A Theoretical Analysis of Entropic Separability Criterion Using Generalized q-Conditional Entropies

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Mr. ANANTHA. S. NAYAK

Under the Guidance of

Dr. SUDHA

Department of Post-Graduate Studies in Physics
Kuvempu University
Shankaraghatta, India.

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TO MY BELOVED

Teachers & Parents

Certificate

This is to certify that the thesis entitled **A Theoretical Analysis of Entropic Separability Criterion Using Generalized q-Conditional Entropies** submitted by **Mr. ANANTHA. S. NAYAK** to Department of Post-Graduate studies in Physics, Kuvempu University, Shankaraghatta, Karnataka, for the award of the degree of DOCTOR OF PHILOSOPHY is a bonafide record of the research work carried out by the candidate. This thesis has not previously formed the basis for the award of any other Degree, Diploma, Associateship, Fellowship or other title of any other University or Institution. The thesis represents independent and original work on the part of the candidate under my guidance.

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I, **Mr. ANANTHA. S. NAYAK** hereby declare that the work reported in this thesis entitled, **A Theoretical Analysis of Entropic Separability Criterion Using Generalized q-Conditional Entropies**, is entirely original and has been carried out by me independently in the Department of Post-Graduate studies in Physics, Kuvempu University, Shankaraghatta, Karnataka under the guidance of Dr. Sudha at Kuvempu University. I also declare that this work has not been the basis for the award of any Degree, Diploma, Associateship or Fellowship or other title of any other University or Institution.

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Pure mathematics is, in its way, the poetry of logical ideas

– *Albert Einstein*

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Contents

Certificate i

Declaration of Authorship ii

Acknowledgements iv

List of Figures ix

List of Tables xiv

1 Introduction 1

1.1 Quantum Entanglement 2

1.2 Entropic Characterization of Separability: A brief review 8

1.3 Outline of thesis 18

2 A new entropic separability criterion using Conditional version of Sandwiched Relative Tsallis Entropy 20

2.1 Sandwiched Tsallis Relative Entropy and its Conditional Version . . 21

2.1.1 Sufficient condition for quantum entanglement in terms of conditional version of sandwiched Tsallis relative entropy . . 26

2.2 Symmetric one-parameter family of noisy W states 27

2.2.1 Bipartite separability in one parameter family of three qubit noisy W-states 29

2.2.1.1 Separability of $\rho_3^{\rm W}(x)$ in its $1:2$ partition 30

2.2.1.2 Separability of $\rho_3^{\rm W}(x)$ in its $2:1$ partition 36

2.2.2 Bipartite separability in one parameter family of four qubit noisy W-states 38

2.2.2.1 Separability of $\rho_4^{\rm W}(x)$ in its $1:3$ partition 39

2.2.2.2 Separability of $\rho_4^{\rm W}(x)$ in its $3:1$ partition 44

2.2.2.3 Separability in $\rho_4^{\rm W}(x)$ in its $2:2$ partition 46

2.3 Symmetric one-parameter family of noisy GHZ states 50
2.3.1 Bipartite separability in symmetric one parameter family of three qubit noisy GHZ states 50
2.3.1.1 Separability of $\rho_3^{\rm GHZ}(x)$ in its $1:2$ partition. 51
2.3.1.2 Separability of $\rho_3^{\rm GHZ}(x)$ in its $2:1$ partition. 55
2.3.2 Bipartite separability in one parameter family of four qubit noisy GHZ-state 57
2.3.2.1 Separability of $\rho_4^{\rm GHZ}(x)$ in its $1:3$ partition 58
2.3.2.2 Separability of $\rho_4^{\rm GHZ}(x)$ in its $3:1$ partition 61
2.3.2.3 Separability of $\rho_4^{\rm GHZ}(x)$ in its $2:2$ partition. 63
2.4 Symmetric one parameter family of noisy states involving equal superposition of W- and obverse W states $|{\rm W\bar{W}}_N\rangle$ 67
2.4.1 Bipartite separability in one parameter family of noisy states involving the states $|{\rm W\bar{W}}_3\rangle$ 67
2.4.1.1 Separability of $\rho_3^{\rm W\bar{W}}(x)$ in its $1:2$ partition 68
2.4.1.2 Separability of $\rho_3^{\rm W\bar{W}}(x)$ in its $2:1$ partition 72
2.4.2 Bipartite separability in one parameter family of noisy states involving the states $|{\rm W\bar{W}}_4\rangle$ 74
2.4.2.1 Separability of $\rho_4^{\rm W\bar{W}}(x)$ in its $1:3$ partition 75
2.4.2.2 Separability of $\rho_4^{\rm W\bar{W}}(x)$ in its $3:1$ partition 77
2.4.2.3 Separability of $\rho_4^{\rm W\bar{W}}(x)$ in its $2:2$ partition 79
2.5 Summary 84

3 Bipartite separability of symmetric N-qubit noisy states using conditional quantum relative Tsallis entropy 86
3.1 Symmetric one parameter families of noisy N-qubit mixed states . . 87
3.2 $1:N-1$ separability in symmetric one-parameter family of noisy W-states 88
3.2.1 $1:4$ separability in symmetric one-parameter family of noisy W-states with 5 qubits 90
3.2.2 $1:5$ separability in symmetric one-parameter family of noisy W-states with 6 qubits 94
3.2.3 Eigenvalues of the sandwiched matrix in the $1:N-1$ partition of $\rho_N^{\rm W}(x)$ and its $1:N-1$ separability 96
3.3 $1:N-1$ separability in symmetric one-parameter family of noisy GHZ-states 101
3.3.1 $1:4$ separability in symmetric one-parameter family of noisy GHZ states with 5 qubits 101
3.3.2 $1:5$ separability in symmetric one-parameter family of noisy GHZ-states with 6 qubits 104
3.3.3 Eigenvalues of the sandwiched matrix in the $1:N-1$ partition of $\rho_N^{\rm GHZ}(x)$ and its $1:N-1$ separability 107

3.4 $1 : N-1$ separability in symmetric one-parameter family involving the states $|\mathrm{W\bar{W}}_N\rangle$ . . . 111
3.4.1 $1 : 4$ separability in symmetric one-parameter family of noisy $\mathrm{W\bar{W}}$-states with 5 qubits . . . 111
3.4.2 $1 : 5$ separability in symmetric one-parameter family of noisy $\mathrm{W\bar{W}}$-states with 6 qubits . . . 113
3.5 Separability of qutrit-qutrit and qubit-qutrit states using CSTRE criterion . . . 117
3.6 Summary . . . 125

4 Biseparability of noisy pseudopure and Werner-like one parameter families of states using conditional quantum relative Tsallis entropy 126
4.1 One parameter family of N qubit Pseudopure states involving W states . . . 127
4.1.1 $1 : 2$ separability in one parameter family of 3-qubit pseudopure W-states . . . 128
4.1.2 $1 : 3$ separability in one parameter family of 4-qubit pseudopure W-states . . . 132
4.1.3 $1 : 4$ separability in one parameter family of 5-qubit pseudopure W-states . . . 136
4.1.4 $1 : 5$ separability in one parameter family of 6-qubit pseudopure W-states . . . 139
4.1.5 $1 : N-1$ separability in pseudopure states involving N-qubit W states . . . 141
4.1.6 Necessary and sufficient condition for $1 : N-1$ separability of $\rho^{\mathrm{PP}}_{\mathrm{W}_N}(x)$ . . . 144
4.2 One parameter family of N-qubit pseudopure GHZ-states . . . 144
4.2.1 $1 : 2$ separability in one parameter family of 3-qubit pseudopure GHZ-states . . . 145
4.2.2 $1 : 3$ separability in one parameter family of four qubit pseudopure GHZ-states . . . 147
4.2.3 1:4 separability in one parameter family of five qubit pseudopure GHZ-states . . . 150
4.2.4 $1 : 5$ separability in one parameter family of six qubit pseudopure GHZ-states . . . 152
4.2.5 $1 : N-1$ separability in $\rho^{\mathrm{PP}}_{\mathrm{GHZ}_N}(x)$ . . . 154
4.3 Werner-like one parameter family of N qubit W states . . . 156
4.3.1 Separability in the $1 : 2$ partition of three qubit Werner-like family involving W-states . . . 157
4.3.2 Separability in the $1 : 3$ partition of four qubit Werner-like family involving W-states . . . 158

4.3.3 Separability in the 1 : 4 partition of five qubit Werner-like family involving W-states 160
4.3.4 Separability in the 1 : 5 partition of six qubit Werner-like family involving W-states 162
4.3.5 $1 : N-1$ separability in Werner like one parameter family involving N-qubit W states 164
4.3.6 $1 : N-1$ separability in Werner-like noisy states involving N-qubit GHZ states 166
4.4 Summary 169

5 One parameter family of N-qudit Werner-Popescu states: Bipartite separability using Conditional quantum relative Tsallis entropy 170
5.1 N-qudit Werner-Popescu states 171
5.2 $1 : N-1$ separability of $\rho_N^d(x)$ using CSTRE 172
5.3 Summary 175

6 Non-Spectral nature of the Conditional version of Sandwiched Tsallis Relative Entropy 179
6.1 Illustration of Nonspectral nature of the CSTRE criterion 180
6.2 Bound Entangled States: An attempt to detect entanglement using CSTRE criterion 183
6.3 Summary 192

7 Conclusion 194
7.1 Summary of the Thesis 194
7.2 Future Directions... 198

Bibliography 200

Publications 205

List of Figures

2.1 The von-Neumann conditional entropy $S(A|B)$ as a function of x, for the state $\rho_3^{\mathrm{W}}(x)$, in its $1:2$ partition. 31

2.2 The AR q-conditional entropy $S_q^T(A|B)$ as a function of x for different values of q, in the $1:2$ partition of the state $\rho_3^{\mathrm{W}}(x)$. 32

2.3 The CSTRE $\tilde{D}_q^T(\rho_3^{\mathrm{W}}(x)||\rho_B)$ as a function of x for different values of q, in the $1:2$ partition of the state $\rho_3^{\mathrm{W}}(x)$. 34

2.4 Implicit plot of $\tilde{D}_q^T(\rho_3^{\mathrm{W}}(x)||\rho_B) = 0$ as a function of q (solid line) indicating that $x \to 0.1547$ as $q \to \infty$ in the $1:2$ partition of the state $\rho_3^{\mathrm{W}}(x)$. In contrast, for the same partition of $\rho_3^{\mathrm{W}}(x)$, the implicit plot of Abe-Rajagopal q-conditional entropy $S_q^T(A|B) = 0$(dashed line), leads to $x \to 0.2$ as $q \to \infty$. 35

2.5 The plot of negativity of partial transpose as a function of x, for $\rho_3^{\mathrm{W}}(x)$ in its $1:2$ partition. 36

2.6 The AR q-conditional entropy $S_q^T(A|B)$ as a function of x for different values of q, for $\rho_3^{\mathrm{W}}(x)$, in its $2:1$ partition. 37

2.7 The CSTRE $\tilde{D}_q^T(\rho_3^{\mathrm{W}}(x)||\rho_B)$ as a function of x for different values of q, for $\rho_3^{\mathrm{W}}(x)$, in its $2:1$ partition. 39

2.8 The von-Neumann conditional entropy $S(A|B)$ as a function of x, for the state $\rho_4^{\mathrm{W}}(x)$, in its $1:3$ partition. 41

2.9 The AR q-conditional entropy $S_q^T(A|B)$ of $\rho_4^{\mathrm{W}}(x)$, in its $1:3$ partition, as a function of x for different values of q. 41

2.10 The CSTRE $\tilde{D}_q^T(\rho_4^{\mathrm{W}}(x)||\rho_B)$ as a function of x for different values of q, in the $1:3$ partition of the state $\rho_4^{\mathrm{W}}(x)$. 43

2.11 The implicit plots of $\tilde{D}_q^T(\rho_4^{\mathrm{W}}(x)||\rho_B) = 0$ (solid line) and $S_q^T(A|B) = 0$ (dashed line) as a function of q for the state $\rho_4^{\mathrm{W}}(x)$, in its $1:3$ partition. 43

2.12 The plot of negativity of partial transpose as a function of x, for the state $\rho_4^{\mathrm{W}}(x)$, in its $1:3$ partition. 44

2.13 The AR q-conditional entropy $S_q^T(A|B)$ of the state $\rho_4^{\mathrm{W}}(x)$, in its $3:1$ partition, as a function of x for different values of q. 46

