bn^2 f_1(n-1) + af_1(n-1) + 3bn^2 f_2 - an^2 f_1](n-1)} rg(Z,Y).$$

Hence, we have the following:

Theorem 2.8.3. *If n-dimensional generalized complex space form $M(f_1, f_2)$ satisfies $P^* \cdot S = 0$ then it is an Einstein manifold.*

2.9 Bochner Ricci-generalized pseudosymmetric generalized complex space form

Let us consider the Ricci-generalized Bochner pseudosymmetric generalized complex space form $M(f_1, f_2)$. Then we have

$$(R(U,W) \cdot D)(X,Y,Z) = L_D((U\Lambda_S W \cdot D)(X,Y,Z)). \tag{2.9.1}$$

This leads to

$$R(U,W)D(X,Y)Z - D(R(U,W)X,Y)Z - D(X,R(U,W)Y)Z - D(X,Y)R(U,W)Z$$
$$= L_D[(U\Lambda_S W)D(X,Y)Z - D((U\Lambda_S W)X,Y)Z - D(X,(U\Lambda_S W)Y)Z - D(X,Y)(U\Lambda_S W)Z].$$

Taking inner product the above with T we have,

$$\begin{aligned} & g(R(U,W)D(X,Y)Z,T) - g(D(R(U,W)X,Y)Z,T) - g(D(X,R(U,W)Y)Z,T) \\ - \;& g(D(X,Y)R(U,W)Z,T) = L_D[g((U\Lambda_S W)D(X,Y)Z,T) - g(D((U\Lambda_S W)X,Y)Z,T) \\ - \;& g(D(X,(U\Lambda_S W)Y)Z,T) - g(D(X,Y)(U\Lambda_S W)Z,T)]. \end{aligned} \tag{2.9.2}$$

Using equations (1.1.12), (1.3.4) and (1.3.5) in (2.9.2) and substituting $U = Y = e_i$, further again substituting $W = T = e_i$ in the resulting equation and summing over i $(i = 1, 2, \dots, n)$, we get

$$\begin{aligned} f_2\{\frac{2n-8}{2n+4}S(X,Z) &- \frac{5n+2}{(2n+4)(2n+2)}rg(X,Z)\} \\ &= L_B[\frac{4((n-1)f_1+3f_2-1)-n(r+1)}{2n+4}S(X,Z) \\ &+ \frac{r(n+2)-(n+4)}{(2n+2)(2n+4)}rg(X,Z)]. \end{aligned} \tag{2.9.3}$$

This implies that

$$\begin{aligned} &[\frac{f_2(2n-8)-L_B(4((n-1)f_1+3f_2-1)-n(r+1))}{2n+4}]S(X,Z) \\ &-[\frac{f_2(5n+2)+L_B(r(n+2)-(n+4))}{(2n+4)(2n+2)}]rg(X,Z) = 0. \end{aligned} \tag{2.9.4}$$

The above equation leads to

$$\alpha_8 S(X,Z) - \beta_8 rg(X,Z) = 0. \tag{2.9.5}$$

where $\alpha_8 = [\frac{f_2(2n-8)-L_B(4((n-1)f_1+3f_2-1)-n(r+1))}{2n+4}]$ and $\beta_8 = [\frac{f_2(5n+2)+L_B(r(n+2)-(n+4))}{(2n+4)(2n+2)}]$. The last equation implies

$$S(X,Z) = \frac{\beta_8 r}{\alpha_8}g(X,Z). \tag{2.9.6}$$

Thus, we state the following:

Theorem 2.9.1. *A Bochner Ricci-generalized pseudosymmetric generalized complex space Form is an Einstein manifold.*

Using equation (2.9.6) in (1.2.1), we get

$$(L_V g)(X,Z) + 2\frac{\beta_8 r}{\alpha_8}g(X,Z) + 2\lambda g(X,Z) = 0. \tag{2.9.7}$$

Contraction (2.9.7) over X and Z gives

$$divV + \frac{\beta_8 r}{\alpha_8} n + \lambda n = 0. \tag{2.9.8}$$

If V is solenoidal then $divV = 0$. Therefore the equation (2.9.8) is reduced to

$$\lambda = \frac{-\beta_8 r}{\alpha_8}. \tag{2.9.9}$$

Thus, we can write the following:

Corollary 2.9.2. *Let (g, V, λ) be a Ricci soliton in a generalized complex space form satisfying Bochner Ricci-generalized pseudosymmetric. Then V is solenoidal if and only if it is shrinking, steady and expanding depending upon the sign of scalar curvature.*

2.10 Generalized complex space form satisfying $D \cdot W_2 = L_1 Q(g, W_2)$

We assume that $D \cdot W_2 = L_1 Q(g, W_2)$ holds on $M(f_1, f_2)$, then we have

$$(D(U, V) \cdot W_2)(X, Y, Z) = L_1[((U \wedge V) \cdot W_2)(X, Y)Z], \tag{2.10.1}$$

where U, V, X, Y, Z are tangent vectors. The above equation leads to,

$$D(U, V)W_2(X, Y)Z - W_2(D(U, V)X, Y)Z - W_2(X, D(U, V)Y)Z - W_2(X, Y)D(U, V)Z$$

$$= L_1[(U\Lambda_g V)W_2(X, Y)Z - W_2((U\Lambda_g V)X, Y)Z - W_2(X, (U\Lambda_g V)Y)Z - W_2(X, Y)(U\Lambda_g V)Z].$$

Taking the above with inner product T we have,

$$\begin{aligned} g(D(U, V)W_2(X, Y)Z, T) \;&-\; g(W_2(D(U, V)X, Y)Z, T) - g(W_2(X, D(U, V)Y)Z, T) \\ -g(W_2(X, Y)D(U, V)Z, T) \;&=\; L_1[g((U\Lambda_g V)W_2(X, Y)Z, T) - g(W_2((U\Lambda_g V)X, Y)Z, T) \\ -g(W_2(X, (U\Lambda_g V)Y)Z, T \;&-\; g(W_2(X, Y)(U\Lambda_g V)Z, T)]. \end{aligned} \tag{2.10.2}$$

Using equations (1.1.12), (1.1.18) and (1.3.3) in (2.10.2) and putting $U = Y = e_i$, further again putting $V = T = e_i$ in the simplified equation we get

$$\frac{\gamma_1}{(n-1)}S(X,Z) + \frac{\delta_1}{(n-1)}rg(X,Z) = L_1[\frac{1}{n-1}[nS(X,Z) - rg(X,Z)]], \quad (2.10.3)$$

where $\gamma_1 = \frac{(2n+2)[(6n^3-8n^2-39n-22)f_2-2(n^3+4n^2+7n-18)(n+1)f_1]+rn(2n+4)}{(2n+2)(2n+4)}$

and $\delta_1 = \frac{-f_1(2n+2)(4n+2)+6f_2(2n+2)(n+1)+r}{(2n+2)(2n+4)}$.

The equation (2.10.3) implies

$$[\gamma_1 S(X,Z) + \delta_1 rg(X,Z)] = L_1[nS(X,Z) - rg(X,Z)]. \quad (2.10.4)$$

Thus we have

$$S(X,Z) = \alpha_6 rg(X,Z), \quad (2.10.5)$$

where $\alpha_6 = \frac{(L_1+\delta_1)}{L_1 n-\gamma_1}$.

Hence, we state the following:

Theorem 2.10.1. *An n-dimensional generalized complex space form satisfying $D \cdot W_2 = L_1 Q(g, W_2)$ is an Einstein manifold.*

Using equation (2.10.5) in (1.2.1), we get

$$(L_V g)(X,Z) + 2\alpha_6 rg(X,Z) + 2\lambda g(X,Z) = 0. \quad (2.10.6)$$

Taking $X = Z = e_i$ and summing over $i = 1, 2, \ldots, n$ in (2.10.6), we obtain

$$(L_V g)(e_i,e_i) + 2\alpha_6 rg(e_i,e_i) + 2\lambda g(e_i,e_i) = 0. \quad (2.10.7)$$

This implies

$$divV + \alpha_6 rn + \lambda n = 0. \quad (2.10.8)$$

If V is solenoidal then $divV = 0$. Therefore the equation (2.10.8) is reduced to

$$\lambda = -\alpha_6 r. \tag{2.10.9}$$

Thus, we can obtain the following:

Corollary 2.10.2. *Let (g, V, λ) be a Ricci soliton in a generalized complex space form satisfying $D \cdot W_2 = L_1 Q(g, W_2)$. Then V is solenoidal if and only if it is shrinking, steady and expanding depending upon the scalar curvature i.e., r is positive, zero and negative respectively.*

2.11 Generalized complex space form with divD=0

Assume that the Bochner curvature tensor of a generalized complex space form is conservative that is $divD = 0$.

Using equations (1.3.4) and (1.3.5) in (1.1.12), we obtain

$$\begin{aligned} D(U,V,W) &= R(U,V,W) - 2\frac{[(n-1)f_1 + 3f_2]}{2n+4}[g(V,W)U - g(U,W)V + g(JV,W)JU \\ &- g(JU,W)JV - 2g(JU,V)JW] \\ &+ \frac{r}{(2n+2)(2n+4)}[g(V,W)U - g(U,W)V + g(JV,W)JU \\ &- g(JU,W)JV - 2g(JU,V)JW]. \end{aligned} \tag{2.11.1}$$

Differentiating (2.11.1) covariantly, contracting and by our assumption we have.

$$\begin{aligned} 0 &= (divR)(U,V)W - 2\frac{d[(n-1)f_1 + 3f_2]}{2n+4}[g(V,W)U - g(U,W)V + g(JV,W)JU \\ &- g(JU,W)JV - 2g(JU,V)JW] \\ &+ \frac{dr}{(2n+2)(2n+4)}[g(V,W)U - g(U,W)V + g(JV,W)JU \\ &- g(JU,W)JV - 2g(JU,V)JW]. \end{aligned} \tag{2.11.2}$$

Using equation (1.1.4) in (2.11.2) we obtain

$$\begin{aligned} 0 &= (\nabla_U S)(V,W) - (\nabla_V S)(U,W) - 2\frac{d[(n-1)f_1+3f_2]}{2n+4}[g(V,W)U - g(U,W)V \\ &+ g(JV,W)JU - g(JU,W)JV - 2g(JU,V)JW] \\ &+ \frac{dr}{(2n+2)(2n+4)}[g(V,W)U - g(U,W)V + g(JV,W)JU \\ &- g(JU,W)JV - 2g(JU,V)JW]. \end{aligned} \tag{2.11.3}$$

Taking $[(n-1)f_1+3f_2] = constant = k_1 \neq 0$ in equation (2.11.3) we obtain

$$\begin{aligned} 0 &= (\nabla_U S)(V,W) - (\nabla_V S)(U,W) + \frac{dr}{(2n+2)(2n+4)}[g(V,W)U - g(U,W)V \\ &+ g(JV,W)JU - g(JU,W)JV - 2g(JU,V)JW]. \end{aligned} \tag{2.11.4}$$

Using equation (1.1.4) in (2.11.4) we get

$$\begin{aligned} 0 &= (\nabla_{JW} S)(JV,U) + \frac{dr}{(2n+2)(2n+4)}[g(V,W)U - g(U,W)V + g(JV,W)JU \\ &- g(JU,W)JV - 2g(JU,V)JW]. \end{aligned} \tag{2.11.5}$$

Replace W by JW in the above equation we get

$$\begin{aligned} (\nabla_W S)(JV,U) &= \frac{dr}{(2n+2)(2n+4)}[g(V,JW)U - g(U,JW)V + g(V,W)JU \\ &- g(U,W)JV + 2g(JU,V)W]. \end{aligned} \tag{2.11.6}$$

Contraction (2.11.6) over V and W after simplification we get $dr(JU) = 0$. If $dr(JU) = 0$ then $dr(U) = 0$ so r is constant. Using $r = constant$ in (2.11.6) we get

$$(\nabla_U S)(V,W) = (\nabla_V S)(U,W). \tag{2.11.7}$$

Thus, we can state the following:

Theorem 2.11.1. *A n-dimensional generalized complex space form with conservative Bochner curvature tensor is of constant scalar curvature provided* $[(n-1)f_1 + 3f_2] = k_1(constant)$.

Theorem 2.11.2. *[26] Let M be a Kähler manifold of dimension $n \geq 4$. Then div R=0 and div C=0 are equivalent.*

Using above Theorem we can state the following:

Theorem 2.11.3. *Let $M(f_1, f_2)$ be a generalized complex space form of dimension $n \geq 4$. Then div R=0, div C=0 and div D=0 are equivalent provided* $[(n-1)f_1 + 3f_2] = k_1(constant)$.

2.12 Conclusion

The important results finding of this chapter are as follows:

- If n-dimensional generalized complex space form $M(f_1, f_2)$

  1. satisfy the conditions like $R \cdot P^* = 0$, $R \cdot \bar{P} = 0$, $R \cdot C^* = 0$, $C^* \cdot R = 0$, $P^* \cdot S = 0$, $\bar{P} \cdot S = 0$, $D \cdot W_2 = L_1 Q(g, W_2)$ and $C^* \cdot S = 0$ then it is an Einstein manifold in each case.

  2. satisfies $R \cdot \nabla P^* = 0$ and $R \cdot \nabla \bar{P} = 0$ then the scalar curvature is constant in each case.

  3. is Bochner semisymmetric and Einstein semisymmetric then it is Einstein manifold provided $f_2 \neq 0$ and $f_1 \neq 0$ respectively.

  4. is Bochner Ricci-generalized pseudosymmetric then it is an Einstein manifold.

- Let (g, V, λ) be a Ricci soliton in a generalized complex space form satisfying

  1. $R \cdot D = 0$, $R \cdot C^* = 0$, $C^* \cdot R = 0$, $D \cdot W_2 = L_1 Q(g, W_2)$ and $R \cdot E = 0$. If V is solenoidal then it is shrinking, steady and expanding accordingly scalar curvature is positive, zero and negative respectively in each case.
  2. $C^* \cdot S = 0$. If V is solenoidal then it is expanding.
  3. Bochner Ricci-generalized pseudosymmetric. Then V is solenoidal if and only if it is shrinking.

- An n-dimensional generalized complex space form with conservative Bochner curvature tensor is of constant scalar curvature provided $[(n-1)f_1 + 3f_2]$ is a non zero constant.

Chapter 3

On Symmetric Properties of K$\ddot{a}$hler Manifolds

3.1 Introduction

The Riemannain symmetric spaces were introduced by French mathematician Cartan during nineteenth century and play a main tool in differential geometry. A Riemannian manifold is called locally symmetric [11] if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M, g). During the last five decades the notion of locally symmetric manifolds has been studied by many authors in several ways to a different extent such as semi-symmetric manifolds, weakly symmetric manifolds, weakly Ricci-symmetric manifolds, pseudo symmetric manifolds, pseudo Ricci-symmetric manifolds, almost pseudo symmetric manifold and almost pseudo Ricci-symmetric manifolds.

The notions of almost pseudo symmetric and almost pseudo Ricci-symmetric manifolds were introduced by De and Gazi [25] and Chaki and Kawaguchi [14] respectively. These are extended class of pseudo symmetric and pseudo Ricci-symmetric manifolds introduced by Chaki [15] and Chaki and Kawaguchi [14] respectively. Here we note that the notion of pseudo symmetry in the sense of Chaki is different from that of Deszcz [31]. However, pseudo symmetry defined by Chaki will be pseudo symmetry defined by Deszcz if and only if the non zero 1-form associated with pseudo symmetric is closed. It may be mentioned that the almost pseudo symmetric manifold is not a particular case of a weakly symmetric manifold introduced by Tamassy and Bink [52]. Tamassy et. al., [53] found interesting results on weakly symmetric and weakly Ricci-symmetric Kähler manifolds in 2000. Also Shaikh et. al., [48] discussed on quasi-conformlly flat almost pseudo Ricci-symmetric manifolds in 2010. Chathurvadi and Pandey [18] studied semi-symmetric non-metric connections in Kähler manifolds. Then in 2015, Chathurvadi and Pandey [19] studied special type of semi-symmetric metric connection in a weakly symmetric Kähler manifold. Based on the above work in this chapter, we have made an attempt to study the properties of Bochner and projective curvature tensors in Kähler manifolds and generalized complex space forms.