2.14 The CSTRE $\tilde{D}_q^T(\rho_4^{\mathrm{W}}(x)||\rho_B)$ as a function of x for different values of q, in the $3:1$ partition of $\rho_4^{\mathrm{W}}(x)$. 47

2.15 The AR q-conditional entropy $S_q^T(A|B)$ as a function of x for different values of q, in the $2:2$ partition of the state $\rho_4^{\mathrm{W}}(x)$. 48

2.16 The CSTRE $\tilde{D}_q^T(\rho_4^{\mathrm{W}}(x)||\rho_B)$ of $\rho_4^{\mathrm{W}}(x)$, in its $2:2$ partition. as a function of x for different values of q. 49
2.17 The plot of negativity of partial transpose as a function of x, in the $2:2$ partition of the state $\rho_4^{\mathrm{W}}(x)$. 49
2.18 The von-Neumann conditional entropy $S(A|B)$ of $\rho_3^{\mathrm{GHZ}}(x)$ in its $1:2$ partition, as a function of x. 52
2.19 The CSTRE $\tilde{D}_q^T(\rho_3^{\mathrm{GHZ}}(x)||\rho_B)$ of $\rho_3^{\mathrm{GHZ}}(x)$, in its $1:2$ partition, as a function of x for different values of q. 53
2.20 The plot of negativity of partial transpose as a function of x, for the $1:2$ partition of the state $\rho_3^{\mathrm{GHZ}}(x)$. 55
2.21 The CSTRE $\tilde{D}_q^T(\rho_3^{\mathrm{GHZ}}(x)||\rho_B)$ of $\rho_3^{\mathrm{GHZ}}(x)$, in its $2:1$ partition, as a function of x for different values of q. 57
2.22 The von-Neumann conditional entropy $S(A|B)$ as a function of x, for $\rho_4^{\mathrm{GHZ}}(x)$ state, in its $1:3$ partitions. 59
2.23 The CSTRE $\tilde{D}_q^T(\rho_4^{\mathrm{GHZ}}(x)||\rho_B)$ as a function of x for different values of q in the $1:3$ partition of $\rho_4^{\mathrm{GHZ}}(x)$. 61
2.24 The plot of negativity of partial transpose of the state $\rho_4^{\mathrm{GHZ}}(x)$, in its $1:3$ partition, as a function of x. 62
2.25 The CSTRE $\tilde{D}_q^T(\rho_4^{\mathrm{GHZ}}(x)||\rho_B)$ as a function of x for different values of q in the $3:1$ partition of $\rho_4^{\mathrm{GHZ}}(x)$. 64
2.26 The CSTRE $\tilde{D}_q^T(\rho_4^{\mathrm{GHZ}}(x)||\rho_B)$ of $\rho_4^{\mathrm{GHZ}}(x)$, in its $2:2$ partition, as a function of x for different values of q. 66
2.27 The plot of negativity of partial transpose in the $2:2$ partition of the state $\rho_4^{\mathrm{GHZ}}(x)$ as a function of x. 66
2.28 The AR q-conditional entropy $S_q^T(A|B)$ in the $1:2$ partition of the state $\rho_3^{\mathrm{W\bar{W}}}(x)$, as a function of x for different values of q. 70
2.29 The CSTRE $\tilde{D}_q^T(\rho_3^{\mathrm{W\bar{W}}}(x)||\rho_B)$ as a function of x for different values of q, in the $1:2$ partition of $\rho_3^{\mathrm{W\bar{W}}}(x)$. 71
2.30 The plot of negativity of partial transpose in the $1:2$ partition of the state $\rho_3^{\mathrm{W\bar{W}}}(x)$ as a function of x. 72
2.31 The CSTRE $\tilde{D}_q^T(\rho_3^{\mathrm{W\bar{W}}}(x)||\rho_B)$ as a function of x for different values of q in the $2:1$ partition of $\rho_3^{\mathrm{W\bar{W}}}(x)$. 74
2.32 The CSTRE $\tilde{D}_q^T(\rho_4^{\mathrm{W\bar{W}}}(x)||\rho_B)$ as a function of x for different values of q in the $1:3$ partition of $\rho_4^{\mathrm{W\bar{W}}}(x)$. 77
2.33 The plot of negativity of partial transpose in the $1:3$ partition of the state $\rho_4^{\mathrm{W\bar{W}}}(x)$ as a function of x. 77
2.34 The CSTRE $\tilde{D}_q^T(\rho_4^{\mathrm{W\bar{W}}}(x)||\rho_B)$ as a function of x for different values of q in the $3:1$ partition of $\rho_4^{\mathrm{W\bar{W}}}(x)$. 79
2.35 The CSTRE $\tilde{D}_q^T(\rho_4^{\mathrm{W\bar{W}}}(x)||\rho_B)$ as a function of x in the $2:2$ partition of $\rho_4^{\mathrm{W\bar{W}}}(x)$ 81
2.36 The plot of negativity of partial transpose of $\rho_4^{\mathrm{W\bar{W}}}(x)$, in its $2:2$ partition as a function of x 81

3.1 Plot of conditional entropies, $\lim_{q\to\infty} \tilde{D}_q^T(\rho_5^{\mathrm{W}}(x)||\rho_B)$, $\lim_{q\to\infty} S_q^T(A|B)$, $S(A|B)$ of $\rho_5^{\mathrm{W}}(x)$, in its $1:4$ partition, as a function of x. 93
3.2 The variation of negativity of partial transpose of the state $\rho_5^{\mathrm{W}}(x)$, as a function of x. 94
3.3 Plot of conditional entropies, $\lim_{q\to\infty} \tilde{D}_q^T(\rho_6^{\mathrm{W}}(x)||\rho_B)$, $\lim_{q\to\infty} S_q^T(A|B)$, $S(A|B)$ of $\rho_6^{\mathrm{W}}(x)$, in its $1:5$ partition. 96
3.4 The plot of negativity of partial transpose of the state $\rho_6^{\mathrm{W}}(x)$ in its $1:5$ partition as a function of x. 98
3.5 Implicit plot of $\tilde{D}_q^T(\rho_8^{\mathrm{W}}(x)||\rho_B) = 0$ as a function of q (solid line) indicating that $x \to 0.0538$ as $q \to \infty$. In contrast, the implicit plot of Abe-Rajagopal q-conditional entropy $S_q^T(A|B) = 0$ of the state $\rho_8^{\mathrm{W}}(x)$, in its $1:7$ partition(dashed line), leads to $x \to 0.1$ as $q \to \infty$.100
3.6 Illustration of the reduction of $1:N-1$ CSTRE separability range in $\rho_N^{\mathrm{W}}(x)$ with increase in the number of qubits N from $N=5$ to $N=8$ 100
3.7 The variation of von-Neumann conditional entropy $S(A|B)$ and $\lim_{q\to\infty} \tilde{D}_q^T(\rho_5^{\mathrm{GHZ}}(x)||\rho_B)$ as a function of x for the state $\rho_5^{\mathrm{GHZ}}(x)$, in its $1:4$ partition. 103
3.8 The graph of negativity of partial transpose of the state $\rho_5^{\mathrm{GHZ}}(x)$ as a function of x 104
3.9 The variation of von-Neumann conditional entropy $S(A|B)$ and CSTRE $\lim_{q\to\infty} \tilde{D}_q^T(\rho_6^{\mathrm{GHZ}}(x)||\rho_B)$ as a function of x for $\rho_6^{\mathrm{GHZ}}(x)$, in its $1:5$ partition. 106
3.10 The graph of negativity of partial transpose of $\rho_6^{\mathrm{GHZ}}(x)$, in its $1:5$ partition, versus x. 107
3.11 Implicit plots of $\tilde{D}_q^T(\rho_6^{\mathrm{GHZ}}(x)||\rho_B) = 0$ as a function of q (solid line) and Abe-Rajagopal q-conditional entropy $S_q^T(A|B) = 0$ (dashed line) of the state $\rho_6^{\mathrm{GHZ}}(x)$, in its $1:5$ partition. The relatively slow convergence of the parameter x to 0.04545 with the increase of q in the case of CSTRE criterion is readily seen. 110
3.12 Illustration of the reduction of $1:N-1$ CSTRE separability range in $\rho_N^{\mathrm{GHZ}}(x)$ with increase in the number of qubits N from $N=5$ to $N=8$ 110
3.13 Both von-Neumann conditional entropy $S(A|B)$ and CSTRE $\lim_{q\to\infty} \tilde{D}_q^T(\rho_5^{\mathrm{W\bar{W}}}(x)||\rho_B)$ varying as a function of x for $\rho_5^{\mathrm{W\bar{W}}}(x)$ in its $1:4$ partition. It can be readily seen that the $1:4$ separability range obtained through von-Neumann conditional entropy is weaker than the corresponding CSTRE separability range. 113
3.14 The plot of negativity as a function of x for the state $\rho_5^{\mathrm{W\bar{W}}}(x)$. . . . 113
3.15 A plot illustrating the variation of conditional entropies $S(A|B)$, $\lim_{q\to\infty} \tilde{D}_q^T(\rho_6^{\mathrm{W\bar{W}}}(x)||\rho_B)$ of $\rho_6^{\mathrm{W\bar{W}}}(x)$, in its $1:5$ partition as a function of x. 115

3.16 The plot of negativity of partial transpose of the state $\rho_6^{\mathrm{W}\bar{\mathrm{W}}}(x)$ in its $1:5$ partition, as a function of x. 115
3.17 The AR q-conditional entropy $S_q^T(a|b)$ as a function of x for different values of q. 119
3.18 The CSTRE $\tilde{D}_q^T(\rho_{ab}(x)||\rho_a)$ as a function of x for different values of q. 119
3.19 The plot of negativity as a function of x, for the state $\rho_{ab}(x)$. 120
3.20 The plot of negativity of partial transpose of the state $\rho_{ab}^X(x)$ as a function of x. 122
3.21 The CSTRE $\tilde{D}_q^T(\rho_{ab}^X(x)||\rho_b)$ as a function of x for $\rho_{ab}^X(x)$. 123
3.22 The CSTRE $\tilde{D}_q^T(\rho_{ab}^X(x)||\rho_a)$ as a function of x for $\rho_{ab}^X(x)$. 124

4.1 Plot illustrating the monotonic decreasing nature of $S(A|B)$, $\lim_{q\to\infty} S_q^T(A|B)$, $\lim_{q\to\infty} \tilde{D}_q^T(\rho_{\mathrm{W}_3}^{\mathrm{PP}}(x)||\rho_B)$ with x, in the $1:2$ partition of $\rho_{\mathrm{W}_3}^{\mathrm{PP}}(x)$. . . 131
4.2 Plot of the conditional entropies $\lim_{q\to\infty} \tilde{D}_q^T(\rho_{\mathrm{W}_4}^{\mathrm{PP}}(x)||\rho_B)$, $\lim_{q\to\infty} S_q^T(A|B)$, $S(A|B)$ of $\rho_{\mathrm{W}_4}^{\mathrm{PP}}(x)$, in its $1:3$ partition, as a function of x. 135
4.3 Identification of the zeroes of $\lim_{q\to\infty} \tilde{D}_q^T(\rho_{\mathrm{W}_5}^{\mathrm{PP}}(x)||\rho_B)$, $\lim_{q\to\infty} S_q^T(A|B)$ and $S(A|B)$ to obtain the $1:4$ separability range of $\rho_{\mathrm{W}_5}^{\mathrm{PP}}(x)$ using different entropic criteria 138
4.4 The conditional entropies $\lim_{q\to\infty} \tilde{D}_q^T(\rho_{\mathrm{W}_6}^{\mathrm{PP}}(x)||\rho_B)$, $\lim_{q\to\infty} S_q^T(A|B)$, $S(A|B)$ as a function of x for $\rho_{\mathrm{W}_6}^{\mathrm{PP}}(x)$, in its $1:5$ partition. 141
4.5 Variation of the Von-Neumann conditional entropy $S(A|B)$ and the CSTRE $\tilde{D}_q^T(\rho_{\mathrm{GHZ}_4}^{\mathrm{PP}}(x)||\rho_B)$ in the $1:3$ partition of $\rho_{\mathrm{GHZ}_4}^{\mathrm{PP}}(x)$ when $q\to\infty$, as a function of x. 149
4.6 Variation of the Von-Neumann conditional entropy $S(A|B)$ and the CSTRE $\tilde{D}_q^T(\rho_{\mathrm{GHZ}_5}^{\mathrm{PP}}(x)||\rho_B)$ in the $1:4$ partition of $\rho_{\mathrm{GHZ}_5}^{\mathrm{PP}}(x)$ when $q\to\infty$, as a function of x. 151
4.7 Plot of the conditional entropies $S(A|B)$ and $\lim_{q\to\infty} \tilde{D}_q^T(\rho_{\mathrm{GHZ}_6}^{\mathrm{PP}}(x)||\rho_B)$ of $\rho_{\mathrm{GHZ}_6}^{\mathrm{PP}}(x)$, in its $1:5$ partition, as a function of x. 153
4.8 Implicit plots of $\tilde{D}_q^T(\rho_{\mathrm{GHZ}_6}^{\mathrm{PP}}(x)||\rho_B) = 0$ and the Abe-Rajagopal q-conditional entropy $S_q^T(A|B) = 0$ as a function of q in the $1:5$ partition of the state $\rho_{\mathrm{GHZ}_6}^{\mathrm{PP}}(x)$. This demonstrates the relatively slower convergence of the noisy parameter x to the cut-off value 0.04545 in the case of the CSTRE approach, when compared with that of the AR method. 155
4.9 Plot of the conditional entropies $S(A|B)$ and $\lim_{q\to\infty} \tilde{D}_q^T(\rho_{\mathrm{W}_6}(x)||\rho_B)$ of $\rho_{\mathrm{W}_6}(x)$, in its $1:5$ partition, as a function of x. 163
4.10 Implicit plots of $\tilde{D}_q^T(\rho_{\mathrm{GHZ}_6}(x)||\rho_B) = 0$ as a function of q and the AR q-conditional entropy $S_q^T(A|B) = 0$ for $\rho_{\mathrm{GHZ}_6}(x)$ in its $1:5$ partition. The convergence of the parameter x to its bound 0.0303 under the CSTRE criterion is slower compared to that of the AR method. 168