3.2 Basic concepts

Definition 3.2.1. A non flat Riemannian manifold (M, g) is said to be almost pseudo symmetric manifold [25] if its curvature tensor satisfies the condition

$$(\nabla_U R)(V,W,X,Y) = [E(U) + F(U)]R(V,W,X,Y) + E(V)R(U,W,X,Y)$$
$$+E(W)R(V,U,X,Y) + E(X)R(V,W,U,Y) + E(Y)R(V,W,X,U), \tag{3.2.1}$$

where E, F are 1-forms defined by (1.5.8) and ∇ denotes the operator of covariant differentiation with respect to the metric g.

Definition 3.2.2. A non-flat Riemannian manifold (M, g) is said to be almost pseudo

Ricci-symmetric [14] whose Ricci tensor S of type $(0,2)$ satisfies the condition

$$(\nabla_U S)(V,W) \;=\; [E(U)+F(U)]S(V,W)+E(V)S(U,W)+E(W)S(V,U), \tag{3.2.2}$$

where E, F and ∇ have the meaning already stated.

If $E(U)=F(U)$ in (3.2.1) and (3.2.2) then it reduces to pseudo symmetric and pseudo Riici-symmetric manifolds respectively.

Definition 3.2.3. A Kähler manifold is called an almost pseudo Bochner symmetric manifold if its Bochner curvature tensor D of type (0,4) is not identically zero and satisfies the condition

$$\begin{aligned}(\nabla_U D)(V,W,X,Y) &= [E(U)+F(U)]D(V,W,X,Y)+E(V)D(U,W,X,Y)\\ &+E(W)D(V,U,X,Y)+E(X)D(V,W,U,Y)+E(Y)D(V,W,X,U),\end{aligned} \tag{3.2.3}$$

where E, F and ∇ have the meaning already stated and D is given by (1.1.12).

Definition 3.2.4. A Kähler manifold is called almost pseudo Bochner Ricci-symmetric manifold if its Bochner Ricci tensor K of type $(0,2)$ is not identically zero and satisfies the condition

$$(\nabla_U K)(V,W)=[E(U)+F(U)]K(V,W)+E(V)K(U,W)+E(W)K(V,U), \tag{3.2.4}$$

where E, F are nowhere vanishing 1-forms and K is given by,

$$K(V,W)=\frac{n}{2n+4}[S(V,W)-\frac{r}{2(n+1)}g(V,W)]. \tag{3.2.5}$$

Suppose (M,g) is a Kähler manifold and (g,V,λ) is a Ricci soliton in (M,g). If V is killing vector field, then

$$L_V g=0. \tag{3.2.6}$$

If V is conformal killing vector field, then

$$L_V g = \varphi g, \tag{3.2.7}$$

where φ is some scalar function.

Putting $W = \rho$ in (1.3.4) we get,

$$S(V, \rho) = \{(n-1)f_1 + 3f_2\}E(V). \tag{3.2.8}$$

If ∇ is the Levi-Civita connection of the manifold then a semi-symmetric non-metric connection is given by [1]

$$\tilde{\nabla}_U V = \nabla_U V + E(V)U, \tag{3.2.9}$$

for all vector fields U, V and E is a 1-form defined by (1.5.8). It is called a special type of semi-symmetric non-metric connection if the torsion tensor $\bar{T}$ and the curvature tensor $\tilde{R}$ of the connection $\tilde{\nabla}$ satisfy the following conditions

$$\bar{T}(U, V) = E(V)U - E(U)V, \tag{3.2.10}$$

$$(\tilde{\nabla}_U \bar{T})(V, W) = E(U)\bar{T}(V, W), \tag{3.2.11}$$

$$\textit{and} \quad \tilde{R}(U, V)W = 0. \tag{3.2.12}$$

Agashe and Chafle [1] proved in 1992 that the curvature tensor $\tilde{R}$ with respect to semi-symmetric non-metric connection $\tilde{\nabla}$, (3.2.9) is given by

$$\tilde{R}(U, V)W = R(U, V)W + \alpha(U, W)V - \alpha(V, W)U, \tag{3.2.13}$$

where $\alpha(V, W)$ is a tensor field of type $(0, 2)$ defined by

$$\alpha(V, W) = (\nabla_V E)(W) - E(V)E(W). \tag{3.2.14}$$

Moreover from (3.2.9) we have

$$(\tilde{\nabla}_U E)(V) = (\nabla_U E)(V) - E(U)E(V). \tag{3.2.15}$$

From (3.2.10) we have,

$$(C_1^1 \bar{T})(V) = (n-1)E(V), \tag{3.2.16}$$

where C_1^1 denotes operation of contraction. From (3.2.16) it follows that

$$(\tilde{\nabla}_U C_1^1 \bar{T})(V) = (n-1)(\tilde{\nabla}_U E)(V). \tag{3.2.17}$$

Again from (3.2.11) we get by using (3.2.16)

$$(\tilde{\nabla}_U C_1^1 \bar{T})(V) = E(U)(C_1^1 \bar{T})(V) = (n-1)E(U)E(V). \tag{3.2.18}$$

Hence from (3.2.17) and (3.2.18) we obtain

$$(\tilde{\nabla}_U E)(V) = E(U)E(V). \tag{3.2.19}$$

Using (3.2.19) the equation (3.2.15) can be written as,

$$(\nabla_U E)(V) = 2E(U)E(V). \tag{3.2.20}$$

By virtue of (3.2.20) it follows from (3.2.14) that

$$\alpha(V, W) = E(V)E(W). \tag{3.2.21}$$

Now using (3.2.21) the equation (3.2.13) can be written as follows

$$\tilde{R}(U, V, W, Z) = R(U, V, W, Z) + E(U)E(W)g(V, Z) - E(V)E(W)g(U, Z). \tag{3.2.22}$$

Applying (3.2.12) in (3.2.22), we infer

$$R(U, V, W, Z) = E(V)E(W)g(U, Z) - E(U)E(W)g(V, Z). \tag{3.2.23}$$

Putting $U = Z = e_i$ and summing over i $(1 \leq i \leq n)$ in (3.2.23) we have

$$S(V, W) = (n-1)E(V)E(W). \tag{3.2.24}$$

Contracting (3.2.24) we get

$$r = (n-1)E(\rho), \tag{3.2.25}$$

where r is the scalar curvature. Further, putting $V = W = e_i$ and taking sum over $i(1 \leq i \leq n)$ in (3.2.23) we have

$$S(U, Z) = g(U, Z)E(\rho) - E(U)E(Z), \tag{3.2.26}$$

putting $Z = \rho$ in (3.2.26) and (3.2.23) we get

$$S(U, \rho) = 0. \tag{3.2.27}$$

and

$$R(U, V, W, \rho) = 0, \tag{3.2.28}$$

respectively.

3.3 Almost pseudo symmetric K$\ddot{a}$hler manifolds

In this section we study almost pseudo symmetric K$\ddot{a}$hler manifold. We have

$$R(JV, JW, X, Z) = R(V, W, X, Z). \tag{3.3.1}$$

Taking the covariant derivative of (3.3.1), we get

$$(\nabla_U R)(JV, JW, X, Z) = (\nabla_U R)(V, W, X, Z). \tag{3.3.2}$$

Applying (3.2.1) in (3.3.2), and by virtue of (3.3.1) we get

$$E(V)R(U,W,X,Z)+E(W)R(V,U,X,Z)=E(JV)R(U,JW,X,Z)$$
$$+E(JW)R(JV,U,X,Z). \tag{3.3.3}$$

Putting $W=X=e_i$ in (3.3.3), we obtain

$$E(V)S(U,Z)-R(V,U,Z,\rho)=E(JV)S(U,JZ)+R(JV,U,Z,J\rho). \tag{3.3.4}$$

Again putting $V=\rho=e_i$ in (3.3.4) and summing over $i(1\leq i\leq n)$, we infer

$$(n-2)S(U,Z)=0. \tag{3.3.5}$$

The above equation leads to

$$S(U,Z)=0. \tag{3.3.6}$$

Thus, we have the following:

Theorem 3.3.1. *Let M be an almost pseudo symmetric Kähler manifold then it is Ricci flat.*

Using equation (3.3.6) in (1.2.1), we get

$$(L_Vg)(U,Z)+2\lambda g(U,Z)=0. \tag{3.3.7}$$

Contracting above equation we gain

$$divV+\lambda n=0. \tag{3.3.8}$$

If V is solenoidal then $divV=0$. Therefore the equation (3.3.8) can be reduced to

$$\lambda=0. \tag{3.3.9}$$

Thus, we can write the following:

Corollary 3.3.2. *The Ricci soliton (g, V, λ) in an almost pseudo symmetric Kähler manifold is steady if and only if V is solenoidal.*

Equation (3.3.7) can be written as

$$(L_V g)(U, Z) = -2\lambda g(U, Z), \tag{3.3.10}$$

comparing equation (3.2.7) and (3.3.10) we write the following:

Corollary 3.3.3. *If (g, V, λ) is a Ricci soliton in an almost pseudo symmetric Kähler manifold then V is conformal killing.*

3.4 Almost pseudo Bochner symmetric K$\ddot{a}$hler manifolds

Using equations (1.1.1) and (1.1.12) we find

$$D(JV, JW, X, Z) = D(V, W, X, Z). \tag{3.4.1}$$

In this section we suppose that (M, g) is an almost pseudo Bochner symmetric K$\ddot{a}$hler manifold. Then using equations (1.1.1), (3.2.3) and (3.4.1) we gain

$$(\nabla_U D)(JV, JW, X, Z) = (\nabla_U D)(V, W, X, Z). \tag{3.4.2}$$

Applying (3.2.3) in (3.4.2), we get

$$E(V)D(U, W, X, Z) + E(W)D(V, U, X, Z) = E(JV)D(U, JW, X, Z)$$
$$+E(JW)D(JV, U, X, Z). \tag{3.4.3}$$

Setting $W = X = e_i$ in (3.4.3) and taking summation over i $(1 \leq i \leq n)$ we get

$$E(V)K(U,Z) - E(D(V,U)Z) = -E(JV)K(JU,Z) - E(D(JV,U)JZ). \tag{3.4.4}$$

Again setting $V = \rho = e_i$ in (3.4.4) and taking sum over $i(1 \leq i \leq n)$, we obtain

$$(n-2)K(U,Z) = 0. \tag{3.4.5}$$

This can be written as

$$K(U,Z) = 0. \tag{3.4.6}$$

The above equation in (3.2.5) gives

$$S(U,Z) = \frac{r}{2(n+1)}g(U,Z). \tag{3.4.7}$$

Thus, we can state the following:

Theorem 3.4.1. *If M is an almost pseudo Bochner symmetric Kähler manifold then it is an Einstein manifold.*

Applying equation (3.4.7) in (1.2.1), we get

$$(L_V g)(U,Z) + 2[\frac{r}{2(n+1)} + \lambda]g(U,Z) = 0, \tag{3.4.8}$$

contracting the above equation we get

$$divV + \frac{r}{2(n+1)}n + \lambda n = 0. \tag{3.4.9}$$

If V is solenoidal then $divV = 0$. Therefore the equation (3.4.9) can be reduced to

$$\lambda = -\frac{r}{2(n+1)}. \tag{3.4.10}$$

Thus, we obtain the following:

Corollary 3.4.2. *Let (g, V, λ) be a Ricci soliton in an almost pseudo Bochner symmetric Kähler manifold. Then V is solenoidal if and only if it is shrinking, steady and expanding depending upon $r > 0, r = 0$ and $r < 0$.*

3.5 Almost pseudo Ricci-symmetric K$\ddot{a}$hler manifolds

If the manifold M is an almost pseudo Ricci-symmetric Kähler Manifold, then from (1.1.1), (1.1.2) and (3.2.2) we find

$$(\nabla_U S)(JV, JW) = (\nabla_U S)(V, W). \tag{3.5.1}$$

Using (3.2.2) in (3.5.1), we get

$$E(JV)S(U, JW) + E(JW)S(JV, U) = E(V)S(U, W) + E(W)S(V, U). \tag{3.5.2}$$

By substituting $V = \rho = e_i$, $1 \leq i \leq n$ in (3.5.2) and summing over i, we have

$$(n + 2)S(U, W) = 0. \tag{3.5.3}$$

This equation leads to

$$S(U, W) = 0. \tag{3.5.4}$$

Hence, we have the following:

Theorem 3.5.1. *Let M be an almost pseudo Ricci-symmetric Kähler manifold then it is Ricci flat.*

Using equation (3.5.4) in (1.2.1), we get

$$(L_V g)(U, W) + 2\lambda g(U, W) = 0. \tag{3.5.5}$$

Substituting $U = W = e_i$ in (3.5.5) and then summing over i $(1 \leq i \leq n)$, we obtain

$$divV + \lambda n = 0. \tag{3.5.6}$$

If V is solenoidal then $divV = 0$. Therefore the equation (3.5.6) can be reduced to

$$\lambda = 0. \tag{3.5.7}$$

Thus, we can write the following:

Corollary 3.5.2. *Ricci soliton (g, V, λ) in an almost pseudo Ricci-symmetric Kähler manifold is steady if and only if V is solenoidal.*

3.6 Almost pseudo Bochner Ricci-symmetric K$\ddot{a}$hler manifolds

If the manifold M is an almost pseudo Bochner Ricci-symmetric Kähler manifold, then we can easily write

$$K(JV, JW) = K(V, W). \tag{3.6.1}$$

Taking the covariant derivative of (3.6.1), we get

$$(\nabla_U K)(JV, JW) = (\nabla_U K)(V, W). \tag{3.6.2}$$

Using (3.2.4) in (3.6.2), we get

$$E(JV)K(U, JW) + E(JW)K(JV, U) = E(V)K(U, W) + E(W)K(V, U). \tag{3.6.3}$$

Letting $V = \rho = e_i$ in (3.6.3), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and summing over i $(1 \leq i \leq n)$ we gain

$$K(U, W) = 0. \tag{3.6.4}$$

Applying the above equation in (3.2.5) we write

$$S(U, W) = \frac{r}{2(n+1)} g(U, W). \tag{3.6.5}$$

Thus, we can state the following:

Theorem 3.6.1. *An almost pseudo Bochner Ricci-symmetric Kähler manifold M is an Einstein manifold.*

Using equation (3.6.5) in (1.2.1), we obtain

$$(L_V g)(U, W) + 2\frac{r}{2(n+1)} g(U, W) + 2\lambda g(U, W) = 0. \tag{3.6.6}$$

Contracting (3.6.6), we obtain

$$divV + \frac{r}{2(n+1)} n + \lambda n = 0. \tag{3.6.7}$$

If V is solenoidal then $divV = 0$. Therefore the equation (3.6.7) can be reduced to

$$\lambda = -\frac{r}{2(n+1)}. \tag{3.6.8}$$

Hence, we state the following result:

Corollary 3.6.2. *Let (g, V, λ) be a Ricci soliton in an almost pseudo Bochner Ricci-symmetric Kähler manifold. Then V is solenoidal if and only if it is shrinking, steady and expanding depending upon the sign of scalar curvature.*

Equation (3.6.6) can be written as

$$(L_V g)(U, W) = -2\left[\frac{r}{2(n+1)} + \lambda\right] g(U, W), \tag{3.6.9}$$

comparing equation (3.2.7) and (3.6.9) we can write the following:

Corollary 3.6.3. *If (g, V, λ) is a Ricci soliton in an almost pseudo Ricci-symmetric Kähler manifold then V is conformal killing.*

3.7 Almost pseudo Bochner symmetric generalized complex space form

Let M be an almost pseudo Bochner symmetric generalized complex space form, we know that

$$D(JV, JW, X, Y) = D(V, W, X, Y). \tag{3.7.1}$$

Taking the covariant derivative of (3.7.1), we get

$$(\nabla_U D)(JV, JW, X, Y) = (\nabla_U D)(V, W, X, Y). \tag{3.7.2}$$

Suppose M is an almost pseudo Bochner symmetric, then using (3.2.3) in (3.7.2), we obtain

$$E(V)D(U, W, X, Y) + E(W)D(V, U, X, Y) = E(JV)D(U, JW, X, Y)$$
$$+E(JW)D(JV, U, X, Y). \tag{3.7.3}$$

Substituting $W = X = e_i$ in (3.7.3) and summing over i $(1 \leq i \leq n)$ we have

$$E(V)K(U, Y) - E(D(V, U)Y) = -E(JV)K(JU, Y) - E(D(JV, U)JY). \tag{3.7.4}$$

Applying equations (1.1.12) and (3.2.5) in (3.7.4) and setting $U = Y = e_i$, $(1 \leq i \leq n)$, further using equations (1.3.6) and (3.2.8) in the resulting equation, then we obtain

$$E(V)r = 0. \tag{3.7.5}$$

Thus if $r \neq 0$, then from (3.7.5) we get $E(V) = 0$. Using $E(V) = 0$ in (3.2.3) we have

$$(\nabla_U D)(V, W, X, Y) = F(U)D(V, W, X, Y).$$

That is, an almost pseudo Bochner symmetric generalized complex space forms reduces to recurrent one. Therefore we can write the following:

Theorem 3.7.1. *An almost pseudo Bochner symmetric generalized complex space form with non zero scalar curvature is recurrent.*

Contracting (3.2.3) with respect to the pair of arguments V, Y, we have

$$\begin{aligned}(\nabla_U K)(W,X) &= [E(U)+F(U)]K(W,X)+E(W)K(U,X)+E(X)K(W,U)\\ &+ E(D(U,W)V)-E(D(X,U)W). \qquad (3.7.6)\end{aligned}$$

Applying equations (1.1.12) and (3.2.5) in (3.7.6) and putting $U = X = e_i$, further using equation (1.3.6) and (3.2.8) to the simplified equation we get

$$drZ = \frac{(n+1)}{n^2}[(n+2)rE(W)+2nS(W,\beta)-\frac{nr}{(n+1)}F(W)]. \qquad (3.7.7)$$

Let us suppose that the space form under consideration of non zero constant scalar curvature. Then from (3.7.7) we get,

$$S(W,\beta) = \frac{r}{2(n+1)}F(W)-\frac{(n+2)r}{2n}E(W). \qquad (3.7.8)$$

By virtue of (1.1.6) the above equation shows that $S(W,\beta)$ cannot be of the form $\kappa F(W)$, where κ is a scalar.

Hence β cannot be an eigen vector corresponding to any eigen value κ of S.

This leads to the following theorem:

Theorem 3.7.2. *In an almost pseudo Bochner symmetric generalized complex space form of non zero constant scalar curvature, β cannot be an eigen vector corresponding to any eigen value of S.*

If in particular $E = F$, then from (3.7.8), we have

$$S(W,\beta) = -\frac{(n^2+2n+2)}{2n(n+1)}rF(W). \qquad (3.7.9)$$

Hence, we obtain the following:

Corollary 3.7.3. *In a pseudo Bochner symmetric generalized complex space form of non zero constant scalar curvature,* $-\frac{(n^2+2n+2)}{2n(n+1)}r$ *is an eigen value corresponding to the eigen vector* β.

Using equation (3.7.9) in (1.2.1), we get

$$(L_V g)(W,\beta) + 2(-\frac{(n^2+2n+2)}{2n(n+1)}r)g(W,\beta) + 2\lambda g(W,\beta) = 0. \tag{3.7.10}$$

Suppose V is killing vector field then using equation (3.2.6) in (3.7.10), we get

$$\lambda = \frac{(n^2+2n+2)}{2n(n+1)}r.$$

This we obtain the following:

Corollary 3.7.4. *Let* (g, V, λ) *be a Ricci soliton in a pseudo Bochner symmetric generalized complex space form of non zero constant scalar curvature. If* V *is killing vector field then it is expanding, steady and shrinking when* $r > 0$, $r = 0$ *and* $r < 0$.

3.8 Almost pseudo Bochner Ricci-symmetric generalized complex space form

Suppose the manifold M is an almost pseudo Bochner Ricci-symmetric generalized complex space form, then we have

$$K(JV, JW) = K(V, W). \tag{3.8.1}$$

Taking the covariant derivative of (3.8.1), we get

$$(\nabla_U K)(JV, JW) = (\nabla_U K)(V, W). \tag{3.8.2}$$

Using (3.2.4) in (3.8.2), we get

$$E(JV)K(U,JW) + E(JW)K(JV,U) = E(V)K(U,W) + E(W)K(V,U). \tag{3.8.3}$$

Applying equation (3.2.5) in (3.8.3) and replacing $U = W = e_i$, further using equations (1.3.6) and (3.2.8) in the resulting equation we obtain,

$$E(V)r = 0. \tag{3.8.4}$$

Thus if $r \neq 0$, then from (3.8.4) we get $E(V) = 0$. Using $E(V) = 0$ in (3.2.4) we have

$$(\nabla_U K)(V,W) \quad = \quad F(U)K(V,W). \tag{3.8.5}$$

Hence, we obtain the following:

Theorem 3.8.1. *An almost pseudo Bochner Ricci-symmetric generalized complex space form with non zero scalar curvature reduces to Ricci recurrent one.*

Also from (3.8.5) we get

$$(\nabla_U K)(V,W) - (\nabla_V K)(U,W) \quad = \quad F(U)K(V,W) - F(V)K(U,W). \tag{3.8.6}$$

Contracting (3.8.6) over V and W and using equations (1.3.4), (1.3.6) and (3.2.5), we get

$$dr(U) = r(n+1)F(U) - \frac{n}{2n+4}S(U,\beta). \tag{3.8.7}$$

If the scalar curvature r is constant, then

$$dr(U) = 0. \tag{3.8.8}$$

By virtue of (3.8.7) and (3.8.8) yields

$$S(U,\beta) = \frac{(2n+4)(n+1)}{n}rF(U). \tag{3.8.9}$$

In the other way, we assume that the Bochner curvature tensor of this space form is Codazzi type [35]. Then we have

$$(\nabla_U K)(V,W) - (\nabla_V K)(U,W) = 0. \tag{3.8.10}$$

Applying (3.8.10) in (3.8.6), we get

$$F(U)K(V,W) - F(V)K(U,W) = 0. \tag{3.8.11}$$

Using equations (1.3.4) and (3.2.5) and contraction over V and W, further using equation (1.3.6) to the simplified equation we infer

$$S(U,\beta) = \frac{(2n+4)(n+1)}{n} rF(U). \tag{3.8.12}$$

This leads the following theorem:

Theorem 3.8.2. *In an almost pseudo Bochner Ricci-symmetric generalized complex space form, if*

- *non zero scalar curvature or*
- *Bochner curvature tensor is Codazzi type*

then $\frac{(2n+4)(n+1)}{n}r$ *is an eigenvalue of the Ricci tensor* S *corresponding to the eigenvector* β*.*

Using equation (3.8.9) or (3.8.12) in (1.2.1), we get

$$(L_V g)(U,\beta) + 2(\frac{(2n+4)(n+1)}{n} r)g(U,\beta) + 2\lambda g(U,\beta) = 0. \tag{3.8.13}$$

By virtue of equation (3.2.7) in (3.8.13), we get

$$\lambda = -\frac{n\varphi + 2(2n+4)(n+1)r}{2n}.$$

This leads the following:

Corollary 3.8.3. *Let (g, V, λ) be a Ricci soliton in an almost pseudo Bochner Ricci-symmetric generalized complex space form of non zero scalar curvature or Bochner curvature tensor is Codazzi type. If V is conformal killing vector field then it is shrinking, steady and expanding depending upon the scalar curvature.*

3.9 Bochner flat almost pseudo Ricci-symmetric generalized complex space form

From (3.2.2) we get

$$(\nabla_U S)(V, W) - (\nabla_V S)(U, W) = F(U)S(V, W) - F(V)S(U, W). \tag{3.9.1}$$

Setting $V = W = e_i$ in (3.9.1), then we obtain

$$dr(U) = 2rF(U) - 2S(U, \beta). \tag{3.9.2}$$

Putting $U = JU$ and $V = JV$ in (3.9.1), we get

$$(\nabla_{JU} S)(JV, W) - (\nabla_{JV} S)(JU, W) = F(JU)S(JV, W) - F(JV)S(JU, W). \tag{3.9.3}$$

Again setting $V = W = e_i$ in (3.9.3), where $\{e_i\}, i = 1, 2, 3 \dots n$, is an orthonormal basis of the tangent space at each point of the manifold and summing over i $(\leq i \leq n)$, we get

$$dr(JU) = -2S(U, \beta). \tag{3.9.4}$$

Let the generalized complex space form be Bochner flat. Then we have

$$D(U, V)W = 0. \tag{3.9.5}$$

Using (3.9.5) in (1.1.12) and Differentiating covariantly and contracting we obtain

$$\begin{aligned}
0 &= (divR)(U,V)W - \frac{1}{2n+4}[g(V,W)dr(U) - (\nabla_V S)(U,W) \\
&+ g(JV,W)dr(JU) - (\nabla_{JV} S)(JU,W) + (\nabla_U S)(V,W) - g(U,W)dr(V) \\
&+ (\nabla_{JU} S)(JV,W) - g(JU,W)dr(JV) - 2(\nabla_{JW} S)(V,JU) \\
&- 2g(JU,V)dr(JW)] + \frac{1}{(2n+2)(2n+4)}[g(V,W)dr(U) - g(U,W)dr(V) \\
&+ g(JV,W)dr(JU) - g(JU,W)dr(JV) - 2g(JU,V)dr(JW)]. \qquad (3.9.6)
\end{aligned}$$

Using equation (1.1.4) in (3.9.6), we get

$$\begin{aligned}
&\frac{2n+3}{2n+4}[(\nabla_U S)(V,W) - (\nabla_V S)(U,W)] - \frac{1}{2n+4}[(\nabla_{JU} S)(JV,W) - (\nabla_{JV} S)(JU,W) \\
&-2(\nabla_{JW} S)(V,JU)] = \frac{2n+1}{(2n+2)(2n+4)}[g(V,W)dr(U) - g(U,W)dr(V) \\
&+g(JV,W)dr(JU) - g(JU,W)dr(JV) - 2g(JU,V)dr(JW)]. \qquad (3.9.7)
\end{aligned}$$

Further let the generalized complex space form be almost pseudo Ricci-symmetric. Then applying equations (3.9.1) and (3.9.3) in (3.9.7) we gain

$$\begin{aligned}
&\frac{2n+3}{2n+4}[F(U)S(V,W) - F(V)S(U,W)] - \frac{1}{2n+4}[F(JU)S(JV,W) - F(JV)S(JU,W) \\
&-2(\nabla_{JW} S)(V,JU)] = \frac{2n+1}{(2n+2)(2n+4)}[g(V,W)dr(U) - g(U,W)dr(V) \\
&+g(JV,W)dr(JU) - g(JU,W)dr(JV) - 2g(JU,V)dr(JW)]. \qquad (3.9.8)
\end{aligned}$$

Using equation (1.3.4) in (3.9.8), we have

$$\begin{aligned}
&\frac{2n+3}{2n+4}\{(n-1)f_1 + 3f_2\}[(n-1)F(U)] - \frac{1}{2n+4}[F(JU)S(JV,W) - F(JV)S(JU,W) \\
&-2(\nabla_{JW} S)(V,JU)] = \frac{2n+1}{(2n+2)(2n+4)}[g(V,W)dr(U) - g(U,W)dr(V) \\
&+g(JV,W)dr(JU) - g(JU,W)dr(JV) - 2g(JU,V)dr(JW)]. \qquad (3.9.9)
\end{aligned}$$

Setting $V = W = e_i$ and taking sum over i $(1 \leq i \leq n)$, on further simplification using equations (1.3.6), (3.9.2) and (3.9.4) we get

$$S(U, \beta) = \frac{4n+3}{-2n^2+4n+4}(n-1)\{(n-1)f_1 + 3f_2\}F(U). \tag{3.9.10}$$

$$S(U, \beta) = \phi_1 g(U, \beta), \tag{3.9.11}$$

where $\phi_1 = \frac{4n+3}{-2n^2+4n+4}(n-1)\{(n-1)f_1 + 3f_2\}$.

Then we can state the following theorem:

Theorem 3.9.1. *In a Bochner flat almost pseudo Ricci-symmetric generalized complex space form, ϕ_1 is an eigen value corresponding to the eigen vector β.*

Applying equation (3.9.11) in (1.2.1), we get

$$(L_V g)(U, \beta) + 2\phi_1 g(U, \beta) + 2\lambda g(U, \beta) = 0. \tag{3.9.12}$$

Equation (3.2.7) in (3.9.12), we get

$$\lambda = -\frac{\varphi + 2\phi_1}{2}.$$

Then we can write the following:

Corollary 3.9.2. *Let (g, V, λ) be a Ricci soliton in a Bochner flat almost pseudo Ricci-symmetric generalized complex space form. If V is conformal killing vector field then it is shrinking.*

3.10 Almost pseudo symmetric K$\ddot{a}$hler manifolds admitting a special type of semi-symmetric non-metric connection $\tilde{\nabla}$

Let (M, g) be an almost pseudo symmetric and Kähler manifold. We know

$$R(JV, JW, X, Y) = R(V, W, X, Y). \tag{3.10.1}$$

Taking the covariant derivative of (3.10.1), we get

$$(\nabla_U R)(JV, JW, X, Y) = (\nabla_U R)(V, W, X, Y). \tag{3.10.2}$$

Applying (3.2.1) in (3.10.2), we get

$$\begin{aligned} &E(V)R(U, W, X, Y) + E(W)R(V, U, X, Y) = E(JV)R(U, JW, X, Y) \\ &+E(JW)R(JV, U, X, Y). \end{aligned} \tag{3.10.3}$$

Contracting (3.10.3) we get

$$E(V)S(U, Y) - R(V, U, Y, \rho) = E(JV)S(U, JY) + R(JV, U, Y, J\rho). \tag{3.10.4}$$

By setting $U = Y = e_i$ in (3.10.4) and summing over $i(1 \leq i \leq n)$, we obtain

$$rE(V) - S(V, \rho) = S(V, \rho). \tag{3.10.5}$$

Using (3.2.27) in (3.10.5) we gain,

$$rE(V) = 0. \tag{3.10.6}$$

Thus if $r \neq 0$, then from (3.10.6) we get $E(V) = 0$. Using $E(V) = 0$ in (3.2.3) we have

$$(\nabla_U D)(V, W, X, Y) = F(U)D(V, W, X, Y).$$

Thus, we can write the following:

Theorem 3.10.1. *An almost pseudo symmetric Kähler manifold admitting a special type of semi-symmetric non-metric connection $\tilde{\nabla}$ with non zero scalar curvature is recurrent.*

We can write (3.2.1) as,

$$\begin{aligned}(\nabla_U R)(V,W)X &= [E(U)+F(U)]R(V,W)X + E(V)R(U,W)X + E(W)R(V,U)X \\ &+ E(X)R(V,W)U + R(V,W,X,U)\rho. \end{aligned} \tag{3.10.7}$$

Contracting the above with respect to U, we get

$$\begin{aligned}(divR)(V,W)X &= E(R(V,W)X) + F(R(V,W)X) + E(V)S(W,X) \\ &- E(W)S(X,V) + R(V,W,X,\rho). \end{aligned} \tag{3.10.8}$$

Applying (1.1.4) in (3.10.8) we obtain

$$\begin{aligned}(\nabla_V S)(W,X) - (\nabla_W S)(V,X) &= E(R(V,W)X) + F(R(V,W)X) + E(V)S(W,X) \\ &- E(W)S(X,V) + R(V,W,X,\rho). \end{aligned} \tag{3.10.9}$$

Replacing orthonormal basis $\{e_i\}$ in place of V and X and using equation (1.1.8) in (3.10.9) and taking summation over i $(1 \le i \le n)$ we get

$$\frac{1}{2}dr(W) - dr(W) = -S(W,\rho) - S(W,\beta) + S(W,\rho) - rE(W) - S(W,\rho), \tag{3.10.10}$$

whence by using (3.2.27) and $dr(W) = 0$ as $r = 0$, we get $S(W,\beta) = 0$. The above equation can also be written as

$$S(W,\beta) = 0.g(W,\beta). \tag{3.10.11}$$

Which, by replacing β by ω, leads to

$$S(W,\omega) = 0.g(W,\omega). \tag{3.10.12}$$

Hence, from equations (3.10.11) and (3.10.12), we conclude:

Theorem 3.10.2. *Let M be an almost pseudo symmetric Kähler manifold admitting a special type of semi-symmetric non-metric connection $\tilde{\nabla}$, then it is Ricci flat and hence β and ω are eigenvectors of the Ricci tensor S with respect to the zero eigen value.*