5.1 The variation of conditional form of sandwiched Tsallis relative entropy $\tilde{D}_q^T(\rho_4^{(3)}(x)||\rho_B)$ in the 1 : 3 partition of 4-qutrit Werner-Popescu states $\rho_4^{(3)}(x)$ (N=4, $d = 3$), with respect to x, in the limit $q \to \infty$. 174
5.2 The comparison between implicit plots of $\tilde{D}_q^T(\rho_5^{(3)}(x)||\rho_B) = 0$ and $S_q^T(A|B) = 0$, as a function of q in the 1 : 4 partition of the 5-qutrit ($N = 5$, $d = 3$) state $\rho_5^{(3)}(x)$. A rapid decrease in the value of x, in comparison with $\tilde{D}_q^T(\rho_5^{(3)}(x)||\rho_B)$ can be observed in the case of $S_q^T(A|B)$. 175
5.3 The comparison between implicit plots of $\tilde{D}_q^T(\rho_4^{(5)}(x)||\rho_B) = 0$ and $S_q^T(A|B) = 0$, as a function of q for 4-partite ($N = 4$), 5-level ($d = 5$) Werner-Popescu states $\rho_4^{(5)}(x)$. 175
5.4 The graph of CSTRE $\tilde{D}_q^T(\rho_N^{(3)}(x)||\rho_B) = 0$ versus x for different values of N when $d = 3$. The decrease of the separability range with N, for any given d is clearly seen. 176
5.5 The graph of CSTRE $\tilde{D}_q^T(\rho_N^{(4)}(x)||\rho_B) = 0$ versus x for different values of N when $d = 4$. 176

6.1 A plot of CSTRE $\tilde{D}_q^T(\rho_{AB}||\rho_B)$ of the two qubit entangled state in Eq. (6.1) as a function of q. It is readily seen that CSTRE is negative for all values of $q > 1$ indicating that the state is entangled. 182
6.2 The CSTRE $\tilde{D}_q^T(\rho_{\rm H}(x)||\rho_B)$ as a function of x for different value of q in the state $\rho_{\rm H}(x)$. 186
6.3 The CSTRE $\tilde{D}_q^T(\rho_a||\rho_A)$ as a function of x for different value of q in the state ρ_a. 189
6.4 The CSTRE $\tilde{D}_q^T(\rho_a||\rho_B)$ as a function of x for different value of q in the state ρ_a. 190

List of Tables

2.1 Comparison of separability range of one parameter families of states through positivity of conditional entropies $S(A|B)$, $\lim_{q\to\infty} S_q^T(A|B)$, $\lim_{q\to\infty} \tilde{D}_q^T(\rho_{AB}||\rho_B)$ and Peres' PPT criterion 82

2.2 Comparison of separability ranges of one parameter families of states through positivity of conditional entropies $S(B|A)$, $\lim_{q\to\infty} S_q^T(B|A)$, $\lim_{q\to\infty} \tilde{D}_q^T(\rho_{AB}||\rho_A)$ and Peres' PPT criterion 85

3.1 The non-zero eigenvalues γ_i of the sandwiched matrix $(I_A\otimes\rho_B)^{\frac{1-q}{2q}}\rho_N^{\rm W}(x)(I_A\otimes\rho_B)^{\frac{1-q}{2q}}$ for $N=3$ *to* 6 97

3.2 The eigenvalues γ_i of the sandwiched matrix $(I_A\otimes\rho_B)^{\frac{1-q}{2q}}\rho_N^{\rm GHZ}(x)(I_A\otimes\rho_B)^{\frac{1-q}{2q}}$ for $N=3,\ 4,\ 5,\ 6$ 108

4.1 Comparison of the $1:N-1$ separability range $(0,\ x_0)$ of the state $\rho^{\rm PP}_{{\rm W}_N}(x)$, for $N=3,\ 4,\ 5,\ 6$ through the threshold values x_0 obtained through different separability criteria 143

4.2 The comparison of the $1:N-1$ separability threshold values x_0 of the state $\rho^{\rm PP}_{{\rm GHZ}_N}(x)$, for $N=3,\ 4,\ 5,\ 6$ obtained through different separability criteria. 154

4.3 The $1:N-1$ separability threshold value of the noisy parameter x in the states $\rho_{{\rm W}_N}(x)$ for $N=3,\ 4,\ 5,\ 6$, obtained through different separability criteria. 165

4.4 The non-zero eigenvalues γ_i of the sandwiched matrix $(I_2\otimes\rho_B)^{\frac{1-q}{2q}}\rho_{{\rm GHZ}_N}(x)(I_2\otimes\rho_B)^{\frac{1-q}{2q}}$ for $N=3$ *to* 6 167

4.5 Comparison of the $1:N-1$ separability threshold value of the noisy parameter x in the states $\rho_{{\rm GHZ}_N}(x)$ using different separability criteria 168

5.1 The non-zero eigenvalues γ_i of the sandwiched matrix $(I_A\otimes\rho_B)^{\frac{1-q}{2q}}\,\rho_N^d(x)\ (I_A\otimes\rho_B)^{\frac{1-q}{2q}}$ 177

5.2 The comparison of the $1:N-1$ separability range of the state $\rho_N^d(x)$, for various compositions of d and N obtained through CSTRE criterion. 178

5.3 The comparison of the value of x for $q=2$, obtained through different criteria 178

Chapter 1

Introduction

Entanglement is a trick that quantum magicians use to produce phenomena that cannot be imitated by classical magicians.

– A. Peres

A set of postulates that govern the atomic scaled phenomena are the fundamental components of the quantum theory. The quantum uncertainty or the Heisenberg uncertainty is at the core of the quantum theory. This uncertainty is not due to the loss or lack of information or because of imprecise measurement, but rather, it is a fundamental uncertainty inherent in the nature itself. The most accurate and complete description of all known physical systems are affirmed by the basic mathematical principles of quantum theory. These mathematical principles form the basis of many new fledgling fields like Quantum Information [1]. The quantum information theory is the study of the ultimate capability of noisy physical systems, governed by the laws of quantum mechanics, to preserve information and correlations. The different branches of sciences like material sciences, theoretical and experimental physics, mathematics, computer science provide resource inputs towards the vast independent research area of quantum information.

At the first sight the quantum information theory may look like the quantum extension of classical information theory developed by Claude Shannon, with a groundbreaking paper in the year 1948 [2]. But there exist many fundamental differences between them. In some sense, the classical information theory is merely an application of probability theory. Its main task is to quantify the ultimate compressibility of information and the ultimate ability for a sender to transmit information reliably to a receiver. Classical uncertainty, arising from our lack of total information about any given scenario, omnipresent throughout all information processing tasks makes classical information theory to rely upon probability theory. In classical information theory the uncertainty is due to imprecise knowledge. Whereas the uncertainty in quantum theory is inherent in nature itself and not intuitive as classical uncertainty. The concepts like single particle interference, the quantum uncertainty principle, the superposition principle (a consequence of the linearity of quantum theory) separates the classical information theory from its quantum counterpart. Apart from the above mentioned concepts a strange non-classical phenomenon known as **quantum entanglement** forms the core of quantum information theory [3]. The quantum entanglement is a phenomenon that occurs within a system of two or more particles in such a way that the quantum state of each particle can not be described independently. Even the large spacial separation between the particles will not destroy the entanglement of the whole system. Quantum entanglement is seen to be a key resource in many contemporary research fields like quantum computation [4–7], quantum error correction [6, 7], quantum communication [8], quantum key distribution [9], quantum teleportation [10], quantum cryptography [9, 11], quantum dense coding [12] etc.,

1.1 Quantum Entanglement

A ground breaking article [13] published by Albert Einstein, Boris Podolsky and Nathan Rosen, together known as EPR, in the year 1935 is the usual starting point for a discussion of the debate on quantum entanglement. This research article draws attention to a phenomenon predicted by quantum mechanics, in which measurements on spatially separated quantum systems can instantaneously influence one another. As a result, quantum mechanics violates the principle of locality according to which changes performed on one physical system should have no immediate effect on another spatially separated system. Einstein called it as *a spooky action at a distance* [14]. In 1935 itself, Erwin Schrödinger gave the name

Verschränkung for this strange phenomenon [15–17] and it was then rather loosely translated to **entanglement** [18].

The classification of a given state as separable or entangled can be done through several criteria. First of all, precise mathematical definitions to distinguish between entangled and separable states are required. This is very simple for pure states: a pure bipartite state $|\phi_{ab}\rangle$ is called separable iff it can be written as $|\phi_{ab}\rangle = |a\rangle \otimes |b\rangle$, otherwise it is entangled. In general a multipartite pure state $|\psi\rangle$ (with n subsystems) is separable iff it can be written as the Kronecker product of its subsystems. That is, $|\psi\rangle$ is separable iff

$$|\psi\rangle = |a_1\rangle \otimes |a_2\rangle \otimes |a_3\rangle \cdots \otimes |a_n\rangle$$

and it is non-separable (or entangled) iff

$$|\psi\rangle \neq |a_1\rangle \otimes |a_2\rangle \otimes |a_3\rangle \cdots \otimes |a_n\rangle$$

The symbol $\otimes$ stands for *Kronecker Product* or *Tensor Product* and $|a_1\rangle, |a_2\rangle, \ldots |a_n\rangle$ are respectively pure states of the subsystems 1,2,...n.

Example for pure separable states: The product states $|00\rangle$, $|01\rangle$, $|10\rangle$ and $|11\rangle$ of two spin-1/2 particles, expressible in the form .

$$|00\rangle = |0\rangle \otimes |0\rangle; \qquad |01\rangle = |0\rangle \otimes |1\rangle;$$

$$|10\rangle = |1\rangle \otimes |0\rangle; \qquad |11\rangle = |1\rangle \otimes |1\rangle.$$

Here

$$|0\rangle = |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad |1\rangle = |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

respectively represent the Up and Down states of a spin-1/2 particle. In quantum information theory this spin-1/2 particle is generally referred to as '*qubit*' (Abbreviation for Quantum Bit), a quantum counterpart of the classical bit. In general a d level particle (a particle with spin $(d-1)/2$) is referred to as '*qudit*'.