Using equation (3.10.11) in (1.2.1), we get

$$(L_V g)(W, \beta) + 2\lambda g(W, \beta) = 0. \tag{3.10.13}$$

By virtue of (3.2.6) in (3.10.13), we get

$$\lambda = 0.$$

Then we can state the following:

Corollary 3.10.3. *Let (g, V, λ) be a Ricci soliton in an almost pseudo symmetric Kähler manifold equipped with a special type of semi-symmetric non-metric connection $\tilde{\nabla}$. If V is killing vector field then it is steady.*

By virtue of (3.2.7) in (3.10.13), we get

$$\lambda = -\frac{\varphi}{2}.$$

Then we can write the following:

Corollary 3.10.4. *Let (g, V, λ) be a Ricci soliton in an almost pseudo symmetric Kähler manifold initiating a special type of semi-symmetric non-metric connection $\tilde{\nabla}$. If V is conformal killing vector field then it is shrinking.*

3.11 Projective flat almost pseudo symmetric K$\ddot{a}$hler manifolds admitting a special type of semi-symmetric non-metric connection $\tilde{\nabla}$

Taking orthonormal basis over U and Z in (3.2.22), we get

$$\tilde{S}(V,W) = S(V,W) - (n-1)E(W)E(V). \tag{3.11.1}$$

Now the projective curvature tensor $\tilde{P}$ of connection $\tilde{\nabla}$ is given by

$$\tilde{P}(U,V,W,Z) = \tilde{R}(U,V,W,Z) - \frac{1}{n-1}[\tilde{S}(V,W)g(U,Z) - \tilde{S}(U,W)g(V,Z)]. \tag{3.11.2}$$

Using (3.2.22) and (3.11.1) in (3.11.2), we get [1]

$$\tilde{P}(U,V,W,Z) = P(U,V,W,Z). \tag{3.11.3}$$

If the manifold is projective flat with respect to $\tilde{\nabla}$ then the manifold will be projective flat with respect to the connection ∇ i.e.,

$$\tilde{P}(U,V,W,Z) = 0 \Rightarrow P(U,V,W,Z) = 0. \tag{3.11.4}$$

Now equation (3.11.4) implies,

$$R(U,V,W,Z) = \frac{1}{n-1}[S(V,W)g(U,Z) - S(U,W)g(V,Z)]. \tag{3.11.5}$$

Applying (3.2.24) in (3.11.5), we have

$$R(U,V,W,Z) = E(W)[E(V)g(U,Z) - E(U)g(V,Z)]. \tag{3.11.6}$$

Hence, from the above equation we conclude:

Theorem 3.11.1. *If M is a projective flat Riemannian manifold with respect to a special type of semi-symmetric non-metric connection $\tilde{\nabla}$, then it is the manifold of constant curvature.*

Equation (3.2.23) and (3.11.6) are identical and therefore Theorems (3.10.1), (3.10.2) and Corollaries (3.10.3) and (3.10.4) can be stated as follows:

Theorem 3.11.2. *Let M be an almost pseudo symmetric projective flat Kähler manifold allowing a special type of semi-symmetric non-metric connection $\tilde{\nabla}$ with non zero scalar curvature is recurrent.*

Theorem 3.11.3. *Let M be an almost pseudo symmetric projective flat Kähler manifold admitting a special type of semi-symmetric non-metric connection $\tilde{\nabla}$, then the vector fields β and ω are eigenvector of the Ricci tensor S with respect to $\tilde{\nabla}$.*

Corollary 3.11.4. *Let (g, V, λ) be a Ricci soliton in an almost pseudo symmetric projective flat Kähler manifold equipped with a special type of semi-symmetric non-metric connection $\tilde{\nabla}$. If V is killing vector field then it is steady.*

Corollary 3.11.5. *Let (g, V, λ) be a Ricci soliton in an almost pseudo symmetric projective flat Kähler manifold initiating a special type of semi-symmetric non-metric connection $\tilde{\nabla}$. If V is conformal killing vector field then it is shrinking.*

3.12 Almost pseudo symmetric K$\ddot{a}$hler manifolds admitting a special type of semi-symmetric non-metric connection $\tilde{\nabla}$ with parallel projective curvature tensor

Assume that the projective curvature of an almost pseudo symmetric K$\ddot{a}$hler manifold is parallel $i.e.$, $\nabla P = 0$.

Using the properties of K$\ddot{a}$hler manifolds and using (3.11.3), the equation (3.11.2) we can be expressed as

$$\begin{aligned} P(JU, JV, W, Z) = R(U, V, W, Z) \quad &- \quad \frac{1}{n-1}[S(JV, W)g(JU, Z) \\ &- \quad S(JU, W)g(JV, Z)]. \end{aligned} \tag{3.12.1}$$

Putting $U = W = e_i$ in (3.12.1), where $\{e_i\}, i = 1, 2, 3 \ldots n$, is an orthonormal basis of the tangent space at each point of the manifold and summing over i, $1 \leq i \leq n$, we get

$$g(P(Je_i, JV)e_i, Z) = \frac{n}{n-1}S(V, Z). \tag{3.12.2}$$

Taking covariant differentiation of (3.12.2) and our assumption yields

$$(\nabla_W)S(V, Z) = 0. \tag{3.12.3}$$

In view of (3.11.3), The covariant derivative ∇P can be expressed in the following form:

$$(\nabla_Z P)(U, V)W = (\nabla_Z R)(U, V)W - \frac{1}{n-1}[(\nabla_Z S)(V, W)U - (\nabla_Z S)(U, W)V]. \tag{3.12.4}$$

Using (3.12.3) in (3.12.4), we obtain

$$(\nabla_Z P)(U, V)W = (\nabla_Z R)(U, V)W. \tag{3.12.5}$$

Thus, we can write the following:

Theorem 3.12.1. *An almost pseudo symmetric Kähler manifold M initiate a special type of semi-symmetric non-metric connection $\tilde{\nabla}$, then M is projectively symmetric if and only if it is locally symmetric.*

3.13 Almost pseudo projective symmetric K$\ddot{a}$hler manifolds admitting a special type of semi-symmetric non-metric connection $\tilde{\nabla}$

Definition 2. *A Riemannian manifold (M, g) is called almost pseudo projective symmetric manifold if its Projective curvature tensor P of type $(0, 4)$ is satisfies the condition*

$$(\nabla_U P)(JV, JW, X, Y) = [E(U) + F(U)]P(JV, JW, X, Y) + E(JV)P(U, JW, X, Y)$$
$$+E(JW)P(JV, U, X, Y) + E(X)P(JV, JW, U, Y) + E(Y)P(JV, JW, X, U), \quad (3.13.1)$$

where E, F are two non zero 1-forms and P is given in (3.12.1).

Setting orthonormal basis over V and Y in (3.13.1) we get

$$\begin{aligned}&(\nabla_U S)(W, X)[\frac{n-2}{n-1}] = [E(U) + F(U)][\frac{n-2}{n-1}]S(W, X) - R(U, JW, X, J\rho)\\&-\frac{1}{n-1}[S(JW, X)E(JU) + S(U, X)E(W)] - E(JW)[\frac{n}{n-1}]S(JU, X)\\&+E(X)[\frac{n-2}{n-1}]S(W, U) - R(X, U, W, \rho) - \frac{1}{n-1}[-S(JW, X)E(JU) + S(JX, \rho)g(JW, U)].\end{aligned} \quad (3.13.2)$$

Again we putting orthonormal basis in (3.13.2) place of W and X we obtain

$$[\frac{n-2}{n-1}]dr(U) = r[E(U) + F(U)][\frac{n-2}{n-1}] + S(U, \rho)[4 - \frac{2}{n-1}]. \quad (3.13.3)$$

Applying equation (3.2.27) in (3.13.3) we get

$$dr(U) = r[E(U) + F(U)]. \tag{3.13.4}$$

Hence, we state the following:

Theorem 3.13.1. *The scalar curvature tensor r of an almost pseudo projective symmetric Kähler manifold allowing a special type of semi-symmetric non-metric connection $\tilde{\nabla}$ satisfies the following relation*

$$dr(U) = r[E(U) + F(U)] \;\; for \;\; all \;\; U. \tag{3.13.5}$$

Let us consider an almost pseudo projective symmetric Kähler manifold of constant scalar curvature. Thus from (3.13.5) and r not equal to 0, we get

$$E(U) + F(U) = 0 \;\; for \;\; all \;\; U. \tag{3.13.6}$$

Then we can write the following:

Theorem 3.13.2. *If the two associated 1-forms are linearly independent in an almost pseudo projective symmetric Kähler manifold admitting a special type of semi-symmetric non-metric connection $\tilde{\nabla}$, then it has non-zero constant scalar curvature.*

3.14 Almost pseudo symmetric with recurrent K$\ddot{a}$hler manifolds admitting a special type of semi -symmetric non-metric connection $\tilde{\nabla}$

Recurrent manifolds were introduced by Walkar [59] in 1950, now we have discuss Kähler recurrent manifolds. A Kähler manifold (M, g) is called recurrent if its curvature tensor

R satisfies the condition

$$(\nabla_U R)(V, W, X, Y) = A(U)R(V, W, X, Y), \tag{3.14.1}$$

where E is a non-zero 1-form and it is defined by $A(U) = g(U, \xi)$, where ξ is the associated vector field.

Using equation (3.2.1) in (3.14.1) then we get,

$$[E(U) + F(U)]R(V, W, X, Y) + E(V)R(U, W, X, Y) + E(W)R(V, U, X, Y)$$
$$+E(X)R(V, W, U, Y) + R(V, W, X, U)E(Y) = A(U)R(V, W, X, Y). \tag{3.14.2}$$

Contracting the above equation (3.14.2) over U and Y we obtain

$$E(R(V, W)X) + F(R(V, W)X) + E(V)S(W, X) - E(W)S(V, X)$$
$$+R(V, W, X, \rho) = R(V, W, X, \xi). \tag{3.14.3}$$

Putting $V = X = e_i$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and summing over $i(1 \leq i \leq n)$, we get

$$S(W, \rho) + S(W, \beta) + rE(W) = S(W, \xi). \tag{3.14.4}$$

Using equation (3.2.27) and (3.10.6) in (3.14.4) we obtain

$$S(W, \beta) - S(W, \xi) = 0. \tag{3.14.5}$$

This implies

$$S(W, \beta - \xi) = 0. \tag{3.14.6}$$

Hence, we can yield the following result:

Theorem 3.14.1. *If M is an almost pseudo symmetric with recurrent Kähler manifold admitting a special type of semi-symmetric non-metric connection $\tilde{\nabla}$, then $\beta - \xi$ is the eigenvector as the Ricci tensor S with respect to the zero eigenvalue.*

3.15 Conclusion

The important results finding of this chapter are as follows:

- The almost pseudo symmetric and almost pseudo Ricci-symmetric Kăhler manifolds are Ricci flat.
- The almost pseudo Bochner symmetric and almost pseudo Bochner Ricci-symmetric Kăhler manifolds are Einstein manifold.
- Let (g, V, λ) be Ricci soliton in an almost pseudo Bochner symmetric or an almost pseudo Bochner Ricci-symmetric Kăhler manifold. Then V is solenoidal if and only if it is shrinking, steady and expanding depending upon the sign of scalar curvature.
- Let (g, V, λ) be Ricci soliton in an almost pseudo Ricci-symmetric Kăhler manifold then it is steady if and only if V is solenoidal.
- An almost pseudo Bochner symmetric and an almost pseudo Bochner Ricci-symmetric generalized complex space form with non zero scalar curvature is recurrent and Ricci recurrent respectively.
- Let (g, V, λ) be Ricci soliton in a pseudo Bochner symmetric generalized complex space form of non zero constant scalar curvature. If V is killing vector field then it is expanding, steady and shrinking when $r > 0$, $r = 0$ and $r < 0$ respectively.
- Let (g, V, λ) be Ricci soliton in a Bochner flat almost pseudo Ricci-symmetric generalized complex space form. If V is conformal killing vector field then it is shrinking.
- Let M be an almost pseudo symmetric or projective flat almost pseudo symmetric

Kähler manifold allowing a special type of semi-symmetric non-metric connection $\tilde{\nabla}$ with non zero scalar curvature is recurrent.

- Let M be an almost pseudo symmetric Kähler manifold admitting a special type of semi-symmetric non-metric connection $\tilde{\nabla}$, then it is Ricci flat.

- Let (g, V, λ) be Ricci soliton in an almost pseudo symmetric Kähler manifold or an almost pseudo symmetric projective flat Kähler manifold, equipped with a special type of semi-symmetric non-metric connection $\tilde{\nabla}$ then we have

  1. If V is killing then it is steady.

  2. If V is conformal then it is expanding.

- An almost pseudo symmetric Kähler manifold M initiate a special type of semi-symmetric non-metric connection $\tilde{\nabla}$, then M is projectively symmetric if and only if it is locally symmetric.

Chapter 4

Eisenhart Problem to Ricci Solitons in K$\ddot{a}$hler Manifolds

4.1 Introduction

Eisenhart [34] proved that if a positive definite Riemannian manifold (M, g) admits a second order parallel symmetric covariant tensor other than a constant multiple of the metric tensor then it is reducible. In 1925 [40], Levy obtained the necessary and sufficient conditions for the existence of such tensors. Since then, many others investigated the Eisenhart problem of finding symmetric and skew-symmetric parallel tensors on various spaces and obtained fruitful results. For instance, by giving a global approach based on the Ricci identity. Sharma [49] investigated Eisenhart problem on non-flat real and complex space forms, in 1989.

Using Eisenhart problem the authors Calin and Crasmareanu [9], Bagewadi and Ingalahalli [39, 7], Debnath and Bhattacharyya [29], Hui and Shaikh [17] have studied the existence of Ricci solitons in f-Kenmotsu manifolds, α-Sasakian, Lorentzian α - Sasakian, Trans-Sasakian and $(LCS)_n$ manifolds. Based on the above work, In this chapter we study Ricci solitons of Kähler manifolds using Eisenhart problem and the following results:

Theorem 4.1.1. *[49] A symmetric parallel second order covariant tensor h in a non-flat real space form of dimension $n > 2$ is a scalar multiple of the metric tensor i.e.,*

$$h(U, W) = \frac{tr.h}{n} g(U, W). \tag{4.1.1}$$

Theorem 4.1.2. *[49] A parallel second order covariant tensor h in a non-flat complex space form is a linear combination (with constant coefficients) of the underlying Kählerian metric and Kählerian 2-form i.e.,*

$$h(U, W) = \frac{1}{n}[(tr.h)g(U, W) + tr.(hJ)\Omega(U, W)], \tag{4.1.2}$$

where U, W are vector fields, J is complex structure tensor of type $(1, 1)$, Ω is a Kählerian 2-form.

4.2 Parallel second order covariant tensor and Ricci soliton in a non-flat real space form

We write the following Corollary using Theorem (4.1.1).

Corollary 4.2.1. *A locally Ricci symmetric ($\nabla S = 0$) non-flat real space form is an Einstein manifold.*

Proof: Take $h = S$ in (4.1.1), then $tr.S = r$. Therefore the equation (4.1.1) can be written as

$$S(U, W) = \frac{r}{n} g(U, W). \tag{4.2.1}$$

Remark 4.2.1. The following statements for non-flat real space form are equivalent.

1. Einstein.

2. locally Ricci symmetric.

3. Ricci semisymmetric.

4. Ricci pseudosymmetric i.e., $R \cdot S = L_S Q(g, S)$ and holds on the set $U_S = \{p \in M : S \neq c\frac{r}{n}g \ at \ p\}$, where L_S is some function on U_S.

Proof: The statements (1) → (2) → (3) and (3) → (4) are trivial. Now, we prove the statement (4) → (1) is true.

Here $R \cdot S = L_S Q(g, S)$ means

$$(R(U, V) \cdot S(X, Y)) = L_S Q(g, S)(X, Y; U, V),$$

this leads to

$$S(R(U, V)X, Y) + S(X, R(U, V)Y) = L_S[S((U \wedge V)X, Y) + S(X, (U \wedge V)Y)]. \quad (4.2.2)$$

Using equations (1.3.1) in (4.2.2) and contracting we get

$$-nS(U, Y) + rg(U, Y) = L_S[-nS(U, Y) + rg(U, Y)]. \quad (4.2.3)$$

The above equation can be written as,

$$[L_S - 1][-nS(U, Y) + rg(U, Y)] = 0. \quad (4.2.4)$$

If $L_S - 1 \neq 0$, then (4.2.4) reduced to

$$S(U, Y) = \frac{r}{n}g(U, Y). \quad (4.2.5)$$

Therefore, we conclude the following:

Lemma 4.2.2. *A Ricci pseudosymmetric in a non-flat real space form is an Einstein manifold if* $L_S \neq 1$.

Corollary 4.2.3. *Suppose that in a non-flat real space form, the (0,2) type field $L_V g+2S$ is parallel where V is a given vector field. Then (g, V) yields a Ricci soliton. In particular, if the given a non-flat real space form is Ricci semisymmetric with $L_V g$ parallel, we have same conclusion.*

Proof: The proof follows from Theorem (4.1.1), Corollary (4.2.1) and Remark (4.2.1). Since

$$(L_V g + 2S)(U, Y) = \frac{2}{n}(divV + r)g(U, Y).$$

Using equation (4.2.5) in (1.2.1), we get

$$(L_V g)(U, Y) + 2\frac{r}{n}g(U, Y) + 2\lambda g(U, Y) = 0. \tag{4.2.6}$$

Contracting the above equation, we get

$$divV + r + \lambda n = 0. \tag{4.2.7}$$

If V is solenoidal then $divV = 0$. Therefore the equation (4.2.7) can be reduced to

$$\lambda = \frac{-r}{n}. \tag{4.2.8}$$

Thus, we can state the following:

Corollary 4.2.4. *Let (g, V, λ) be a Ricci soliton in a non-flat real space form of dimension $(n > 2)$. Then V is solenoidal if and only if it is shrinking, steady and expanding depending upon the scalar curvature is $r > 0$, $r = 0$ and $r < 0$.*

4.3 Parallel second order covariant tensor and Ricci soliton in a non-flat complex space form

We write the following Corollary using Theorem (4.1.2).

Corollary 4.3.1. *A locally Ricci symmetric* ($\nabla S = 0$) *non-flat complex space form is an Einstein manifold.*

Proof: Take $h = S$ in (4.1.2). If $H = Q$ then $tr.Q = r$ and $tr.QJ = 0$ by virtue of (1.1.3). Hence (4.1.2) can be written as

$$S(U, W) = \frac{r}{n} g(U, W). \tag{4.3.1}$$

Remark 4.3.1. The following statements for non-flat complex space form are equivalent.

1. Einstein.

2. locally Ricci symmetric.

3. Ricci semisymmetric.

4. Ricci pseudosymmetric i.e., $R \cdot S = L_S Q(g, S)$ and holds on the set $U_S = \{p \in M : S \neq c\frac{r}{n}g \; at \; p\}$, where L_S is some function on U_S.

Proof: The statements (1) → (2) → (3) and (3) → (4) are trivial. Now, we prove the statement (4) → (1) is true.

Here $R \cdot S = L_S Q(g, S)$ means

$$(R(U, V) \cdot S(X, Y)) = L_S Q(g, S)(X, Y; U, V).$$

Which implies

$$S(R(U, V)X, Y) + S(X, R(U, V)Y) = L_S[S((U \wedge V)X, Y) + S(X, (U \wedge V)Y)]. \tag{4.3.2}$$

Applying equations (1.3.2) in (4.3.2) and putting orthonormal basis over V and X, we get after simplification that

$$nS(U, Y) - rg(U, Y) = L_S[nS(U, Y) - rg(U, Y)]. \tag{4.3.3}$$

The above equation implies,

$$[L_S - 1][nS(U,Y) - rg(U,Y)] = 0. \tag{4.3.4}$$

If $L_S - 1 \neq 0$, then (4.3.4) reduced to

$$S(U,Y) = \frac{r}{n} g(U,Y). \tag{4.3.5}$$

Therefore, we conclude the following:

Lemma 4.3.2. *A Ricci pseudosymmetric non-flat complex space form is an Einstein manifold if $L_S \neq 1$.*

Corollary 4.3.3. *Suppose that on a non-flat complex space form, the (0,2) type field $L_V g + 2S$ is parallel where V is a given vector field. Then (g,V) yields a Ricci soliton if JV is solenoidal. In particular, if the given non-flat complex space form is Ricci semisymmetric with $L_V g$ parallel, we have same conclusion.*

Proof: From Theorem (4.1.2), Remark (4.3.1) and Corollary (4.3.1), we have $\lambda = -\frac{r}{n}$ as seen below:

$$\begin{aligned}(L_V g + 2S)(U,Y) &= \frac{1}{n}[tr(L_V g + 2S)g(U,Y) + tr.((L_V g + 2S)J)\Omega(U,Y)] \\ &= \frac{1}{n}[2(divV + r)g(U,Y) + 2(divJV)\Omega(U,Y) \\ &+ 2(tr.SJ)\Omega(U,Y)],\end{aligned} \tag{4.3.6}$$

by virtue of (1.1.3) the above equation becomes

$$(L_V g + 2S)(U,Y) = \frac{2}{n}[(divV + r)g(U,Y) + (divJV)\Omega(U,Y)]. \tag{4.3.7}$$

By definition (g,V,λ) yields Ricci soliton. If $divJV = 0$ then $divV = 0$ because $JV = iV$ i.e.,

$$(L_V g + 2S)(U,Y) = \frac{2r}{n} g(U,Y) = -2\lambda g(U,Y). \tag{4.3.8}$$

Therefore $\lambda = -\frac{r}{n}$.

Corollary 4.3.4. *Let (g, V, λ) be a Ricci soliton in a non-flat complex space form of dimension $(n > 2)$. Then V is solenoidal if and only if it is shrinking, steady and expanding depending upon the scalar curvature is positive, zero and negative respectively.*

Proof: Using equation (4.3.5) in (1.2.1) we get

$$(L_V g)(U, Y) + 2\frac{r}{n} g(U, Y) + 2\lambda g(U, Y) = 0. \tag{4.3.9}$$

Taking orthonormal basis over U and Y of the above equation, we get

$$divV + r + \lambda n = 0. \tag{4.3.10}$$

If V is solenoidal then $divV = 0$. Therefore the equation (4.3.10) can be reduces to

$$\lambda = \frac{-r}{n}.$$

4.4 Parallel second order covariant tensor and Ricci soliton in a non-flat generalized complex space form

Let h be a $(0, 2)$-tensor which is parallel with respect to ∇ that is $\nabla h = 0$. Applying the Ricci identity [49]

$$\nabla^2 h(U, Y; Z, W) - \nabla^2 h(U, Y; W, Z) = 0, \tag{4.4.1}$$

we obtain the relation

$$h(R(U, Y)Z, W) + h(Z, R(U, Y)W) = 0. \tag{4.4.2}$$

Applying equation (1.3.3) in (4.4.2) and letting $U = W = e_i$, $1 \leq i \leq n$ after simplification, we get

$$\begin{aligned} f_1\{g(Y,Z)(tr.H) - h(Y,Z)\} &+ f_2\{h(JY,JZ) - g(Y,JZ)(tr.HJ) + 2h(JZ,JY)\} \\ &- \{(n-1)f_1 - 3f_2\}h(Z,Y) = 0, \end{aligned} \tag{4.4.3}$$

where H is a $(1,1)$ tensor metrically equivalent to h. Symmetrization and anti-symmetrization of (4.4.3) yield

$$\frac{[nf_1 - 3f_2]}{f_1}h(Z,Y) - \frac{3f_2}{f_1}h(JY,JZ) = (tr.H)g(Y,Z). \tag{4.4.4}$$

$$\frac{[(n-2)f_1 - 3f_2]}{f_2}h(Y,Z) + h(JZ,JY) = g(Y,JZ(tr.HJ)). \tag{4.4.5}$$

Replacing Y, Z by JY, JZ respectively in (4.4.4) and adding the resultant equation from (4.4.4), we obtain

$$h_s(Y,Z) = \pi(tr.H)g(Y,Z), \tag{4.4.6}$$

where $\pi = \frac{f_1}{[nf_1 - 6f_2]}$.

Replacing Y, Z by JY, JZ respectively in (4.4.5) and adding the resultant equation from (4.4.5), we obtain

$$h_a(Y,Z) = \frac{f_2}{[(n-2)f_1 - 4f_2]}(tr.HJ)g(Y,JZ). \tag{4.4.7}$$

By summing up (4.4.6) and (4.4.7) we obtain the expression

$$h = \pi(tr.H)g + \rho_1(tr.HJ)\Omega, \tag{4.4.8}$$

where $\rho_1 = \frac{f_2}{[(n-2)f_1 - 4f_2]}$. Hence we can state the following:

Theorem 4.4.1. *A second order parallel tensor in a non-flat generalized complex space form is a linear combination (with constant coefficients) of the underlying Kählerian metric and Kählerian 2-form.*

Corollary 4.4.2. *The only symmetric (anti-symmetric) parallel tensor of type* $(0, 2)$ *in a non-flat generalized complex space form is the Kählerian metric (Kählerian 2-form) up to a constant multiple.*

Corollary 4.4.3. *A locally Ricci symmetric* $(\nabla S = 0)$ *non-flat generalized complex space form is an Einstein manifold.*

Proof: If $H = S$ in (4.4.8) then $tr.H = r$ and $tr.HJ = 0$ by virtue of (1.1.3). Equation (4.4.8) can be written as

$$S(Y, Z) = \pi r g(Y, Z). \tag{4.4.9}$$

Remark 4.4.1. The following statements for non-flat generalized complex space form are equivalent.

1. Einstein.

2. locally Ricci symmetric.

3. Ricci semisymmetric that is $R \cdot S = 0$ if $f_1 \neq 0$.

Proof: The statements (1) $\to$ (2) $\to$ (3) are trivial. Now, we prove the statement (3) $\to$ (1) is true.

Here $R \cdot S = 0$ means

$$(R(U, V) \cdot S)(X, Y) = 0.$$

Which implies

$$S(R(U, V)X, Y) + S(X, R(U, V)Y) = 0. \tag{4.4.10}$$

Using equations (1.3.3) in (4.4.10) and setting $V = X = e_i$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and summing over i $(1 \leq i \leq n)$

we yield after simplification that

$$f_1\{nS(U,Y) - rg(U,Y)\} = 0. \tag{4.4.11}$$

If $f_1 \neq 0$, then (4.4.11) reduces to

$$S(U,Y) = \frac{r}{n} g(U,Y). \tag{4.4.12}$$

Therefore, we conclude the following:

Lemma 4.4.4. *A Ricci semisymmetric non-flat generalized complex space form is an Einstein manifold if $f_1 \neq 0$.*

Corollary 4.4.5. *Suppose that on a non-flat generalized complex space form, the (0,2) type field $L_V g + 2S$ is parallel where V is a given vector field. Then (g, V) yield a Ricci soliton if JV is solenoidal. In particular, if the given non-flat generalized complex space form is Ricci semisymmetric with $L_V g$ parallel, we have same conclusion.*

Proof: From Theorem (4.4.1) and corollary (4.4.3), we have $\lambda = -\pi r$ as seen below:

$$\begin{aligned}(L_V g + 2S)(U,Y) &= [\pi.tr(L_V g + 2S)g(U,Y) + \rho_1.tr((L_V g + 2S)J)\Omega(U,Y)] \\ &= [\pi 2(divV + r)g(U,Y) + \rho_1[2(divJV)\Omega(U,Y) \\ &+ 2(tr.SJ)\Omega(U,Y)]],\end{aligned} \tag{4.4.13}$$

by virtue of (1.1.3) the above equation becomes

$$(L_V g + 2S)(U,Y) = [2\pi(divV + r)g(U,Y) + 2\rho_1(divJV)\Omega(U,Y)]. \tag{4.4.14}$$

By definition (g, V, λ) yields Ricci soliton if $divJV = 0$ then $divV = 0$ because $JV = iV$ i.e.,

$$(L_V g + 2S)(U,Y) = 2\pi r g(U,Y) = -2\lambda g(U,Y). \tag{4.4.15}$$

Therefore $\lambda = -\pi r$

Corollary 4.4.6. *Let (g, V, λ) be a Ricci soliton in a non-flat generalized complex space form. Then V is solenoidal if and only if it is shrinking, steady and expanding depending upon the sign of scalar curvature.*

Proof: Using equation (4.4.12) in (1.2.1) we get

$$(L_V g)(U, Y) + 2\frac{r}{n}g(U, Y) + 2\lambda g(U, Y) = 0. \tag{4.4.16}$$

Setting $U = Y = e_i$, i $(1 \leq i \leq n)$ in the above equation we obtain

$$divV + r + \lambda n = 0. \tag{4.4.17}$$

If V is solenoidal then $divV = 0$. Therefore the equation (4.4.17) can be reduced to

$$\lambda = \frac{-r}{n}.$$

4.5 Conclusion

The important results finding of this chapter are as follows:

- A second order parallel tensor in a non-flat generalized complex space form is a linear combination (with constant coefficients) of the underlying Kählerian metric and Kählerian 2-form.

- The only symmetric (anti-symmetric) parallel tensor of type $(0, 2)$ in a non-flat generalized complex space form is the Kählerian metric (Kählerian 2-form) up to a constant multiple.

- The following statements for non-flat real or complex space form are equivalent.

    1. Einstein.

    2. Locally Ricci symmetric.

    3. Ricci semisymmetric.

    4. Ricci pseudosymmetric i.e., $R \cdot S = L_S Q(g, S)$ and holds on the set $U_S = \{p \in \tilde{M} : S \neq c\frac{r}{n}g \ at \ p\}$, where L_S is some function on U_S.

- From Corollaries (4.2.3), (4.3.3) and (4.4.5) we conclude the following:

  According to Corollary (4.2.3) we have

$$(L_V g + 2S)(U, Y) = \frac{2}{n}(divV + r)g(U, Y)$$

  and in the above equation the solenoidal condition does not affect for the existence of Ricci soliton in non-flat real space form, but according to Corollaries (4.3.3) and (4.4.5) we have

$$(L_V g + 2S)(U, Y) = \frac{2}{n}[(divV + r)g(U, Y) + (divJV)\Omega(U, Y)],$$
$$(L_V g + 2S)(U, Y) = [2\pi(divV + r)g(U, Y) + 2\rho_1(divJV)\Omega(U, Y)],$$

  and in these equations solenoidal condition affects for the existence of Ricci solitons in non-flat complex and generalized complex space form, unless $divJV = 0$.

- Let (g, V, λ) be a Ricci soliton in a non-flat real or complex or generalized space forms of dimension $n(> 2)$. Then V is solenoidal if and only if it is shrinking, steady and expanding depending upon the scalar curvature $r > 0, r = 0$ and $r < 0$.

Chapter 5

Submanifolds in Real and Complex Space Forms

5.1 Introduction

Riemannian invariants play the most fundamental role in Riemannian geometry. They provide the intrinsic characteristics of Riemannian manifolds moreover, they affect the behavior of Riemannian manifolds in general. Classically, among Riemannian curvature invariants people have studied Sectional, Ricci and Scalar curvatures intensively since Riemann.