Examples for pure entangled states:

$$|\phi_1\rangle = \frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right], \qquad |\phi_2\rangle = \frac{1}{\sqrt{2}}\left[|00\rangle - |11\rangle\right]$$
$$|\phi_3\rangle = \frac{1}{\sqrt{2}}\left[|01\rangle + |10\rangle\right], \qquad |\phi_4\rangle = \frac{1}{\sqrt{2}}\left[|01\rangle - |10\rangle\right] \tag{1.1}$$

The states $|\phi_1\rangle$, $|\phi_2\rangle$, $|\phi_3\rangle$, $|\phi_4\rangle$ are the so-called Bell states [1] and are the maximally entangled two-qubit pure states.

For a bipartite mixed state, criterion for separability has been proposed by Reinhard F. Werner in 1989 [19] and is called Werner's Separability criterion. According to this criterion, a bipartite state is separable if it can be expressed as a convex combination of product states i.e.,

$$\rho_{AB}^{(\mathrm{sep})} = \sum_k p_k \left(\rho_k^A \otimes \rho_k^B\right) \; ; \quad 0 \leq p_k \leq 1, \; \sum_k p_k = 1, \tag{1.2}$$

where ρ_k^A and ρ_k^B, $k = 1, 2, \ldots$ are density matrices of the subsystems A and B respectively. The coefficients p_k indicate probabilities. For entangled states such a convex combination is not possible. That is, one can call a mixed state ρ to be entangled iff

$$\rho_{AB}^{(\mathrm{ent})} \neq \sum_k p_k \left(\rho_k^A \otimes \rho_k^B\right) \tag{1.3}$$

Example for mixed separable state:

$$\rho_{sep} = \frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right).$$

In fact using the relation

$$|ab\rangle\langle cd| = |a\rangle\langle c| \otimes |b\rangle\langle d|$$

it can be readily seen that ρ_{sep} can be written as

$$\begin{aligned}\rho_{sep} &= \frac{1}{2}\left(|0\rangle\langle 0| \otimes |0\rangle\langle 0|+\right) + \frac{1}{2}\left(|1\rangle\langle 1| \otimes |1\rangle\langle 1|\right) \\ &= \frac{1}{2}\left(\rho_{A_1} \otimes \rho_{B_1}\right) + \frac{1}{2}\left(\rho_{A_2} \otimes \rho_{B_2}\right)\end{aligned}$$

and it is in the Werner separable form Eq. (1.2) with

$$\rho_{A_1} = \rho_{B_1} = |0\rangle\langle 0|, \quad \rho_{A_2} = \rho_{B_2} = |1\rangle\langle 1| \tag{1.4}$$

and $p_1 = p_2 = 1/2$.

Example for mixed entangled state: The so-called Werner state [19, 20], an admixture of a Bell state to the Identity, given by

$$\rho_w = \frac{(1-x)I_4}{4} + x|\phi_1\rangle\langle\phi_1|, \quad |\phi_1\rangle = \frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right] \tag{1.5}$$

is entangled when $\frac{1}{3} < x \leq 1$ and is separable when $0 \leq x \leq \frac{1}{3}$.

The separability criteria defined so far are not so 'user-friendly' in identifying entanglement in the mixed states. In fact, even in composite pure states the identification of separability is not as trivial as is seen through the examples given above. For example, the state

$$|\psi\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

does not look like a separable pure state at the first instance. But it is a product state and one can see that

$$|\psi\rangle = \left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right] \otimes \left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right]$$

The above form of the state $|\psi\rangle$ is obtained through the so-called *Schmidt decomposition* [1] which is a useful tool to identify separable/entangled pure states. The above example thus brings out the need to make use of the so-called operational criteria in establishing entanglement or separability in composite states. An operational criterion is a recipe that can be applied to an explicit density matrix

ρ, giving some immediate answer like ρ is entangled, or ρ is separable, or this criterion is not strong enough to decide whether ρ is separable or entangled. In 1996, Asher Peres [21] proposed an operational criterion for separability in bipartite quantum states. According to him, the act of obtaining the partial transpose of a density matrix is equivalent to the application of positive trace preserving map on the original density matrix. If the partially transposed density matrix is not positive semi-definite it indicates the presence of entanglement. Here, the partial transpose of a composite density matrix is obtained by transposing only one of the subsystems. For a bipartite system, the partial transpose, transposed with respect to the first subsystem[1] is given by $(\rho^T)_{m\mu,n\nu} = \rho_{n\mu,m\nu}$ where the Latin indices refer to the first subsystem and the Greek indices refer to the second subsystem. It follows from the definition of a separable (mixed) state that the Partial Transpose (ρ^T) (either transposed with respect to first system or second system) of a separable state is again a density matrix. As the eigenvalues of a density matrix are non-negative, the partial transpose of a separable state must have non-negative eigenvalues. In other words, Peres Partial Transpose (PPT) criterion implies that *the negative eigen values of the partially transposed density matrix necessarily implies entanglement in the bipartite quantum system.*

Horodecki *et al,* [22] in the same year, showed that PPT criterion forms a necessary and sufficient condition for only (2×2) and (2×3)-dimensional systems. This implies that if the partial transpose of a qubit-qubit system or a qubit-qutrit system are non-negative definite, then the states are *not entangled* and hence are *separable*. This combined criterion known as *Peres-Horodecki criterion* is extremely useful in deciding whether a quantum system is separable or not. In higher dimensional quantum systems, negativity under partial transpose is only a sufficient condition for entanglement. That is, there exist entangled states with positive partial transpose in higher dimensional systems [23]. The entangled states with positive partial transpose are referred to as bound entangled states [23, 24].

Using Peres Partial Transpose (PPT) criterion, G. Vidal and R. F. Werner [25] came up with a *computable* measure of entanglement which is based on the *trace norm* of the partial transpose ρ^T of the bipartite mixed state ρ.

[1]The partial transpose of density matrix of any bipartite system transposed with respect to the second subsystem is given by $(\rho^T)_{m\mu,n\nu} = \rho_{m\nu,n\mu}$.

Trace norm of any operator $\hat{A}$ is given as $\|\hat{A}\| = \mathrm{Tr}\sqrt{\hat{A}^\dagger \hat{A}}$. The trace norm of a partially transposed density operator ρ^T_{AB} can be simplified as follows:

$$\begin{aligned}
\|\rho^T_{AB}\| &= \mathrm{Tr}\sqrt{\rho^T_{AB}(\rho^T_{AB})^\dagger} = \mathrm{Tr}\sqrt{(\rho^T_{AB})^2} \quad \text{as} \quad (\rho^T_{AB})^\dagger = \rho^T_{AB} \\
&= \mathrm{Tr}\begin{pmatrix} \lambda_1^2 & 0 & 0 & \ldots \\ 0 & \lambda_2^2 & 0 & \ldots \\ 0 & 0 & \lambda_3^2 & \ldots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}^{\frac{1}{2}}
\end{aligned}$$

where λ_i denotes the eigenvalues of ρ^T_{AB}. Thus, one can write

$$\begin{aligned}
\|\rho^T_{AB}\| &= \mathrm{Tr}\begin{pmatrix} |\lambda_1| & 0 & 0 & \ldots \\ 0 & |\lambda_2| & 0 & \ldots \\ 0 & 0 & |\lambda_3| & \ldots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \\
&= \sum_i |\lambda_i| \\
&= \sum_k \lambda_{pk} + \sum_l |\lambda_{nl}|
\end{aligned}$$

where λ_{pk} and λ_{nl} denotes the positive and negative eigenvalues of ρ^T_{AB} respectively. As $\mathrm{Tr}\,\rho^T_{AB} = 1$ i.e., $\sum_i |\lambda_i| = 1$, it can readily be shown that

$$\sum_k \lambda_{pk} - \sum_l |\lambda_{nl}| = 1$$

and this leads to

$$\begin{aligned}
\|\rho^T_{AB}\| &= 1 + 2\sum_l |\lambda_{nl}| = 1 + 2\,N(\rho) \\
N(\rho) &= \sum_l |\lambda_{nl}| = \text{Sum of absolute values of negative eigenvalues of } \rho^T.
\end{aligned}$$

Here $\|\rho^T\|$ denotes the trace norm of the partial transpose ρ^T and a useful quantity, called *negativity of partial transpose* is constructed [25]. Negativity of Partial Transpose measures the degree to which ρ^T, the partial transpose of the density matrix of a bipartite system fails to be positive. It is denoted by $N(\rho)$ and is given by

$$N(\rho) = \frac{\|\rho^T\| - 1}{2}. \tag{1.6}$$

$N(\rho)$ corresponds to the sum of absolute values of negative eigenvalues of the partially transposed density matrix ρ^T and is shown to be a good measure of entanglement [25] for mixed bipartite systems. It can be readily seen that Bell states are the maximally entangled pure states with $N(\rho) = 1/2$, the maximum value of $N(\rho)$ for two qubit states. In this thesis, while evaluating $N(\rho)$ of any state under consideration, Eq. (1.6) is made use of.

1.2 Entropic Characterization of Separability: A brief review

In recent years a surge of activity has been noticed towards the characterization of entanglement in bipartite quantum systems. Among various methods, the entropic characterization of separability of composite quantum systems has gathered significant attention [26–41]. In this section, a brief review on entropic characterization of separability of composite quantum systems is given.

In classical information theory, the uncertainty associated with the values that a random variable X can take, is given by the Shannon entropy. If X takes values $x_1, x_2, \ldots x_n$ with corresponding probabilities p_1, $p_2, \ldots p_n$, the Shannon entropy associated with this probability distribution is defined as [1]

$$H(X) = H(p_1, p_2 \ldots p_n) = -\sum_i p_i \log_2 p_i$$

Here, logarithms are taken to base two and it is defined that $0 \log_2 0 \equiv 0$. The quantum version of the Shannon entropy, the so-called *v*on-Neumann entropy [8, 42], is described in a similar fashion, with density operators replacing probability distribution. That is, for a quantum state ρ the von Neumann entropy is defined as

$$S(\rho) = -\text{Tr}(\rho\ \log_2\ \rho) \tag{1.7}$$

If λ_i are the eigenvalues of ρ then the von Neumann entropy can be re-expressed as

$$S(\rho) = -\sum_i \lambda_i\ \log_2\ \lambda_i \tag{1.8}$$

and is useful for evaluating the entropy of any given state ρ. For instance, the completely mixed state I_d/d of a qudit, I_d being a $d \times d$ identity matrix, has all its d eigenvalues equal, given by $\lambda_i = \frac{1}{d}$. Thus, the maximum possible entropy of a qudit, the entropy of a maximally mixed state I_d/d is $\log_2\ d$

$$S(I_d/d) = -d\,\frac{1}{d}\ \log_2 \frac{1}{d} = -\log_2 \frac{1}{d} = \log_2\ d$$

When $d = 2$, the maximum possible von-Neumann entropy for a qubit turns out to be $\log_2\ 2 = 1$.

It is to be observed that a pure state is a perfectly ordered state because in the density matrix representation of any quantum state, a pure state $|\psi\rangle$ is represented by $\rho_{\text{pure}} = |\psi\rangle\langle\psi|$ and hence occurs with probability 1. A pure state therefore has only one non-zero eigenvalue $\lambda = 1$. Thus the von Neumann entropy of a pure state is $\log_2\ 1 = 0$. It can therefore be concluded that the definition of von-Neumann entropy gives physically intuitive results that a pure state has zero entropy whereas a totally random state I_d/d has maximum entropy of $\log_2\ d$.

Further for a composite quantum system, by analogy with the Shannon entropy, it is possible to define quantum joint and conditional entropies. The joint entropy $S(A, B)$ for a composite system with two components A and B is defined as [1],

$$S(A,\,B) \equiv -\text{Tr}\ (\rho_{AB}\ \log_2\ \rho_{AB}) \tag{1.9}$$

where ρ_{AB} is the density matrix of the composite system with subsystems ρ_A, ρ_B. $S(A,\,B)$ quantifies the disorder or randomness in the bipartite state ρ_{AB}.

Quite similar to the definitions

$$H(A|B) = H(A,\,B) - H(B), \quad H(B|A) = H(A,\,B) - H(A)$$

of conditional Shannon entropies, the corresponding quantum conditional entropies for the composite system AB are defined as [1]

$$S(A|B) \equiv S(A,B) - S(B) \quad \text{and} \quad S(B|A) \equiv S(A,B) - S(A) \tag{1.10}$$

While $S(A|B)$ quantifies the information content of the composite system ρ_{AB} which is not contained in its subsystem ρ_B, $S(B|A)$ quantifies the information content in ρ_{AB} not accounted for by the information contained in the subsystem ρ_A.