The study of submanifolds was initiated by Darboux and Nash. In 1971, Chern et. al., [23] studied submanifold with parallelism of the second fundamental form. Minimal surfaces of an Euclidean m-space E^m and minimal surfaces of hyperspheres of E^m are surfaces with parallel mean curvature vector of E^m, i.e. $\nabla H = 0$. In [20] Chen and [65] Yau independently studied the surface with parallel mean curvature vector in real space form. In 2004, Turkay [57] classified the surfaces immersed in E^5 satisfying the condition $R^{\perp}(U,V) \cdot H = 0$. Then in 2011, Kadri et. al. [4], found interesting results on H-recurrent surfaces in Euclidean space E^m. The authors Yano and Kon [61], Chen [20], Alegre and Carriazo [3], De and Shaikh [24], Hasan Shahid [30], Özgur [50], Bagewadi [5] et. al. studied submanifolds in different structures of manifold. Sharma [49] investigated Eisenhart problem on non-flat real and complex space forms, in 1989. Using the result of Sharma we study the geometric properties of submanifold M of non-flat real and complex space form. In this chapter we study submanifold of real and complex space forms whose second fundamental forms are parallel, semi-parallel, recurrent and using the

Theorems (4.1.1): (4.1.2):

Theorem 5.1.1. *[20, 65] Let M be a smooth surface in m-dimensional real space form $R^m(k)$ of constant sectional curvature k. If H is parallel in the normal bundle, then M is one of the following surfaces:*

1. *M is minimal surface in $R^m(k)$.*
2. *M is minimal surface in a small hypersphere of $R^m(k)$.*
3. *M is a surface with constant mean curvature $\|H\|$ in S^3 of $R^m(k)$.*

Theorem 5.1.2. *[4] Let M be a smooth submanifold in E^n. If M satisfies the H-recurrent condition $\nabla_U H = B(U)H$, then M is $R^\perp$-parallel. Where B is a 1-form.*

Theorem 5.1.3. *[57] If M is a surface satisfying $R^\perp(U,V)\cdot H = 0$, then M is either minimal or totally umbilical or normally flat i.e., $R^\perp = 0$.*

Theorem 5.1.4. *[32] Every semi-parallel surface is H-parallel.*

5.2 Basic concepts

If M is an immersion of space form $\widetilde{M}(k)$, then we know that ∇ and $\widetilde{\nabla}$ are Levi-Civita connections on M and $\widetilde{M}(k)$ respectively and σ is a second fundamental form on M. The second fundamental form σ of the imbedding is said to be

- parallel if $(\widetilde{\nabla}_U\sigma)(V,W) = 0$.
- recurrent if $(\widetilde{\nabla}_U\sigma)(V,W) = B(U)\sigma(V,W)$.
- semi-parallel if $\tilde{R}\cdot\sigma = 0$.
- pseudoparallel if $\tilde{R}\cdot.\sigma = L_1 Q(g,\sigma)$
- Ricci-generalized pseudoparallel if $\tilde{R}\cdot\sigma = L_1 Q(S,\sigma)$,

where L_1 and L_2 are function depending on σ.

We know that $\sigma(U,JV) = J\sigma(U,V)$ [21], where σ is second order covariant tensor on Kähler manifold. Then

$$\frac{1}{n}(tr.(\sigma J)) = J\frac{1}{n}(tr.\sigma) = HJ. \tag{5.2.1}$$

5.3 Parallel and semi-parallel submanifolds in a non-flat real space form

Let σ be parallel, i.e., $(\widetilde{\nabla}_U\sigma)(V,W) = 0$.

Using (1.4.4) in the above equation implies

$$\nabla^{\perp}_U\sigma(V,W) - \sigma(\nabla_U V,W) - \sigma(V,\nabla_U W) = 0. \tag{5.3.1}$$

Since σ is symmetric, covariant tensor of order 2 we have by virtue (4.1.1)

$$\nabla^{\perp}_{U}\{\frac{tr.\sigma}{n}g(V,W)\} - \frac{tr.\sigma}{n}g(\nabla_U V,W) - \frac{tr.\sigma}{n}g(V,\nabla_U W) = 0. \qquad (5.3.2)$$

Using equation (1.4.7) in (5.3.2) we get

$$\nabla^{\perp}_{U}\{Hg(V,W)\} - H(\nabla_U V,W) - H(V,\nabla_U W) = 0. \qquad (5.3.3)$$

This implies

$$(\nabla^{\perp}_{U}H)g(V,W) + H\nabla^{\perp}_{U}g(V,W) = HXg(V,W). \qquad (5.3.4)$$

Putting an orthonormal basis over V and W in the above equation implies

$$(\nabla^{\perp}_{U}H)n = 0. \qquad (5.3.5)$$

The above equation becomes

$$\nabla^{\perp}_{U}H = 0. \qquad (5.3.6)$$

This implies H is parallel in normal bundle. Then we have the following result from Theorem (5.1.1):

Theorem 5.3.1. *Let M be a submanifold of a non flat real space form. If σ is parallel, then M is one of the following surfaces:*

1. *M is minimal surface in $R^m(k)$.*

2. *M is minimal surface in a small hypersphere of $R^m(k)$.*

3. *M is a surface with constant mean curvature $\|H\|$ in S^3 of $R^m(k)$.*

Let $\tilde{R}$ and σ satisfy the equation $\tilde{R}\cdot\sigma = 0$, i.e., M be semi-parallel.

The above equation implies

$$R^{\perp}(U,V)\sigma(X,Y) - \sigma(R(U,V)X,Y) - \sigma(X,R(U,V)Y) = 0. \qquad (5.3.7)$$

Using equation (1.3.1) and Gauss equation (1.4.3) in (5.3.7) and setting $X = Y = e_i$ we obtain,

$$R^{\perp}(U, V)H = 0. \tag{5.3.8}$$

This implies H is either constant or zero or $R^{\perp} = 0$, hence we state the following from Theorem (5.1.3):

Theorem 5.3.2. *Let M be a submanifold of a non-flat real space form. If σ is semi-parallel then M is either totally umbilical or minimal or normal flat.*

Then M must be either part of an extrinsic sphere or a plane which is totally umbilical [12, 21]. Hence we state the following:

Corollary 5.3.3. *Let M be a connected and compact submanifold of a non-flat real space form then σ is semi-parallel if and only if M is either an extrinsic sphere or a plane.*

Every semi-parallel surface is H-parallel in [32]. Thus We state the following:

Theorem 5.3.4. *Let M be a submanifold of a non-flat real space form. If σ is semi-parallel then M is H-parallel.*

The calculation for pseudoparallel or Ricci-generalized pseudoparallel in a non-flat real space form will lead to the (5.3.8), so we state the following:

Theorem 5.3.5. *Let M be a submanifold of a non-flat real space form. If σ is pseudoparallel or Ricci-generalized pseudoparallel then M is either totally umbilical or minimal or normal flat.*

Corollary 5.3.6. *Let M be a connected and compact submanifold of a non-flat real space form then σ is pseudoparallel or Ricci-generalized pseudoparallel if and only if M is either an extrinsic sphere or a plane.*

Theorem 5.3.7. *Let M be a submanifold of a non-flat real space form. If σ is pseudoparallel or Ricci-generalized pseudoparallel then M is H-parallel.*

5.4 Recurrent submanifolds in a non-flat real space form

Consider σ is recurrent, from (1.4.6) we get

$$(\widetilde{\nabla}_U \sigma)(V, W) = B(U)\sigma(V, W). \tag{5.4.1}$$

Using (1.4.4) in the above equation implies

$$\nabla^{\perp}_U \sigma(V, W) - \sigma(\nabla_U V, W) - \sigma(V, \nabla_U W)) = B(U)\sigma(V, W). \tag{5.4.2}$$

Setting $V = W = e_i$ in (5.4.2), then we obtain

$$(\nabla^{\perp}_U H) = -B(U)H. \tag{5.4.3}$$

We state the following:

Theorem 5.4.1. *Let M be a submanifold of a non-flat real space form. If σ is recurrent then the mean curvature vector is recurrent in the normal space.*

If M satisfies the H-recurrent condition $\nabla_U H = B(U)H$, then M is $R^{\perp}$-parallel. Hence we state the following:

Corollary 5.4.2. *Let M be a submanifold of a non-flat real space form. If σ is recurrent, then M is $R^{\perp}$-parallel.*

5.5 Parallel and semi-parallel submanifolds in a non-flat complex space form

Let σ be parallel, i.e., $(\widetilde{\nabla}_U \sigma)(V, W) = 0$.

Using (1.4.4) above equation implies

$$\nabla^{\perp}_U \sigma(V, W) - \sigma(\nabla_U V, W) - \sigma(V, \nabla_U W) = 0. \tag{5.5.1}$$

Since σ is covariant tensor of order 2 we have by virtue (4.1.2)

$$\nabla^{\perp}_U\{\frac{1}{n}[(tr.\sigma)g(V, W) + (tr.(\sigma J))g(V, JW)]\} - \frac{1}{n}[(tr.\sigma)g(\nabla_U V, W) + (tr.(\sigma J))g(\nabla_U V, JW)]$$
$$-\frac{1}{n}[(tr.\sigma)g(V, \nabla_U W) + (tr.(\sigma J))g(V, \nabla_U JW)] = 0. \tag{5.5.2}$$

Using equation (1.4.7) and (5.2.1) in (5.5.2) we get

$$\nabla^{\perp}_U\{Hg(V, W) + HJg(V, JW)]\} - [Hg(\nabla_U V, W) + HJg(\nabla_U V, JW)]$$
$$-[Hg(V, \nabla_U W) + HJg(V, \nabla_U JW)] = 0. \tag{5.5.3}$$

The above equation implies

$$\begin{aligned}[(\nabla^{\perp}_U H)g(V, W) + H\nabla^{\perp}_U g(V, W)] &+ [(\nabla^{\perp}_U HJ)g(V, JW) + HJ(\nabla^{\perp}_U)g(V, JW)] \\ &= HXg(V, W) + HJXg(V, JW).\end{aligned} \tag{5.5.4}$$

Taking $V = W = e_i$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i $(1 \leq i \leq n)$, after simplification we yield

$$(\nabla^{\perp}_U H)n = 0. \tag{5.5.5}$$

The above equation implies

$$\nabla^{\perp}_U H = 0. \tag{5.5.6}$$

This implies H is parallel in normal bundle. Then we have the following result:

Theorem 5.5.1. *Let M be a submanifold of a non-flat complex space form. If σ is parallel then M is minimal.*

Let $\tilde{R}$ and σ satisfy the equation $\tilde{R} \cdot \sigma = 0$, i.e., M be semi-parallel.

The above equation implies

$$R^{\perp}(U,V)\sigma(X,Y) - \sigma(R(U,V)X,Y) - \sigma(X,R(U,V)Y) = 0. \tag{5.5.7}$$

Using equation (1.3.2) and Gauss equation (1.4.3) in (5.5.7) and setting $X = Y = e_i$ we obtain,

$$R^{\perp}(U,V)HJ = k\Omega(U,V)HJ. \tag{5.5.8}$$

Thus, we have the following result:

Theorem 5.5.2. *Let M be a submanifold of a non-flat complex space form. If σ is semi-parallel then the mean curvature vector in Kähler space is the eigen vector of the normal transformation and the corresponding eigen value is the product of holomorphic sectional curvature and Kähler metric.*

5.6 Conclusion

The important results finding of this chapter are as follows:

- Let M be a submanifold of a non flat real space form. If σ is parallel, then M is one of the following surfaces:

    1. M is minimal surface in $R^m(k)$.

2. M is minimal surface in a small hypersphere of $R^m(k)$.

3. M is a surface with constant mean curvature $\|H\|$ in S^3 of $R^m(k)$.

- Let M be a submanifold of a non-flat real space form. If σ is semi-parallel or pseudoparallel or Ricci generalized pseudoparallel then M is either totally umbilical or minimal or normal flat.

- Let M be a connected and compact submanifold of a non flat real space form then σ is semi-parallel or pseudoparallel or Ricci generalized pseudoparallel if and only if M is either an extrinsic sphere or a plane.

- Let M be a submanifold of a non-flat real space form. If σ is recurrent, then M is $R^{\perp}$-parallel.

- Let M be a submanifold of a non-flat complex space form. If σ is parallel then M is minimal.

- Let M be a submanifold of a non-flat complex space form. If σ is semi-parallel then the mean curvature vector in Kähler space is the eigen vector of the normal transformation and the corresponding eigen value is the product of holomorphic sectional curvature and Kähler metric.

Chapter 6

Ricci Solitons in Quaternion Space Forms

6.1 Introduction

In this chapter we extend the work of Sharma [49] and Bagewadi [7, 8] to quaternion space form.

6.2 Basic concepts

Let U be a unit vector tangent to the quaternion Kahlerian manifold $\bar{M}$, then U, JU, KU and LU form an orthonormal frame. We denote by $Q(U)$ the 4-plane spanned by them, and call it the quaternion 4-plane determined by U. Every plane in a quaternion 4-plane is called a quaternion plane. The sectional curvature for a quaternion plane is called a quaternion sectional curvature.

A quaternion Kählerian manifold is called a quaternion space form $\bar{M}(k)$ if its quaternion sectional curvatures are equal to a constant k. It is known that a quaternion Kählerian manifold is a quaternion space form if and only if its curvature tensor R is of the following form [37, 22]:

$$\begin{aligned}
R(U,Y)Z &= \frac{k}{4}[g(Y,Z)U - g(U,Z)Y + g(JY,Z)JU - g(JU,Z)JY + 2g(U,JY)JZ \\
&+ g(KY,Z)KU - g(KU,Z)KY + 2g(U,KY)KZ + g(LY,Z)LU \\
&- g(LU,Z)LY + 2g(U,LY)LZ]. \qquad (6.2.1)
\end{aligned}$$

6.3 Parallel second order covariant tensor and Ricci soliton in a non-flat quaternion space form

If h is a parallel $(0,2)$ covariant tensor in a non-flat quaternion space form, then using (6.2.1) in (4.4.2) and contracting over U and W, we get

$$\begin{aligned}
&g(Y,Z)(tr.H) - h(Y,Z) - (n+8)h(Z,Y) + g(JY,Z)tr.(HJ) + h(JY,JZ) \\
&+2h(JZ,JY) + g(KY,Z)tr.(HK) + h(KY,KZ) + 2h(KZ,KY) \\
&+g(LY,Z)tr.(HL) + h(LY,LZ) + 2h(LZ,LY) = 0, \qquad (6.3.1)
\end{aligned}$$

where H is a $(1,1)$ tensor metrically equivalent to h. Symmetrization and anti-symmetrization of (6.3.1) yield:

$$\begin{aligned}
g(Y,Z)(tr.H) &= (n+9)h(Y,Z) - 3h(JY,JZ) - 3h(KY,KZ) \\
&- 3h(LY,LZ), \qquad (6.3.2) \\
g(JY,Z)tr.(HJ) &+ g(KY,Z)tr.(HK) + g(LY,Z)tr.(HL) = (n+7)h(Z,Y) \\
&- h(JZ,JY) - h(KZ,KY) - h(LZ,LY). \qquad (6.3.3)
\end{aligned}$$

Replacing Y and Z by JY and JZ, KY and KZ, LY and LZ respectively in (6.3.2) and adding the resultant equations from (6.3.2), then we get

$$g(Y,Z)(tr.H) = (n+6)h_s(Y,Z) - 3h_s(KY,KZ) - 3h_s(LY,LZ), \tag{6.3.4}$$

$$g(Y,Z)(tr.H) = (n+6)h_s(Y,Z) - 3h_s(JY,JZ) - 3h_s(LY,LZ), \tag{6.3.5}$$

$$g(Y,Z)(tr.H) = (n+6)h_s(Y,Z) - 3h_s(JY,JZ) - 3h_s(KY,KZ). \tag{6.3.6}$$

Again changing Y, Z by KY, KZ respectively in (6.3.4) and adding the resultant equation from (6.3.4), we obtain

$$(n+9)g(Y,Z)(tr.H) = ((n+6)^2-9)h_s(Y,Z) - 3(n+6)h_s(LY,LZ) - 9h_s(JY,JZ). \tag{6.3.7}$$

Multiply -3 to equation (6.3.5) and adding the resultant equation from (6.3.7) we obtain the expression

$$g(Y,Z)(tr.H) = \frac{(n^2+9n+9)}{(n+6)}h_s(Y,Z) - \frac{(3n+9))}{(n+6)}h_s(LY,LZ). \tag{6.3.8}$$

Substituting Y, Z by LY, LZ respectively in (6.3.8) and adding the resultant equation from (6.3.8), the relation

$$h_s(Y,Z) = \frac{(tr.H)}{n}g(Y,Z). \tag{6.3.9}$$

Likewise: changing Y and Z by JY and JZ, KY and KZ, LY and LZ respectively in (6.3.3) and adding the resultant equations from (6.3.3), then we obtain

$$g(Y,JZ)tr.(HJ) + g(Y,KZ)tr.(HK) + g(Y,LZ)tr.(HL) = (n+6)h_a(Y,Z)$$
$$-h_a(KY,KZ) - h_a(LY,LZ), \tag{6.3.10}$$

$$g(Y,JZ)tr.(HJ) + g(Y,KZ)tr.(HK) + g(Y,LZ)tr.(HL) = (n+6)h_a(Y,Z)$$
$$-h_a(JY,JZ) - h_a(LY,LZ), \tag{6.3.11}$$

$$\begin{aligned} g(Y,JZ)tr.(HJ) &+ g(Y,KZ)tr.(HK)+g(Y,LZ)tr.(HL)=(n+6)h_a(Y,Z) \\ &- h_a(KY,KZ)-h_a(JY,JZ). \end{aligned} \tag{6.3.12}$$

Again replacing Y, Z by KY, KZ respectively in (6.3.10) and adding the resultant equation from (6.3.10), we get

$$\begin{aligned} &(n+7)[g(Y,JZ)tr.(HJ)+g(Y,KZ)tr.(HK)+g(Y,LZ)tr.(HL)] \\ &=[(n+6)^2-1]h_a(Y,Z)-(n+6)h_a(LY,LZ)-h_a(JY,JZ). \end{aligned} \tag{6.3.13}$$

Multiply -1 to equation (6.3.11) and adding the resultant equation from (6.3.13) we obtain the expression

$$\begin{aligned} g(Y,JZ)tr.(HJ) &+ g(Y,KZ)tr.(HK)+g(Y,LZ)tr.(HL)=\frac{n^2+11n-29}{(n+6)}h_a(Y,Z) \\ &- \frac{(n+5)}{(n+6)}h_a(LY,LZ). \end{aligned} \tag{6.3.14}$$

Substituting Y, Z by LY, LZ respectively in (6.3.14) and adding the resultant equation from (6.3.14), then we get

$$\begin{aligned} h_a(Y,Z) &= \frac{(n+6)}{(n^2+10n-34)}[g(Y,JZ)tr.(HJ)+g(Y,KZ)tr.(HK) \\ &+ g(Y,LZ)tr.(HL)]. \end{aligned} \tag{6.3.15}$$

By summing up (6.3.9) and (6.3.15) we obtain the expression

$$h=[\frac{1}{n}(tr.H)g+\varrho[g(Y,JZ)tr.(HJ)+g(Y,KZ)tr.(HK)+g(Y,LZ)tr.(HL)], \tag{6.3.16}$$

where $\varrho=\frac{(n+6)}{(n^2+10n-34)}$. The above equation implies

$$h=\left[\frac{1}{n}(tr.H)g+\varrho[tr.(HJ)\Omega_1+tr.(HK)\Omega_2+tr.(HL)\Omega_3]\right]. \tag{6.3.17}$$

Thus, we can state the following:

Theorem 6.3.1. *A second order parallel tensor in a non-flat quaternion space form is a linear combination (with constant coefficients) of the underlying quaternion Kählerian metric and quaternion Kählerian 2-forms.*

Corollary 6.3.2. *The only symmetric (anti-symmetric) parallel tensor of type* $(0,2)$ *in a non-flat quaternion space form is the quaternion Kählerian metric (quaternion Kählerian 2-forms) up to a constant multiple.*

Corollary 6.3.3. *A locally Ricci symmetric* $(\nabla S = 0)$ *non-flat quaternion space form is an Einstein manifold.*

Proof: If $h = S$ in (6.3.17) then $tr.H = r$, $tr.HJ = 0$, $tr.HK = 0$ and $tr.HL = 0$ by virtue of (1.5.6). Equation (6.3.17) can be written as

$$S(Y,Z) = \frac{r}{n} g(Y,Z). \tag{6.3.18}$$

Remark 6.3.1. The following statements for non-flat quaternion space form are equivalent.

1. Einstein.

2. locally Ricci symmetric.

3. Ricci semisymmetric that is $R \cdot S = 0$.

Proof: The statements (1) $\rightarrow$ (2) $\rightarrow$ (3) are trivial. Now, we prove the statement (3) $\rightarrow$ (1) is true.

Here $R \cdot S = 0$ means

$$(R(U,V) \cdot S(Y,Z)) = 0. \tag{6.3.19}$$

Which implies

$$S(R(U,V)Y,Z) + S(Y,R(U,V)Z) = 0. \tag{6.3.20}$$

Using equations (6.2.1) in (6.3.20) and taking orthonormal basis over V and Y, we get after simplification that

$$\frac{k}{4}\{nS(U,Z) - rg(U,Z)\} = 0. \tag{6.3.21}$$

The above equation implies

$$S(U,Z) = \frac{r}{n}g(U,Z). \tag{6.3.22}$$

Therefore, we conclude the following:

Lemma 6.3.4. *A Ricci semisymmetric non-flat quaternion space form is an Einstein manifold.*

Corollary 6.3.5. *Suppose that on a non-flat quaternion space form, the (0,2) type field $L_V g + 2S$ is parallel where V is a given vector field. Then (g, V) yields a Ricci soliton if JV, KV and LV are solenoidal. In particular, if the given non-flat quaternion space form is Ricci semisymmetric with $L_V g$ parallel, we have same conclusion.*

Proof: From Theorem (6.3.1) and Corollary (6.3.3), we have $\lambda = -\frac{r}{n}$ as seen below:

$$\begin{aligned}(L_V g + 2S)(Y,Z) &= [\frac{1}{n}tr(L_V g + 2S)g(Y,Z) + \varrho[tr.((L_V g + 2S)J)\Omega_1(Y,Z)\\ &+ tr.((L_V g + 2S)K)\Omega_2(Y,Z) + tr.((L_V g + 2S)L)\Omega_3(Y,Z)]],\end{aligned}$$

$$\begin{aligned}(L_V g + 2S)(Y,Z) &= [\frac{1}{n}2(divV + r)g(Y,Z) + \varrho[2(divJV)\Omega_1(Y,Z) + 2(tr.SJ)\Omega_1(Y,Z)\\ &+ 2(divKV)\Omega_2(Y,Z) + 2(tr.SK)\Omega_2(Y,Z)\\ &+ 2(divLV)\Omega_3(Y,Z) + 2(tr.SL)\Omega_3(Y,Z)]],\end{aligned} \tag{6.3.23}$$

by virtue of (1.5.6) the above equation becomes

$$\begin{aligned}(L_V g + 2S)(Y,Z) &= [\frac{2}{n}[(divV + r)g(Y,Z) + 2\varrho[(divJV)\Omega_1(Y,Z)\\ &+ (divKV)\Omega_2(Y,Z) + (divLV)\Omega_3(Y,Z)]].\end{aligned} \tag{6.3.24}$$

By definition (g, V, λ) yields Ricci soliton. If $divJV = 0$, $divKV = 0$ and $divLV = 0$ then $divV = 0$ because $JV = KV = LV = iV$ i.e.,

$$(L_V g + 2S)(Y, Z) = \frac{2r}{n} g(Y, Z) = -2\lambda g(Y, Z). \tag{6.3.25}$$

Therefore $\lambda = -\frac{r}{n}$.

Lemma 6.3.6. *Let (g, V, λ) be a Ricci soliton in a non-flat quaternion space form. Then V is solenoidal if and only if it is shrinking, steady and expanding depending upon the scalar curvature is positive, zero and negative respectively.*

Proof: Using equation (6.3.22) in (1.2.1) we obtain

$$(L_V g)(Y, Z) + 2\frac{r}{n} g(Y, Z) + 2\lambda g(Y, Z) = 0. \tag{6.3.26}$$

Putting orthonormal basis over Y and Z of the above equation, we get

$$divV + r + \lambda n = 0. \tag{6.3.27}$$

If V is solenoidal then $divV = 0$. Therefore the equation (6.3.27) can be reduced to

$$\lambda = \frac{-r}{n}.$$

6.4 Semisymmetric quaternion space form

Let us consider the semisymmetric conditions in quaternion space form i.e.,

$$(R(U, V) \cdot R)(X, Y, Z) = 0. \tag{6.4.1}$$

This implies

$$\begin{aligned} R(U, V)R(X, Y)Z &- R(R(U, V)X, Y)Z - R(X, R(U, V)Y)Z \\ &- R(X, Y)R(U, V)Z = 0. \end{aligned} \tag{6.4.2}$$

Taking inner product with T we have,

$$\begin{aligned} g(R(U,V)R(X,Y)Z,T) &- g(R(R(U,V)X,Y)Z,T) - g(R(X,R(U,V)Y)Z,T) \\ &- g(R(X,Y)R(U,V)Z,T) = 0. \end{aligned} \tag{6.4.3}$$

Using equation (6.2.1) in (6.4.3) and putting $U = Y = e_i$, further again putting $V = T = e_i$ in the resultant equation, we get

$$S(X,Z) = k\left(-12n^2 - 51n - \frac{131}{2}\right) g(X,Z). \tag{6.4.4}$$

This implies,

$$S(X,Z) = \beta_3 g(X,Z). \tag{6.4.5}$$

where $\beta_3 = k\left(-12n^2 - 51n - \frac{131}{2}\right)$. That is quaternion space form is an Einstein manifold. Hence, we can state the following result:

Theorem 6.4.1. *A quaternion space form satisfying $R \cdot R = 0$ is an Einstein manifold.*

Using equation (6.4.5) in (1.2.1), we get

$$(L_V g)(X,Z) + 2\beta_3 g(X,Z) + 2\lambda g(X,Z) = 0. \tag{6.4.6}$$

Putting $X = Z = e_i$ in the above equation and taking summation over i $(1 \leq i \leq n)$, we get

$$divV + \beta_3 + \lambda = 0. \tag{6.4.7}$$

If V is solenoidal then $divV = 0$. Therefore the equation (6.4.7) can be reduced to

$$\lambda = -\beta_3. \tag{6.4.8}$$

Hence, we obtain the following result:

Corollary 6.4.2. *Let (g, V, λ) be a Ricci soliton in a quaternion space form satisfying semisymmetric condition, if V is solenoidal then it is shrinking.*

6.5 Quaternion space form satisfying $R \cdot B = 0$

Let $\bar{M}(k)$ be a quaternion space form satisfy $(R(U,V)\cdot B)(X,Y,Z) = 0$, then U, V, X, Y, Z are any tangent vectors. This equation turns into

$$R(U,V)B(X,Y)Z - B(R(U,V)X,Y)Z - B(X,R(U,V)Y)Z - B(X,Y)R(U,V)Z = 0.$$

Taking inner product with T we have

$$g(R(U,V)B(X,Y)Z,T) - g(B(R(U,V)X,Y)Z,T) - g(B(X,R(U,V)Y)Z,T)$$
$$-g(B(X,Y)R(U,V)Z,T) = 0. \tag{6.5.1}$$

Using equations (6.2.1) and (1.1.15) in (6.5.1) and setting $U = Y = e_i$, further again setting $V = T = e_i$ in the resultant equation we obtain

$$S(X,Z) = \frac{\alpha_4}{\beta_4} g(X,Z), \tag{6.5.2}$$

where $\alpha_4 = [k(3n-36)x_0 + (2n-9)4x_1 r + (2n^2+6n-17)8x_2 r$ and

$\beta_4 = 4((3-2n)x_0 - n(2n+7)x_1)$. That is $\bar{M}(k)$ is an Einstein manifold.

Hence we have the following result:

Theorem 6.5.1. *A quaternion space form satisfying $R \cdot B = 0$ is an Einstein manifold.*

Using equation (6.5.2) in (1.2.1), we get

$$(L_V g)(X,Z) + 2\frac{\alpha_4}{\beta_4} g(X,Z) + 2\lambda g(X,Z) = 0, \tag{6.5.3}$$

setting $X = Z = e_i$ in (6.5.3) and taking summation over i $(1 \le i \le n)$, we obtain

$$divV + \frac{\alpha_4}{\beta_4} n + \lambda n = 0. \tag{6.5.4}$$

If V is solenoidal then $divV = 0$. Therefore the equation (6.5.4) can be reduced to

$$\lambda = -\frac{\alpha_4}{\beta_4}. \tag{6.5.5}$$

Thus, we can write the following:

Corollary 6.5.2. *Let (g, V, λ) be a Ricci soliton in a quaternion space form satisfying $R \cdot B = 0$. If V is solenoidal then it is shrinking.*

The particular cases of Theorem (6.5.1) and Corollary (6.5.2) for different curvature tensors are as follows:

Curvature tensors	Einstein	Ricci tensor $S = \frac{\alpha_4}{\beta_4}$	Ricci solitons
quasi-conformal curvature tensor C^*	Einstein	$\alpha_4 = 3(n^3 - 13n^2 + 12n)a$ $+4br(2n^3 - 11n^2 + 9n)$ $+8r(2n^2 + 6n - 17)$ $(a + 2b(n-1))$ $\beta_4 = 4n(n-1)((3-2n)a$ $-n(2n+7)b)$	shrinking
weyl-conformal curvature tensor C	Einstein	$\alpha_4 = k(3n^3 - 45n^2 + 114n - 72)$ $+4r(17n - 26)$, $\beta_4 = 4(14n - 6)(n-1)$	shrinking
concircular curvature tensor $\tilde{C}$	Einstein	$\alpha_4 = 2nk(3n^2 - 39n + 36)$ $+8r(2n^2 + 6n - 17)$ $\beta_4 = 8(3-2n)n(n-1)$	shrinking
conharmonic curvature tensor L^*	Einstein	$\alpha_4 = k(3n^2 - 42n + 72) - 4r(2n - 9)$ $\beta_4 = 4(14n - 16)$	shrinking

6.6 Quaternion space form satisfying $B \cdot R = 0$

Let B and R be satisfy the equation $B \cdot R = 0$ in $\bar{M}(k)$. Then for any tangent vectors U, V, X, Y and Z, the above implies

$$(B(U, V) \cdot R)(X, Y, Z) = 0. \tag{6.6.1}$$

This implies

$$\begin{aligned} B(U,V)R(X,Y)Z &- R(B(U,V)X,Y)Z - R(X,B(U,V)Y)Z \\ &- R(X,Y)B(U,V)Z = 0. \end{aligned} \tag{6.6.2}$$

Taking inner product with T we have,

$$\begin{aligned} g(B(U,V)R(X,Y)Z,T) &- g(R(B(U,V)X,Y)Z,T) - g(R(X,B(U,V)Y)Z,T) \\ &- g(R(X,Y)B(U,V)Z,T) = 0. \end{aligned} \tag{6.6.3}$$

Using equations (6.2.1) and (1.1.15) in (6.6.3) and contracting over U and Y, further again contracting over V and T in the simplified equation we gain

$$S(X,Z) = \frac{\alpha_2}{\beta_2} g(X,Z), \tag{6.6.4}$$

where $\alpha_2 = [-3x_0 k(n+8) + 16x_1 r + 16n(1-n)x_2 r]$ and

$\beta_2 = 4(-2x_0 + 3(2-n)x_1)$. That is $\bar{M}(k)$ is an Einstein manifold.

Hence we obtain the following result:

Theorem 6.6.1. *A quaternion space form satisfying $B \cdot R = 0$ is an Einstein manifold.*

Using equation (6.6.4) in (1.2.1), we get

$$(L_V g)(X,Z) + 2\frac{\alpha_2}{\beta_2} g(X,Z) + 2\lambda g(X,Z) = 0, \tag{6.6.5}$$

setting $X = Z = e_i$ in (6.6.5) and taking summation over i $(1 \le i \le n)$, we obtain

$$divV + \frac{\alpha_2}{\beta_2} n + \lambda n = 0. \tag{6.6.6}$$

If V is solenoidal then $divV = 0$. Therefore the equation (6.6.6) can be reduced to

$$\lambda = -\frac{\alpha_2}{\beta_2}. \tag{6.6.7}$$

Thus, we have state the following:

Corollary 6.6.2. *Let (g, V, λ) be a Ricci soliton in a quaternion space form satisfying $B \cdot R = 0$. If V is solenoidal then it is shrinking.*

The particular cases of Corollary (6.6.2) for different curvature tensors is as follows:

Corollary 6.6.3. *Let (g, V, λ) be a Ricci soliton in a quaternion space form satisfying $C^* \cdot R = 0$, $C \cdot R = 0$, $V \cdot R = 0$ and $L \cdot R = 0$. If V is solenoidal then in all these conditions the space form is shrinking.*

6.7 Hypersurface of a quaternion space form

The notion of quasi-Einstein manifold was studied in [15, 16] by Chaki and Maity. Sular and $\ddot{O}$zgur [50] have proved that a quasi-umbilical hypersurface of Kenmotsu space forms is generalized quasi-Einstein hypersurface. Also the authors Bagewadi and Bharathi [6] have studied hypersurface of complex space form.

Let M be a hypersurface of a quaternion K$\breve{a}$hler manifold $\bar{M}$. If $T\bar{M}$ and TM denote the Lie algebra of vector fields on $\bar{M}$ and M respectively and $T^{\perp}M$, is the set of all vector fields normal to M, then Gauss and weingarten formulae are respectively, given by (1.4.1) and (1.4.2).