The concept of joint and conditional entropies combined with the definition of separability leads to a necessary and sufficient criterion for entanglement for bipartite pure states. In fact, by the definition of bipartite separable pure states as

$$|\psi_{AB}^{sep}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$$

it can be readily seen that the subsystems of a separable pure state also correspond to pure states thus having zero subsystem entropies. In view of the fact that $S(|\psi_{AB}^{sep}\rangle) = S(A,\, B) = 0$ and as $S(|\psi_A\rangle) = S(A) = 0$, $S(|\psi_B\rangle) = S(B) = 0$, the conditional entropies $S(A,\, B) - S(A) = S(A,\, B) - S(B)$ of the separable pure state $|\psi_{AB}^{sep}\rangle$ are zero. It can thus be concluded that if the subsystem entropies of a pure state are non-zero (possible only when the subsystems are mixed states), then the state is entangled. The conditional entropy $S(A,\, B) - S(A) = S(A,\, B) - S(B)$ of an entangled pure state is therefore negative as $S(A,\, B) = 0$, $S(A) = S(B) \neq 0$. Here, the fact that $S(A) = S(B)$ follows from the possibility of Schmidt decomposition for bipartite pure states [1]. Negative conditional entropy is therefore a necessary and sufficient criterion[2] for bipartite pure states to be entangled.

For instance, consider the Bell state $|\phi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and its density matrix

$$\rho_{AB} = |\phi_1\rangle\langle\phi_1| = \frac{(|00\rangle + |11\rangle)(\langle 00| + \langle 11|)}{2}$$

[2]The definition of von-Neumann entropy of any of the reduced density matrices of a bipartite pure state as a measure of entanglement is based on this criterion. More mixedness implies larger entropy in the subsystem implying larger entanglement in the pure state.

The partial trace operation over one of the subsystems is nothing but taking inner product over the basis states of the subsystem being traced out. Explicitly, the partial trace operation over the first subsystem can be obtained using

$$\mathrm{Tr}_1 |a_i b_j\rangle\langle a_r b_s| = \langle a_i|a_r\rangle |b_j\rangle\langle b_s| = \delta_{ir} |b_j\rangle\langle b_s|$$

where $|a\rangle$, $|b\rangle$ denote the orthonormal basis states of the first, second subsystems respectively, $\langle\ |\ \rangle$ denotes the inner product and δ denoting the *Kronecker delta*. One can readily obtain the reduced density matrix ρ_B of ρ_{AB} as

$$\begin{aligned}
\rho_B &= \mathrm{Tr}_A\, \rho_{AB} = \frac{1}{2}\mathrm{Tr}_A\, [(|00\rangle + |11\rangle)(\langle 00| + \langle 11|)] \\
&= \frac{1}{2}\mathrm{Tr}_A\, [|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|] \\
&= \frac{1}{2}(\langle 0|0\rangle\, |0\rangle\langle 0| + \langle 0|1\rangle\, |0\rangle\langle 1| + \langle 1|0\rangle\, |1\rangle\langle 0| + \langle 1|1\rangle\, |1\rangle\langle 1|) \\
&= \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{I_2}{2}
\end{aligned}$$

One can similarly obtain $\rho_A = I_2/2$ implying that the subsystems of the Bell state $|\phi_1\rangle$ correspond to maximally mixed states. In fact, all the four Bell states have maximally disordered subsystems $I_2/2$. The subsystems of all the Bell states thus have maximum entropy $S(A) = S(B) = 1$ leading to the conditional entropies

$$\begin{aligned}
S(A|B) &= S(A, B) - S(B) = -S(B) = -1, \\
S(B|A) &= S(A, B) - S(A) = -S(A) = -1
\end{aligned}$$

owing to the fact that $S(A, B)$, the entropy of the pure Bell states are zero. In general, an arbitrary pure entangled state satisfies the inequality[3]

$$S(B|A) = S(A, B) - S(A) \leq 0 \tag{1.11}$$

[3]Notice that $S(A|B) = S(B|A)$ for pure states as $S(A, B) = 0$ and as $S(A) = S(B)$ owing to Schmidt decomposition.

reflecting the remarkable fact that pure entangled states are more disordered locally than globally.

While the pure separable states have zero conditional entropies, it is of interest to examine the nature of conditional entropies of mixed separable states. It has been observed that the subsystems of a mixed composite separable state $\rho_{AB}^{sep} = \sum_i p_i \, (\rho_A \otimes \rho_B)$ with $0 \leq p_i \leq 1$, $\sum_i p_i = 1$ are more ordered than the whole system [40, 43], i.e., $S(\rho_{AB}^{sep}) \geq S(\rho_A), S(\rho_B)$ leading to non-negative conditional entropies. This makes intuitive sense because composite separable states (whether pure or mixed) can be thought of as classical systems with the whole system having more entropy (global entropy) than its subsystems (local entropy). But this physical intuition cannot be extended to conclude that all entangled states must have their subsystem entropies greater than the global system entropy leading to negative conditional entropies. This is because, though all pure entangled states have negative conditional entropies, not all mixed entangled states satisfy the relations $S(\rho_{AB}^{ent}) \leq S(\rho_A), S(\rho_B)$. It is to be observed here that while negative conditional entropy of a composite state definitely implies that the state is entangled, it is not possible to conclude that all composite mixed states with non-negative conditional entropies are separable. Negative conditional entropies, implied by the inequality Eq.(1.11), provide sufficient (but not necessary) criterion to characterize mixed entangled states. This fact has been illustrated through the example of the two-qubit Werner state.

In the case of two qubit Werner state (See Eq. (1.5))

$$\rho_w = \frac{(1-x)I_4}{4} + x|\phi_1\rangle\langle\phi_1|, \quad 0 \leq x \leq 1$$

the conditional entropy is positive when $0 \leq x \leq 0.747$. But this range of separability is weaker compared to $0 \leq x \leq \frac{1}{3}$ obtained through Peres partial transpose criterion [21, 23], which is a necessary and sufficient criteria for entanglement in 2×2 and 2×3 systems. This example brings out the limitation of the entropic criterion in characterizing entanglement in mixed composite states. Generalized entropic measures [31–39, 44] provide more sophisticated tools to explore global vs local disorder in mixed states and lead to more stringent limitation on separability than that obtained using positivity of the conditional von Neumann entropy. In this context, the quantum counterparts of Rényi entropy [31, 32]

$$S_q^R(\rho) = \frac{1}{1-q} \log_2 \, \mathrm{Tr}[\rho^q] \tag{1.12}$$

and Tsallis entropy [45, 46]

$$S_q^T(\rho) = \frac{1}{1-q} \left(\mathrm{Tr}[\rho^q] - 1\right) \tag{1.13}$$

have often been employed. In the limit $q \to 1$ both these generalized entropies reduce to the von Neumann entropy. Horodecki et al. [31, 32] recognized that

$$S_q^R(\rho_{AB}^{sep}) \geq S_q^R(\rho_A), \;\; S_q^R(\rho_B) \tag{1.14}$$

for separable states showing that negative values of the conditional Rényi entropy $S_q^R(B|A) = S_q^R(\rho_{AB}) - S_q^R(\rho_A)$ is a signature of quantum entanglement in mixed composite states.

On the other hand, Abe and Rajagopal [33, 44] argued that the limitation of the von Neumann entropic criterion in characterizing entanglement, in mixed composite states, is a consequence of direct generalization of the von Neumann conditional entropy from its classical counterpart, the Shannon conditional entropy. The summary of their argument is as follows.

The definition of classical Shannon conditional entropy is based on classical conditional probability distribution. That is, the Shannon conditional entropy

$$H(B|A) \equiv -\sum_j p_{ij}(B|A) \, \log_2 \, p_{ij}(B|A) \tag{1.15}$$

with $p_{ij}(B|A) = p_{ij}(A,B)/p_i(A)$ is the conditional probability of B in its j^{th} state with A found in its i^{th} state. Here $p_i(A)$ is the marginal probability distribution i.e., $p_i(A) = \sum_j p_{ij}(A,B)$. From Eq.(1.15) one can obtain the Shannon conditional entropy $H(B|A) = H(A,B) - H(A)$.

In the particular case when A and B are statistically independent, $p_{ij}(B|A)$ is equal to $p_j(B)$ and therefore $H(B|A) = H(B)$, implying the additivity law

$H(A,B) = H(A) + H(B)$. Notice that

$$H(A,B) \equiv -\sum_{ij} p_{ij}(A,B) \log_2 p_{ij}(A,\, B) \tag{1.16}$$

and there is a natural correspondence relation between multiplication law and the additivity law:

$$p_{ij}(A,B) = p_i(A)\; p_{ij}(B|A) \Leftrightarrow S(A,B) = S(A) + S(B|A). \tag{1.17}$$

At this point Abe and Rajagopal (in Ref. [33]) argued that there is a profound difference between classical and quantum probability concepts. They observed that in the quantum scenario, one of Kolmogorov's axioms, namely the additivity of the probability measure, is violated in general [47]. Thus, Abe and Rajagopal suggested that the measure of quantum entanglement may not be additive. Further, their argument is supported by the theoretical observations [24, 48] that formal correspondences exist between thermodynamics and quantum entanglement. As nonadditivity or nonextensivity, is an important concept in the field of statistical thermodynamics it should also appear in quantum mechanics. A statistical system is nonextensive if it contains long-range interaction, long-range memory, or (multi)fractal structure. In such a system, a nonextensive generalization of Boltzmann-Gibbs statistical mechanics is formulated by Tsallis [45, 46]. In this formalism, referred to as nonextensive statistical mechanics, the Shannon entropy in Eq. (1.16) is generalized as follows:

$$S_q^T(A,B) = \frac{1}{1-q}\left\{\sum_{ij}[p_{ij}(A,B)]^q - 1\right\} \tag{1.18}$$

where q is a positive parameter. The quantity converges to the Boltzmann Shannon entropy in the limit $q \to 1$. From this, Tsallis entropy in Eq.(1.18), a nonextensive conditional entropy is obtained as follows

$$\begin{aligned} S_q(B|A_i) &= \frac{1}{1-q}\left[\sum_j (p_{ij}(B|A))^q - 1\right] \\ &= \frac{1}{1-q}\left[\frac{\sum_{ij}(p_{ij}(A,B))^q}{\sum_i (p_i(A))^q} - 1\right]. \end{aligned} \tag{1.19}$$

The equation for the joint system entropy is thus obtained as

$$S_q(A,B) = S_q(A) + S_q(B|A) + (1-q)S_q(A)S_q(B|A), \tag{1.20}$$

which is nonadditive in nature.

Based on Tsallis entropy and the form invariant structures of Khinchin's axioms, Abe and Rajagopal [33, 44] generalized the concept of statistical conditional entropy in Eq. (1.19) to its quantum version as

$$S_q^T(B|A) = \frac{1}{1-q}\left(\frac{\mathrm{Tr}[\rho_{AB}^q]}{\mathrm{Tr}[\rho_A^q]} - 1\right) \tag{1.21}$$

The Abe-Rajagopal (AR) q-conditional entropy, given by Eq.(1.21) is non-negative for a separable state, but may assume negative value in a quantum entangled state, suggesting its importance in the characterization of quantum entanglement in mixed composite states. In fact, the superiority of Abe-Rajagopal q-conditional entropy in the limit $q \longrightarrow \infty$ over the von-Neumann conditional entropy is seen through the identification of separability range in the two-qubit Werner state. While the von-Neumann conditional entropy yielded the weaker separability range $0 \leq x \leq 0.747$ for two-qubit Werner state, it was found that [33] $\lim_{q\longrightarrow\infty} S_q^T(B|A)$ is non-negative in the range $0 \leq x \leq \frac{1}{3}$ thus matching with the strictest possible separability range obtained using Peres-Horodecki criterion.

Prabhu et al. [40] employed Eq.(1.21) to find out the separability range in N-qubit symmetric one parameter family of noisy states involving W and GHZ states [49, 50] in their different partitions. The symmetric N-qubit noisy mixed states considered by Prabhu et al [40] are given by

$$\rho_N^{(\mathrm{W/GHZ})}(x) = \left(\frac{1-x}{N+1}\right) P_N + x|\psi_{\mathrm{W/GHZ}}\rangle_N\langle\psi_{\mathrm{W/GHZ}}|. \tag{1.22}$$

where $|\psi_{\mathrm{W/GHZ}}\rangle$ is the N qubit W [51] or GHZ state [52, 53] and P_N is the projector onto the symmetric subspace of the collective angular momentum. They found that the separability range obtained by using AR criterion, in the limit $q \to \infty$, is stricter compared to the one obtained using traditional von Neumann entropy. Using the AR criterion it was found that the $1 : N-1$ separability range of one parameter family of mixed GHZ states

$$\rho_N^{\mathrm{GHZ}}(x) = \left(\frac{1-x}{N+1}\right) P_N + x|\psi_{\mathrm{GHZ}}\rangle_N\langle\psi_{\mathrm{GHZ}}|$$

matches with the one obtained by employing PPT criterion, the strictest available separability criterion. But for the one-parameter family of W states

$$\rho_N^{\mathrm{W}}(x) = \left(\frac{1-x}{N+1}\right) P_N + x|\psi_{\mathrm{W}}\rangle_N\langle\psi_{\mathrm{W}}|,$$

the PPT criterion provides a stricter separability range than that through AR criterion. Thus, for the state $\rho_N^{\mathrm{W}}(x)$, though AR criterion gives a better separability range than the one obtained using von-Neumann conditional entropy, it is found to be weaker compared to the PPT criterion [40].

Sumiyoshi Abe [35] also employed AR-criterion to find out the separability range of one parameter family of asymmetric d-dimensional, N-partite Werner-Popescu-type of states [19, 20, 35] in their $1 : N-1$ partition. It was observed in Ref. [35] that the $1 : N-1$ separability range of Werner-Popescu states obtained by AR-criterion is stricter than that obtained using von Neumann conditional entropy and matches with the separability range obtained by the algebraic method, a necessary and sufficient criterion [54, 55].

Recently, a different quantum generalization of Rényi relative entropy was introduced [56, 57], termed sandwiched Rényi relative entropy, and there has been a surge of activity in establishing several properties of this recent version of Rényi entropy [56–60]. The sandwiched Rényi relative entropy is defined as [56, 57]

$$D_q^R(\rho||\sigma) = \frac{\log_2\left[\mathrm{Tr}\left\{\left(\sigma^{\frac{1-q}{2q}}\,\rho\,\sigma^{\frac{1-q}{2q}}\right)^q\right\}\right]}{1-q} \tag{1.23}$$

This generalized Rényi relative entropy of a pair of density operators (ρ, σ) reduces to the traditional one when the two density operators commute with each other [61]. It is thus natural to anticipate that this quantity is more effective than its traditional version when non-commuting density matrices are involved.

The literature survey on entropic characterization of entanglement summarized above provided the motivation for the work detailed in this thesis. In fact, the observation that the AR-criterion defined using traditional Tsallis entropy does not yield separability ranges matching with that through PPT criterion in symmetric one-parameter families of noisy states involving W states urges one to think of a better entropic separability criterion than AR criterion. The definition of non-commuting version of Rényi relative entropy led to the identification of an analogous non-commuting form of Tsallis relative entropy and its conditional version is also arrived at. The entropic separability criterion based on the conditional version of sandwiched Tsallis relative entropy (CSTRE criterion) forms the basis of the investigations carried out in the present thesis. Several one parameter families of N qubit mixed symmetric and nonsymmetric states are investigated using this newly defined entropic separability criterion. The nonspectral nature of this criterion is established through identifying entanglement in an isospectral state [43]. Prompted by the nonspectral nature of the CSTRE criterion, an attempt to identify entanglement in bound entangled states has been carried out. It has been established in this thesis that the entropic separability criterion based on conditional version of sandwiched Tsallis relative entropy fares better than the AR-criterion. It is shown that due to non-commuting nature of a composite density matrix and its subsystem density matrices, CSTRE criterion is the best available entropic separability criteria yielding strictest separability ranges in several one parameter families of noisy states.

1.3 Outline of thesis

The thesis is divided into seven chapters including the Introductory chapter which gives the outline of the thesis and concluding chapter which provides a comprehensive summary of the results.

The introductory chapter gives an outline of the concept of quantum entanglement and its characterization through different criteria. Along with a short review of literature on entropic separability criteria, a brief outline of the thesis is provided in this chapter.

In the second chapter, the quantum relative Tsallis entropy of two non-commuting density matrices is defined and its conditional version is arrived at. It has been established that the negative values of the Conditional Sandwiched Tsallis Relative Entropy (CSTRE) necessarily imply entanglement in bipartite states. The separability range in all possible bipartitions of *symmetric* noisy one-parameter family of W-, Greenberger-Horne-Zeilinger(GHZ)- and equal superposition of W, obverse W states ($\mathrm{W\bar{W}}$) with three and four qubits is explored in this chapter. It is shown that the results inferred from negative values of CSTRE in a particular bipartition matches with that obtained through Peres's partial transpose criterion. The non-commuting nature of CSTRE and its supremacy over its commuting version, the Abe-Rajagopal criterion, is also established.

In Chapter 3, the exploration of bipartite separability ranges in three-, four-qubit *symmetric* one parameter families of noisy states involving W-, GHZ-, $\mathrm{W\bar{W}}$ states using CSTRE criterion is extended to corresponding N qubit states, in their $1 : N-1$ partition. It is shown that $1 : N-1$ CSTRE separability range matches exactly with the range obtained through PPT criterion, for all N. The advantages of using non-commuting version of q-conditional relative Tsallis entropy are clearly brought out through the identification of bipartite separability ranges in *symmetric* one parameter families of noisy states.

In the fourth chapter, the CSTRE criterion is employed to determine the $1 : N-1$ separability range in the *non-symmetric* noisy one-parameter families of pseudopure and Werner-like N-qubit states containing W-, GHZ- states. For both these non-symmetric families of N-qubit mixed states, the $1 : N-1$ CSTRE separability range is explicitly obtained for any N and it is seen to be in perfect agreement with the necessary and sufficient criterion as well as PPT separability range for these states.

In Chapter 5, CSTRE criterion is employed to study the bipartite separability of one parameter family of Werner-Popescu state containing N- qudits, in its $1 : N-1$ partition. For all N, the $1 : N-1$ separability range is found to match exactly with the corresponding separability range obtained using a necessary and sufficient condition based on an algebraic method. The superiority of using CSTRE criterion over Abe-Rajagopal (AR) criterion is shown by comparing the convergence of the parameter x as $q \longrightarrow \infty$ in the implicit plots of CSTRE and AR- q conditional entropy.

Chapter 6 gives an account of the non-spectral nature of the CSTRE criterion which differentiates it from the other entropic criteria. With the knowledge that only non-spectral witnesses can help in the identification of bound entangled states, an attempt has been made to identify entanglement in bound entangled states using CSTRE criterion and the details of this investigation is given in Chapter 6.

Finally, in the seventh chapter, the concluding remarks of the investigations carried out in the thesis are given along with a lead to future directions of research in this area.

The end of the thesis contains a Bibliography, the list of Publications and a list of Seminars/Conferences/Workshops participated during the period of research.The Bibliography contains a list of all reference materials cited in the thesis.

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Chapter 2

A new entropic separability criterion using Conditional version of Sandwiched Relative Tsallis Entropy

In this chapter, a non-commuting version (the so-called sandwiched version) of Tsallis relative entropy is introduced [62]. Using this sandwiched Tsallis relative entropy, its conditional form is identified [62]. It is established that whenever the conditional version of sandwiched Tsallis relative entropy (CSTRE) is negative, the state under consideration is entangled [63] indicating its usefulness in detecting entanglement when a density matrix and its marginals are non-commuting [62]. The CSTRE is shown to reduce to the Abe-Rajagopal (AR) q-conditional entropy when the reduced density matrix of a state is maximally mixed.

The contents of the chapter are organized into five sections. Sec. 2.1 introduces the sandwiched relative Tsallis entropy and its conditional version. It is proved that the negative values of this conditional form necessarily imply quantum entanglement. The various separability ranges in symmetric one parameter families of three- and four-qubit noisy states involving W-, Greenberger-Horne-Zeilinger (GHZ) states and an equal superposition of W-, obverse W states are identified through the conditional form of the sandwiched Tsallis relative entropy in Sections

2.2, 2.3, 2.4. A comparison of these separability ranges with those obtained using von-Neumann conditional entropy, Abe-Rajagopal q-conditional entropy and Peres partial transpose criterion is carried out for each state under consideration. Section 2.5 contains a summary of the results in the chapter.

2.1 Sandwiched Tsallis Relative Entropy and its Conditional Version

The generalized entropies, the Rényi and Tsallis entropies, denoted respectively by $S_q^R(\rho)$, $S_q^T(\rho)$ are given by [31–33, 45, 46]

$$\begin{aligned} S_q^R(\rho) &= \frac{1}{1-q}\log \mathrm{Tr}[\rho^q] \\ S_q^T(\rho) &= \frac{\mathrm{Tr}[\rho^q]-1}{1-q}. \end{aligned}$$

Here, q is a real positive parameter. Both these reduce to von Neumann entropy in the limit $q \to 1$.

The traditional quantum relative Rényi entropy for a pair of density operators ρ and σ is defined, by ignoring the ordering of the density matrices, as [61]

$$\begin{aligned} D_q^R(\rho||\sigma) &= \frac{\log \mathrm{Tr}\left(\rho^q\sigma^{1-q}\right)}{q-1} \quad \text{if } q \in (0,1)\cup(1,\infty) \\ &= \mathrm{Tr}\left[\rho(\log\rho - \log\sigma)\right] \quad \text{when} \quad q \to 1; \end{aligned} \tag{2.1}$$

Recently a generalized version of quantum relative Rényi entropy was introduced by Wilde et al. [56] and Müller-Lennert et al. [57] independently:

$$\tilde{D}_q^R(\rho||\sigma) = \frac{1}{q-1}\log \mathrm{Tr}\left[\left(\sigma^{\frac{1-q}{2q}}\rho\sigma^{\frac{1-q}{2q}}\right)^q\right] \quad \text{when } q \in (0,1)\cup(1,\infty). \tag{2.2}$$

The quantum relative Rényi entropy in Eq. (2.2) reduces to the traditional one given in Eq.(2.1) when the density matrices ρ and σ commute and hence the new version is an extension to non-commutative case.

It can be recalled that the traditional form of Tsallis relative entropy is given by

$$D_q^T(\rho||\sigma) = \frac{\mathrm{Tr}\left(\rho^q \sigma^{1-q}\right) - 1}{q-1}, \tag{2.3}$$

Quite analogous to the non-commuting form of relative Rényi entropy given in Eq. (2.2), the non-commuting or *sandwiched* Tsallis relative entropy is defined and is given by [62],

$$\tilde{D}_q^T(\rho||\sigma) = \frac{\mathrm{Tr}\left\{\left(\sigma^{\frac{1-q}{2q}}\rho\,\sigma^{\frac{1-q}{2q}}\right)^q\right\} - 1}{q-1} \tag{2.4}$$

It can be verified that when $\sigma = I$, I being the identity matrix, the sandwiched Tsallis relative entropy given in Eq. (2.4) reduces to the Tsallis entropy $S_q^T(\rho)$. Also, in the limit $q \to 1$, it reduces to the von-Neumann relative entropy [1]. That is,

$$\lim_{q\to 1} \tilde{D}_q^T(\rho||\sigma) = \mathrm{Tr}(\rho(\log\rho - \log\sigma)).$$

In order to make use of the sandwiched Tsallis relative entropy $\tilde{D}_q^T(\rho||\sigma)$ to detect entanglement in a bipartite state ρ_{AB}, its conditional version is to be defined. This can be accomplished by taking $\rho = \rho_{AB}$ and $\sigma \equiv I_A \otimes \rho_B$ (or $\rho_A \otimes I_B$) in Eq. (2.4) with $\rho_B = \mathrm{Tr}_A[\rho_{AB}]$ ($\rho_A = \mathrm{Tr}_B[\rho_{AB}]$) being the subsystem density matrices of the bipartite state ρ_{AB} [62]. The conditional versions of sandwiched Tsallis relative entropy (CSTRE) are given by [62]

$$\tilde{D}_q^T(\rho_{AB}||\rho_B) = \frac{\mathrm{Tr}\left\{\left[(I_A\otimes\rho_B)^{\frac{1-q}{2q}}\rho_{AB}(I_A\otimes\rho_B)^{\frac{1-q}{2q}}\right]^q\right\} - 1}{1-q} \tag{2.5}$$

$$\tilde{D}_q^T(\rho_{AB}||\rho_A) = \frac{\mathrm{Tr}\left\{\left[(\rho_A\otimes I_B)^{\frac{1-q}{2q}}\rho_{AB}(\rho_A\otimes I_B)^{\frac{1-q}{2q}}\right]^q\right\} - 1}{1-q} \tag{2.6}$$

[1] It may be noted that when $q = 1/2$, Eq. (2.4) reduces to $\tilde{D}_{\frac{1}{2}}^T(\rho||\sigma) = 2(F(\rho|\sigma) - 1)$, the negative of the Bures metric, with $F(\rho|\sigma) = \mathrm{Tr}\left[\left(\sigma^{\frac{1}{2}}\rho\sigma^{\frac{1}{2}}\right)^{\frac{1}{2}}\right]$ denoting the fidelity [1].