The Gauss equation is

$$\widetilde{R}(U, V, W, X) = R(U, V, W, X) - g(\sigma(U, X), \sigma(V, W)) + g(\sigma(V, X), \sigma(U, W)). \quad (6.7.1)$$

Definition 6.7.1. A hypersurface of a quaternion K$\breve{a}$hler manifold $\bar{M}$ is said to be

- quasi umbilical if its second fundamental tensor has the form

$$\sigma(U, V) = \alpha_1 g(U, V) + \beta_1 E(U)E(V), \quad (6.7.2)$$

- generalized quasi-umbilical if its second fundamental tensor has the form

$$\sigma(U,V) = \alpha_1 g(U,V) + \beta_1 E(U)E(V) + \gamma_1 F(U)F(V), \tag{6.7.3}$$

where α_1, β_1 and γ_1 are scalars and E, F are 1-forms defined in (1.5.8).

Theorem 6.7.1. *Let $\bar{M}(k)$ be a quaternion space form.*

1. *Let (g,V,λ) be a Ricci soliton in Quasi-umbilical hypersurface of $\bar{M}(k)$ then it is shrinking if and only if V is solenoidal.*

2. *Let (g,V,λ) be a Ricci soliton in generalized Quasi-umbilical hypersurface of $\bar{M}(k)$ then it is shrinking if and only if V is solenoidal.*

Proof: 1.

Putting equation (6.7.2) in (6.7.1) and using (6.2.1), we have

$$\begin{aligned}
&\frac{k}{4}[g(X,Y)U - g(U,Y)X + g(JX,Y)JU - g(JU,Y)JX + 2g(U,JX)JY \\
&+g(KX,Y)KU - g(KU,Y)KX + 2g(U,KX)KY + g(LX,Y)LU - g(LU,Y)LX \\
&+2g(U,LX)LY] = R(U,X,Y,Z) + \alpha_1^2[g(U,Y)g(X,Z) - g(X,Y)g(U,Z)] \\
&+\alpha_1\beta_1[g(U,Y)E(X)E(Z) + g(X,Z)E(U)E(Y) - g(X,Y)E(U)E(Z) \\
&-g(U,Z)E(X)E(Y)].
\end{aligned} \tag{6.7.4}$$

Setting $U = Z = e_i$ and taking sum over $i (1 \le i \le n)$ in equation (6.7.4), where $\{e_i\}$ is orthonormal basis of the given space form we have

$$S(X,Y) = \kappa g(X,Y) + \tau E(X)E(Y), \tag{6.7.5}$$

where $\kappa = [(n-1)\alpha_1^2 + \frac{k}{4}(n+8) + \alpha_1\beta_1]$ and $\tau = (n-2)\alpha_1\beta_1$.

Equation (6.7.5) in (1.2.1), we get

$$(L_X g)(X,Y) + 2\kappa g(X,Y) + 2\tau E(X)E(Y) + 2\lambda g(X,Y) = 0. \tag{6.7.6}$$

Contracting the above equation, we get

$$2divV + 2n\kappa + 2\tau + 2\lambda n = 0. \tag{6.7.7}$$

If V is solenoidal then $divV = 0$. Therefore the equation (6.7.7) can be reduced to

$$n\kappa + \tau + \lambda n = 0, \tag{6.7.8}$$

this implies

$$\lambda = -\frac{n\kappa + \tau}{n}. \tag{6.7.9}$$

Thus, we have proved the result one.

Proof: 2. *Similar proofs for statement* (2) *is obtained by using equations (6.7.3) and (6.2.1) in (6.7.1) and putting* $V = Y = e_i$ *we get mixed generalized quasi-Einstein manifold. The use of resulting equation in (1.2.1), contraction and solenoidal property will give* λ *is negative.*

6.8 Conclusion

The important results finding of this chapter are as follows:

- A second order parallel tensor in a non-flat quaternion space form is a linear combination (with constant coefficients) of the underlying quaternion Kählerian metric and quaternion Kählerian 2-forms.

- The only symmetric (anti-symmetric) parallel tensor of type $(0, 2)$ in a non-flat quaternion space form is the quaternion Kählerian metric (quaternion Kählerian 2-forms) up to a constant multiple.

- The following statements for a non-flat quaternion space form are equivalent.

  1) Einstein.

  2) Locally Ricci symmetric.

  3) Ricci semisymmetric that is $R \cdot S = 0$.

- Suppose that on a non-flat quaternion space form, the (0,2) type field $L_V g + 2S$ is parallel where V is a given vector field. Then (g, V) yields a Ricci soliton if JV, KV and LV are solenoidal. In particular, if the given non-flat quaternion space form is Ricci semisymmetric with $L_V g$ parallel and JV, KV and LV are solenoidal, then it is also Ricci soliton.

- Let (g, V, λ) be a Ricci soliton in a non-flat quaternion space form. Then V is solenoidal if and only if it is shrinking, steady and expanding depending upon the sign of scalar curvature.

- Let (g, V, λ) be a Ricci soliton in an queternion space form satisfying semisymmetric conditions like $R \cdot R = 0$, $R \cdot B = 0$ and $B \cdot R = 0$. If V is solenoidal then the space is shrinking in each case.

- Let $\tilde{M}(k)$ be quaternion space form

  1. The Ricci soliton (g, V, λ) in quasi-umbilical hypersurface of $\bar{M}(k)$ is shrinking if and only if V is solenoidal.

  2. The Ricci soliton (g, V, λ) in generalized quasi-umbilical hypersurface of $\bar{M}(k)$ is shrinking if and only if V is solenoidal.

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Bibliography

[1] N. S. Agashe and M. R. Chafle, *A semi-symmetric non-metric connections on a Riemannian manifold,* Indian Journal of Pure and Applied Mathematics, 23 (1992), 399-409.

[2] P. Alegre, D. E. Blair and A. Carriazo, *Generalized Sasakian-space forms,* Isrel. J. Math., 141 (2004), 151-183.

[3] P. Alegre and A. Carriazo, *Submanifolds of generalized sasakian space forms,* Taiwanese Journal Of Mathematics, 13(3) (2009), 923-941.

[4] K. Arslan[1], C. Murathan[2], G. Ozturk[3] and S. Turkay[4], *H-recurrent surfaces in Euclidean space E^m,* Seluk J. Appl. Math., 12(2)(2011), 35-41.

[5] C. S. Bagewadi, *On totally real submanifolds of a Kählerian manifold admitting semi-symmetric metric f-connection,* Indian J. Pure Appl. Math., 13(5) (1982), 528-536.

[6] C. S. Bagewadi and M. C. Bharathi, *A study on hypersurface of complex space form,* Acta Et Commentationes Universitatis Tartuensis De Mathematica, 17(1), June (2013), 65-70.

[7] C. S. Bagewadi and G. Ingalahalli, *Ricci soltons in Lorentzian α-Sasakian manifolds,* Acta Mathematica Academiae Paedagogicas Nyiregyhaziensis, 28(1) (2012), 59-68.

[8] M. C. Bharathi and C. S. Bagewadi, *On generalized complex space forms,* IOSR Journal of Mathematics, 10 (2014), 44-46.

[9] C. Calin and M. Crasmareanu, *From the Eisenhart problem to Ricci solitons in f-Kenmotsu manifolds,* Bull. Malays. Math. Sci. Soc., (2), 33(3) (2010), 361-368.

[10] H. D. Cao, *Deformation of Kähler metrics to Kahler-Einstein metrics on compact Kähler manifolds,* Invent. Math., 81(2) (1985), 359-372.

[11] E. Cartan, *Surune classe remarquable d'esspace de Riemannian,* Bulletin de la Societe Mathematique de France, 54 (1926), 214-264.

[12] E. Cartan, *Lecons sur geometrie des espaces de Riemann,* Second ed. Gauthier-Villars, Paries, (1946).

[13] M. C. Chaki, *On pseudo symmetric manifolds,* Analele Stiintifice ale Universitati-iAl.I. Cuza din Iasi, 33 (1987), 53-58.

[14] M. C. Chaki and T. Kawaguchi, *On almost pseudo Ricci symmetric manifolds,* Tensor N. S, 68 (2007), 10-14.

[15] M. C. Chaki, *On super quasi-Einstein manifolds,* Publ. Math. Debrecen., 64 (2004), 481-488.

[16] M. C. Chaki and R. K. Maity, *On quasi-Einstein manifolds,* Publ. Math. Debrecen., 57 (2000), 297-306.

[17] S. Chandra, S. K. Hui, and A. A. Shaikh, *Second order parallel tensors and ricci solitons on (LCS)n-manifolds,* Commun. Korean Math. Soc., 30(2) (2015), 123-130.

[18] B. B. Chaturvadi and P. N. Pandey, *Semi-symmetric non-metric connections on Kaehler manifolds,* Soochow Journal of Mathematics, 23(1) (1997), 381-389.

[19] B. B. Chaturvadi and P. N. Pandey, *Study on special type of a weakly symmetric Kaehler manifold,* Differential Geometry and Dynomical Systems, 17 (2015), 22-37.

[20] B. Y. Chen, *Surfaces with parallel mean curvature vector,* Bull. Amer. Math. Soc., 78(5), 1972.

[21] B. Y. Chen, *Totally umbilical submanifolds of Cayley plane,* Soochow Journal Of Mathematical and Natural Sciences-3, December 1977.

[22] B. Y. Chen, *Totally umbilical submanifolds of quaternion space forms,* J. Austral. Math. Soc. (Series A), 26 (1978), 154-162.

[23] S. S. Chern, M. Carmo, S. Kobayashi, *Minimal submanifolds of a sphere with second fundamental form of constant length,* Functional Analysis and Related Fields, Springer-Verlag (1970), 59-75.

[24] U. C. De, Adnan Al-Aqeel and A. A. Shaikh, *Submanifolds of a Lorentzian Para-Sasakian Manifold,* Bull. Malays. Math. Sci. Soc. (2), 28(2) (2005), 223-227.

[25] U. C. De and A. K. Gazi, *On almost pseudo symmetric manifolds,* Annales Univ. Sci. Budapest, 51 (2008), 53-68.

[26] U. C. De, *A note on Kaehler Manifolds,* European Journal Of Pure And Applied Mathematics, 3, 6(2010), 1137-1140.

[27] U. C. De and P. Majhi, *Certain curvature properties of generalized-Sasakian-space forms,* Facta Universitatis(NIS), Ser. Math. Inform., 27(3) (2012), 271-282.

[28] A. De, *On almost pseudo symmetric manifolds admitting a type of semi-symmetric non-metric connection,* Acta Math. Hungar, 125(1-2) (2009), 183-190.

[29] S. Debnath and A. Bhattacharyya, *Second order parallel tensor in Trans-Sasakian manifolds and connection with Ricci soliton,* Lobachevskii Journal of Mathematics, 33(4) 2012, 312-316.

[30] S. Deshmukh and M. H. Shahid, *Extrinsic spheres in a real space form,* Bull. Belg. Math. Soc. Simon Stevin, 15 (2008), 269-275.

[31] R. Deszcz, *On pseudosymmetric spaces,* Bull. Soc. Math., Belg. Ser., 44 (1992), 1-34.

[32] F. Dillen, J. Fantenakels, S. Haesen, J. Van der Veken, and P. Verstraelen, *Submanifold theory and the parallel transport of levi civita,* Preprint.

[33] L. P. Eisenhart, *Riemannian Geometry,* Princeton University Press, (1949).

[34] L. P. Eisenhart, *Symmetrtice tensoros of the second order whose first covariant derivatives are zero,* Trans. Amer. Math, Soc., 25(2) (1923), 297-306.

[35] D. Ferus, *A remark on Codazzi tensors on constant curvature space, Lecture Notes Math,* Global Differential Geometry and Global Analysis, Springer-Verlag, New York, 838(1981), 1477-1488.

[36] R. S. Hamilton, *The Ricci flow on surfaces,* Mathematics and general relativity (Santa Cruz, CA, 1986), Contemp. Math., 71 (1988), American Math. Soc., 237-262.

[37] S. Ishihara, *Quaternion Kaehlerian manifolds,* J. Differential Geometry, 9(1974), 483-500.

[38] Y. Ishii, *On conharmonic transformations,* Tensor (N.S.), 7 (1957), 73-80.

[39] G. Ingalahalli and C. S. Bagewadi, *Ricci soltons in α-Sasakian manifolds,* ISRN Geometry, (2012), 13 page.

[40] H. Levy, *Symmetric tensors of the second order whose covariant derivatives vanish,* Ann. of Math.,(2), 2 (1925), 91-98.

[41] H. G. Nagaraja, G. Somashekhara and S. Shashidhar, *On generalized Sasakian space forms,* International Scholarly Research Network Geometry, 2012.

[42] Z. Olszak, *On The existance of generlized complex space form,* Isrel. J. Math., 65 (1989), 214-218.

[43] G. Perelman, *Ricci flow with surgery on three-manifolds,* arXiv:math/0303109 [math.DG], 2008.

[44] G. P. Pokhariyal and R. S. Mishra, *Curvature tensors and their relativistic significance,* Yokohama Math. J., 18 (1970), 105-108.

[45] B. Prasad, *A pseudo projective curvature tensor on a Riemannian manifold,* Bull. Calcutta Math. Soc., 94(3) (2002), 163-166.

[46] W. Roter, *On conformally recurrent Ricci-recurrent manifolds,* Colloq Math, 46(1) (1982), 45-57.

[47] A. A. Shaikh and H. Kundu, *On equivalency of various geometric structures,* J. Geom., 105 (2014), 139-165.

[48] A. A. Shaikh, S. K. Hui and C. S. Bagewadi, *On quasi-conformally flat almost pseudo Ricci symmetric manifolds,* Tamsui Oxford Journal of Mathematical Science Aletheia University, 26(2) (2010), 203-219.

[49] R. Sharma, *Second order parallel tensor in real and complex space forms,* Internat. J. Math. and Math. Sci., 12(4) (1989), 787-790.

[50] S. Sular and C. *Özgür*, *On some submanifolds of Kenmotsu manifolds,* Chaos, Solitons and Fractals, 42 (2009), 1990-1995.

[51] Z. I. Szabo, *Structure theorem on Riemannian spaces satisfying $R(X,Y)\cdot R=0$,* The local version, J. Diff. Geo., 17 (1982), 531-582.

[52] L. Tamsassy and T. Q. Bink, *On weakly symmetric and weakly projective symmetric Riemannian manifolds,* Colloquia Mathematica Societatis Janos Bolyai, 50 (1989), 663-670.

[53] L. Tamsassy, U. C. De and T. Q. Bink, *On weak symmetries of Kaehler manifolds,* Balkan. J. Geom. Appl., 5 (2000), 149-155.

[54] P. Topping, *Lectures on the Ricci flow,* (Cambridge Univ. Press, 2006).

[55] F. Tricerri and L. Vanhecke, *Curvature tensors on almost Hermitian manifolds,* Transactions of the American Mathematical Society, 267 (1981), 365-398.

[56] M. M. Tripathi and P. Gupta, *τ-curvature tensor on a semi-Riemannian manifold,* J. Adv. Math. Stud., 4 (1) (2011), 117-129.

[57] S. Trkay, *Biharmonic curves and surfaces,* MSc Thesis, Uludag University, Bursa, 2004.

[58] L. Vanhecke, *Almost Hermitian manifolds with J-invariant Riemann curvature tensor,* Rendiconti del Seminario Mathematico della Universit e Politecnico di Torino, 34 (1975-76), 487-498.

[59] A. G. Walkar, *On Ruses spaces of recurrent curvature,* Proc. London Math. Soc., 52 (1950), 36-64.

[60] S. Yadav, D. L. Suthar and A. K. Srivastava, *Some results on $M(f_1, f_2, f_3)_{2n+1}$-manifolds,* International Journal of Pure and Applied Mathematics, 70(3) (2011), 415-423.

[61] K. Yano and M. Kon, *Structures on manifolds,* Series In Pure Mathematics-3.

[62] K. Yano, *Differential geometry on complex and almost complex spaces,* Pergamon Press, New York, (1965).

[63] K. Yano and S. Sawaki, *Riemannian manifolds admitting a conformal transformation group,* J. Diff. Geom., 2 (1968), 161-184.

[64] K. Yano and S. Bochner, *Curvature and Betti numbers,* Annals of Mathematics Studies 32, Princeton University Press, 1953.

[65] S. T. Yau, *Submanifolds with constant mean curvature I,* Amer. J. Math. 96, (1974), 346-366.