In a more concise manner, one can write $\tilde{D}_q^T(\rho_{AB}||\rho_B)$ as,

$$\tilde{D}_q^T(\rho_{AB}||\rho_B) = \frac{\tilde{Q}_q(\rho_{AB}||\rho_B) - 1}{1-q} \tag{2.7}$$

where

$$\tilde{Q}_q(\rho_{AB}||\rho_B) = \mathrm{Tr}\left\{\left[(I_A \otimes \rho_B)^{\frac{1-q}{2q}}\rho_{AB}(I_A \otimes \rho_B)^{\frac{1-q}{2q}}\right]^q\right\}. \tag{2.8}$$

While the evaluation of the expression $\tilde{Q}_q(\rho_{AB}||\rho_B)$ does not seem trivial, construction of the unitary matrix that diagonalizes the subsystem density matrix ρ_B makes the calculation a feasible one. The details of evaluation of $\tilde{Q}_q(\rho_{AB}||\rho_B)$ are as given below:

Let U_B be the unitary matrix that diagonalizes ρ_B i.e., let

$$U_B\rho_B U_B^\dagger = \mathrm{diag}\,(\lambda_1,\,\lambda_2,\ldots,\lambda_n) \tag{2.9}$$

where λ_i, $i = 1,\,2,\,3\ldots,n$ are the eigenvalues of ρ_B. In view of the fact that

$$(I_A \otimes \rho_B)^{\frac{1-q}{2q}} \quad = \quad I_A^{\frac{1-q}{2q}} \otimes \rho_B^{\frac{1-q}{2q}} = I_A \otimes \rho_B^{\frac{1-q}{2q}} \quad \text{for any } q,$$

one has,

$$\begin{aligned}(I_A \otimes U_B)(I_A \otimes \rho_B)^{\frac{1-q}{2q}}(I_A \otimes U_B)^\dagger &= I_A \otimes U_B\rho_B^{\frac{1-q}{2q}}U_B^\dagger \\ &= I_A \otimes [\mathrm{diag}\,(\lambda_1,\,\lambda_2,\ldots,\lambda_n)]^{\frac{1-q}{2q}} \qquad (2.10) \\ &= I_A \otimes \mathrm{diag}\left(\lambda_1^{\frac{1-q}{2q}},\,\lambda_2^{\frac{1-q}{2q}},\ldots,\lambda_n^{\frac{1-q}{2q}}\right).\end{aligned}$$

On denoting

$$\Gamma = (I_A \otimes \rho_B)^{\frac{1-q}{2q}}\rho_{AB}(I_A \otimes \rho_B)^{\frac{1-q}{2q}} \tag{2.11}$$

one can write

$$\Gamma = (I_A\otimes\rho_B)^{\frac{1-q}{2q}}(I_A\otimes U_B)^\dagger\,(I_A\otimes U_B)\rho_{AB}\,(I_A\otimes U_B)^\dagger\,(I_A\otimes U_B)(I_A\otimes\rho_B)^{\frac{1-q}{2q}}. \tag{2.12}$$

Notice that Γ is unitarily equivalent to Γ_U where

$$\Gamma_U = (I_A \otimes U_B)\Gamma(I_A \otimes U_B)^\dagger \tag{2.13}$$

and Γ, Γ_U have same eigenvalues. Observing that

$$(I_A \otimes U_B)(I_A \otimes \rho_B)^{\frac{1-q}{2q}}(I_A \otimes U_B)^\dagger = I_A \otimes \mathrm{diag}\left(\lambda_1^{\frac{1-q}{2q}}, \lambda_2^{\frac{1-q}{2q}}, \ldots, \lambda_n^{\frac{1-q}{2q}}\right)$$

and in view of Eqs. (2.12), (2.13), it can be seen that

$$\begin{aligned} \Gamma_U &= \left\{I_A \otimes \mathrm{diag}\left(\lambda_1^{\frac{1-q}{2q}}, \lambda_2^{\frac{1-q}{2q}}, \ldots, \lambda_n^{\frac{1-q}{2q}}\right)\right\}(I_A \otimes U_B)\rho_{AB}\,(I_A \otimes U_B)^\dagger \\ &\quad \left\{I_A \otimes \mathrm{diag}\left(\lambda_1^{\frac{1-q}{2q}}, \lambda_2^{\frac{1-q}{2q}}, \ldots, \lambda_n^{\frac{1-q}{2q}}\right)\right\}. \end{aligned} \tag{2.14}$$

In fact, the construction of the unitary matrix U_B using the orthonormal eigenvectors of ρ_B and knowledge of the eigenvalues λ_i allows one to evaluate Γ_U and because of the unitary equivalence with Γ, the eigenvalues of Γ_U are those of Γ. It is not difficult to see from Eqs.(2.8), (2.11), (2.13) that

$$\tilde{Q}_q(\rho_{AB}||\rho_B) = \mathrm{Tr}\,(\Gamma)^q = \mathrm{Tr}\,(\Gamma_U)^q$$

Thus an evaluation of the eigenvalues γ_i of Γ_U immediately leads us to the quantity

$$\tilde{Q}_q(\rho_{AB}||\rho_B) = \sum_{i=1}^{d} \gamma_i^q, \quad d = \text{dimension of } \rho_{AB}. \tag{2.15}$$

Finally an expression for the conditional form of sandwiched Tsallis relative entropy[2] (CSTRE) $\tilde{D}_q^T(\rho_{AB}||\rho_B)$ is obtained [62] as

$$\tilde{D}_q^T(\rho_{AB}||\rho_B) = \frac{\sum_{i=1}^{d} \gamma_i^q - 1}{1-q}. \tag{2.16}$$

Having defined the CSTRE $\tilde{D}_q^T(\rho_{AB}||\rho_B)$ in an operational manner (Eqs. (2.12), (2.13), (2.15), (2.16)), the next task is to identify its use in detecting entanglement. At this juncture, it is important to notice that the sandwiched conditional Tsallis entropy (See Eq. (2.5)) reduces to AR q-conditional Tsallis entropy [33]

$$S_q^T(A|B) = \frac{1}{q-1}\left(1 - \frac{\mathrm{Tr}(\rho_{AB}^q)}{\mathrm{Tr}(\rho_B^q)}\right)$$

when the subsystem density matrix is a maximally mixed state[3]. It is well known [33–40] that negative values of AR q-conditional entropy indicate entanglement in bipartite states and the so-called AR-criterion based on this fact has been employed as a separability criterion for several classes of composite states [33–40]. Thus, for CSTRE $\tilde{D}_q^T(\rho_{AB}||\rho_B)$ to be useful in detecting entanglement, one needs to prove that whenever $\tilde{D}_q^T(\rho_{AB}||\rho_B)$ is negative, the state ρ_{AB} corresponds to an entangled state. The following theorem establishes this fact.

---

[2]The expression for $\tilde{D}_q^T(\rho_{AB}||\rho_A)$ can be obtained in an analogous manner and it is given by

$$\tilde{D}_q^T(\rho_{AB}||\rho_A) = \frac{\sum_{i=1}^{d} \omega_i^q - 1}{1-q}$$

where ω_i are the eigenvalues of the matrix

$$\Omega_U = (U_A \otimes I_B)\Omega(U_A \otimes I_B)^\dagger$$

which is unitarily equivalent to

$$\Omega = (\rho_A \otimes I_B)^{\frac{1-q}{2q}} \rho_{AB} (\rho_A \otimes I_B)^{\frac{1-q}{2q}}.$$

Here U_A is the matrix that diagonalizes the subsystem density matrix ρ_A of ρ_{AB}.

[3]That is, $\tilde{D}_q^T(\rho_{AB}||\rho_B) \equiv S_q^T(A|B)$ when ρ_B is maximally mixed and $\tilde{D}_q^T(\rho_{AB}||\rho_A) \equiv S_q^T(B|A)$ when ρ_A is maximally mixed.

2.1.1 Sufficient condition for quantum entanglement in terms of conditional version of sandwiched Tsallis relative entropy

Theorem: *Negative values of the conditional version of the sandwiched Tsallis relative entropy (CSTRE)* $\tilde{D}_q^T(\rho_{AB}||\rho_B)$ *with* $q > 1$ **necessarily imply entanglement** *in the state* ρ_{AB} [63].

Proof: For any two positive semi-definite operators ρ and σ, the trace functional [64] $\tilde{Q}_q(\rho||\sigma) = \mathrm{Tr}\left\{\left[\sigma^{\frac{1-q}{2q}}\rho\,\sigma^{\frac{1-q}{2q}}\right]^q\right\}$ satisfies the inequality [64]

$$\tilde{Q}_q(\rho||\sigma) \leq \tilde{Q}_q(\rho||\rho) \;\; \text{for} \;\; q>1 \;\; \text{whenever} \;\; \rho \leq \sigma. \tag{2.17}$$

Notice that when ρ is a density matrix, $\tilde{Q}_q(\rho||\rho) = \mathrm{Tr}\,\rho = 1$ implying that

$$\tilde{Q}_q(\rho||\sigma) \leq 1 \;\; \text{when} \;\; \rho \leq \sigma \;\; \text{and} \;\; q > 1. \tag{2.18}$$

With $\rho = \rho_{AB}$, $\sigma = I_A \otimes \rho_B$ and denoting

$$\tilde{Q}_q(\rho_{AB}||I_A \otimes \rho_B) \equiv \tilde{Q}_q(\rho_{AB}||\rho_B),$$

Eq. (2.18) gives

$$\tilde{Q}_q(\rho_{AB}||\rho_B) = \mathrm{Tr}\left\{\left[(I_A \otimes \rho_B)^{\frac{1-q}{2q}}\rho_{AB}(I_A \otimes \rho_B)^{\frac{1-q}{2q}}\right]^q\right\} \leq 1 \tag{2.19}$$

$$\text{when} \;\; \rho_{AB} \leq I_A \otimes \rho_B \;\; \text{and} \;\; q > 1.$$

It can now be recalled that for all separable states ρ_{AB},

$$\rho_{AB} - (I_A \otimes \rho_B) \leq 0 \tag{2.20}$$

according to reduction criterion [65]. Thus, as $\rho_{AB} \leq I_A \otimes \rho_B$ for all separable states ρ_{AB}, one has

$$\tilde{Q}_q(\rho_{AB}||\rho_B) \leq 1 \;\; \text{whenever} \;\; q > 1. \tag{2.21}$$

It can now be readily seen that

$$\tilde{D}_q^T(\rho_{AB}||\rho_B) = \frac{\tilde{Q}_q(\rho_{AB}||\rho_B) - 1}{1-q}$$

with $q > 1$ is non-negative for all separable states. In other words, negative values of the conditional version of sandwiched Tsallis relative entropy (CSTRE) $\tilde{D}_q^T(\rho_{AB}||\rho_B)$ $(q > 1)$ indicate entanglement in the state ρ_{AB} thus proving the theorem[4].

Through Theorem 1, it is established that '*negativity*' of CSTRE ($\tilde{D}_q^T(\rho_{AB}||\rho_B)$ $(q > 1)$) is a '*sufficient criterion*' for the bipartite state ρ_{AB} *to be entangled*. This fact is used in one-parameter family of mixed symmetric states to identify the value of the parameter x at which $\tilde{D}_q^T(\rho_{AB}||\rho_B)$ $(q > 1)$ changes from positive to negative or vice versa in the limit $q \to \infty$. In other words, identification of the '*zeroes*' of $\tilde{D}_q^T(\rho_{AB}||\rho_B)$ when $q \to \infty$ lead to the separability range(s), the range(s) of the parameter x in which $\lim_{q\to\infty} \tilde{D}_q^T(\rho_{AB}||\rho_B) \geq 0$.

2.2 Symmetric one-parameter family of noisy W states

The symmetric one parameter family of N-qubit mixed states, involving a W-state is given by

$$\rho_N^{\mathrm{W}}(x) = \left(\frac{1-x}{N+1}\right) P_N + x|\mathrm{W}_N\rangle\langle\mathrm{W}_N| \tag{2.22}$$

Here $0 \leq x \leq 1$ and

$$P_N = \sum_M \left|\frac{N}{2},\, M\right\rangle\left\langle\frac{N}{2},\, M\right|, \quad M = \frac{N}{2}, \frac{N}{2}-1, \cdots, -\frac{N}{2} \tag{2.23}$$

denotes the projector onto the symmetric subspace of N-qubits spanned by the $N+1$ angular momentum states $\left|\frac{N}{2},\, M\right\rangle$, $M = \frac{N}{2}, \frac{N}{2}-1, \cdots, -\frac{N}{2}$ belonging to the maximum value $J = \frac{N}{2}$ of total angular momentum. In fact $P_N/(N+1)$ is the

[4] In an analogous manner it can be shown that $\tilde{D}_q^T(\rho_{AB}||\rho_A) \geq 0$ for separable states ρ_{AB} for all $q > 1$.

identity operator in the $N+1$ dimensional maximal multiplicity space and hence the name 'noisy state' is justified for the family of states in Eq. (2.22). Notice that as $x = 1$, the noisy state becomes the pure symmetric N qubit W-state [51]

$$|\mathrm{W}_N\rangle = \frac{1}{\sqrt{N}}\left[|1_1 0_2 \cdots 0_N\rangle + |0_1 1_2 \cdots 0_N\rangle + \cdots + \cdots + |0_1 0_2 0_3 \cdots 1_N\rangle\right] \tag{2.24}$$

and when $x = 0$, the state becomes a completely noisy state $P_N/(N+1)$ in the symmetric subspace. It is not difficult to notice that $|\mathrm{W}_N\rangle \equiv \left|\frac{N}{2}, \frac{N}{2} - 1\right\rangle$ is one among the basis states of the $N+1$ dimensional symmetric subspace. As the maximal multiplicity subspace is spanned by the angular momentum states $|\frac{N}{2}, M\rangle$, $M = \frac{N}{2}, \frac{N}{2} - 1, \cdots, -\frac{N}{2}$ is a symmetric subspace (states belonging to it are invariant under interchange of qubits) the states in Eq. (2.22) are *s*ymmetric states.

A systematic attempt to examine the separability ranges of the noisy one parameter family of W states using the AR-criterion has been carried out in Ref. [40]. While they could obtain a result matching with that of positive partial transpose (PPT) criterion [21] for the 2-qubit states $\rho_2^{\mathrm{W}}(x)$, the range of separability identified by them is weaker than that through PPT criterion, for the states $\rho_N^{\mathrm{W}}(x)$ when $N \geq 3$. In this section, the separability ranges in different partitions of the state $\rho_N^{\mathrm{W}}(x)$ are determined using CSTRE criterion when $N = 3$, $N = 4$. It is identified that the non-maximal mixedness of the subsystem states (and hence non-commuting with the global density matrix) of the density matrix $\rho_3^{\mathrm{W}}(x)$, $\rho_4^{\mathrm{W}}(x)$ plays a major role in the AR-criterion not yielding strictest separability range in all bipartitions. The separability domain inferred through non-negative values of CSTRE is shown to be stricter compared to that obtained from AR-criterion and von-Neumann conditional entropy criterion. It is also shown that in some of the bipartitions, the CSTRE criterion yields a weaker separability range than that obtained through PPT criterion. The next two subsections contain the details on the determination of separability ranges in all possible bipartitions of the states $\rho_3^{\mathrm{W}}(x)$, $\rho_4^{\mathrm{W}}(x)$ through different separability criteria.

2.2.1 Bipartite separability in one parameter family of three qubit noisy W-states

The symmetric one parameter family of 3-qubit mixed W-states are defined as

$$\rho_3^{\rm W}(x) = \left(\frac{1-x}{4}\right) P_3 + x|{\rm W}_3\rangle\langle {\rm W}_3| \tag{2.25}$$

Here $0 \leq x \leq 1$ and $P_3 = \sum_M \left|\frac{3}{2},\, M\right\rangle \left\langle\frac{3}{2},\, M\right|$, with $\left|\frac{3}{2},\, M\right\rangle$, $M = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$ being the basis states of the four dimensional symmetric subspace. These states $\left\{\left|\frac{3}{2}, M\right\rangle\right\}$ are given explicitly in terms of the single qubit basis states $|0\rangle = |\uparrow\rangle$, $|1\rangle = |\downarrow\rangle$ as follows,

$$\begin{aligned}
\left|\frac{3}{2}, \frac{3}{2}\right\rangle &= |\uparrow_A\uparrow_B\uparrow_C\rangle\,, \quad \left|\frac{3}{2}, \frac{-3}{2}\right\rangle = |\downarrow_A\downarrow_B\downarrow_C\rangle\,, \\
\left|\frac{3}{2}, \frac{1}{2}\right\rangle &= |{\rm W}_3\rangle = \frac{1}{\sqrt{3}}\left(|\downarrow_A\uparrow_B\uparrow_C\rangle + |\uparrow_A\downarrow_B\uparrow_C\rangle + |\uparrow_A\uparrow_B\downarrow_C\rangle\right), \\
\left|\frac{3}{2}, \frac{-1}{2}\right\rangle &= |\bar{W}_3\rangle = \frac{1}{\sqrt{3}}\left(|\uparrow_A\downarrow_B\downarrow_C\rangle + |\downarrow_A\uparrow_B\downarrow_C\rangle + |\downarrow_A\downarrow_B\uparrow_C\rangle\right).
\end{aligned} \tag{2.26}$$

The density matrix of the state is explicitly given by

$$\rho_3^{\rm W}(x) = \begin{pmatrix}
\frac{1-x}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{(1+3x)}{12} & \frac{(1+3x)}{12} & 0 & \frac{(1+3x)}{12} & 0 & 0 & 0 \\
0 & \frac{(1+3x)}{12} & \frac{(1+3x)}{12} & 0 & \frac{(1+3x)}{12} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1-x}{12} & 0 & \frac{1-x}{12} & \frac{1-x}{12} & 0 \\
0 & \frac{(1+3x)}{12} & \frac{(1+3x)}{12} & 0 & \frac{(1+3x)}{12} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1-x}{12} & 0 & \frac{1-x}{12} & \frac{1-x}{12} & 0 \\
0 & 0 & 0 & \frac{1-x}{12} & 0 & \frac{1-x}{12} & \frac{1-x}{12} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1-x}{4}
\end{pmatrix} \tag{2.27}$$

The non-zero eigenvalues of $\rho_3^{\rm W}(x)$ are seen to be

$$\lambda_1 = \lambda_2 = \lambda_3 = \frac{1-x}{4} \quad \text{and} \quad \lambda_4 = \frac{1}{4}(1+3x)$$

2.2.1.1 Separability of $\rho_3^{\rm W}(x)$ in its $1:2$ partition

In the $1:2$ partition of $\rho_3^{\rm W}(x)$, one needs to consider the single qubit marginal as the first subsystem i.e., $\rho_A = {\rm Tr}_{23}\, \rho_3^{\rm W}(x)$ and the second subsystem denoted by B is a two-qubit marginal obtained by tracing out the first qubit of $\rho_3^{\rm W}(x)$. In fact, owing to the symmetry of the state $\rho_3^{\rm W}(x)$ under interchange of qubits, the single qubit and two-qubit marginals remain the same irrespective of which qubit/s are traced out.

It can be seen that

$$\rho_A = {\rm Tr}_{23}\, \rho_3^{\rm W}(x) = \begin{pmatrix} \frac{3+x}{6} & 0 \\ 0 & \frac{3-x}{6} \end{pmatrix} \tag{2.28}$$

with eigenvalues $(3 \pm x)/6$ and

$$\rho_B = {\rm Tr}_{1}\, \rho_3^{\rm W}(x) = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1+x}{6} & \frac{1+x}{6} & 0 \\ 0 & \frac{1+x}{6} & \frac{1+x}{6} & 0 \\ 0 & 0 & 0 & \frac{1-x}{3} \end{pmatrix}, \tag{2.29}$$

with non-zero eigenvalues

$$\mu_1 = \frac{1}{3}, \quad \mu_2 = \frac{1-x}{3}, \quad \mu_3 = \frac{1+x}{3}.$$

With the knowledge of the eigenvalues of ρ_B, $\rho_3^{\rm W}(x)$, the respective von-Neumann entropies

$$S(B) = -\sum_i \mu_i \log_2\, \mu_i, \quad S(A,\, B) = -\sum_i \lambda_i \log_2\, \lambda_i$$

are given by

$$S(A,\, B) = -3\, \frac{(1-x)}{4} \log_2 \frac{1-x}{4} - \frac{1+3x}{4} \log_2 \frac{1+3x}{4} \tag{2.30}$$

$$S(B) = -\frac{1}{3} \log_2 \frac{1}{3} - \frac{(1-x)}{3} \log_2 \frac{1-x}{3} - \frac{1+x}{3} \log_2 \frac{1+x}{3} \tag{2.31}$$

The von-Neumann conditional entropy for the state $\rho_3^{\rm W}(x)$ in its $1:2$ partition is given by $S(A|B)=S(A,\,B)-S(B)$ and can be readily evaluated using Eqs. (2.30), (2.31). The plot of $S(A|B)$, as a function of x, is shown in Fig. 2.1. Identifying the zero of the curve $S(A|B)=0$ the separability range of $\rho_3^{\rm W}(x)$, in its $1:2$ partition is obtained as $(0,\ 0.5695)$.

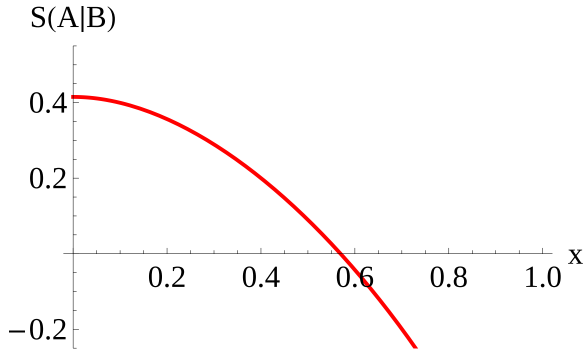

FIGURE 2.1: The von-Neumann conditional entropy $S(A|B)$ as a function of x, for the state $\rho_3^{\rm W}(x)$, in its $1:2$ partition.

In order to evaluate the $1:2$ separability range of $\rho_3^{\rm W}(x)$ using AR-criterion, one needs to evaluate the Abe-Rajagopal (AR) q-conditional entropy

$$S_q^T(A|B)=\frac{1}{q-1}\left[1-\frac{{\rm Tr}\,(\rho_{AB})^q}{{\rm Tr}\,(\rho_B)^q}\right]$$

as a function of x and identify its zero/s. Knowing the eigenvalues λ_i of $\rho_3^{\rm W}(x)$ and μ_i of ρ_B one has

$$\begin{aligned}{\rm Tr}\,\rho_{\rm AB}^{\rm q} &= \sum_i \lambda_i^q = 3\left(\frac{1-x}{4}\right)^q+\left(\frac{1+3x}{4}\right)^q \\ {\rm Tr}\,\rho_{\rm B}^{\rm q} &= \sum_i \mu_i^q = \left(\frac{1}{3}\right)^q+\left(\frac{1-x}{3}\right)^q+\left(\frac{1+x}{3}\right)^q\end{aligned} \tag{2.32}$$

which facilitates the evaluation of $S_q^T(A|B)$ as a function of x and q. The plot of $S_q^T(A|B)$, as a function of x, for different values of q is shown in Fig. 2.2. It can be seen that in the limit $q\to 1$, $\rho_3^{\rm W}(x)$ is separable in the range $(0,\ 0.5695)$ which is the $1:2$ von-Neumann separability range obtained earlier. As $q\to\infty$, the $1:2$ separability range of $\rho_3^{\rm W}(x)$ is seen to be $(0,\ 0.2)$ [40]. It is clear that the 1 : 2 AR-separability range of $\rho_3^{\mathrm{W}}(x)$ is stricter than the 1 : 2 von-Neumann separability range.

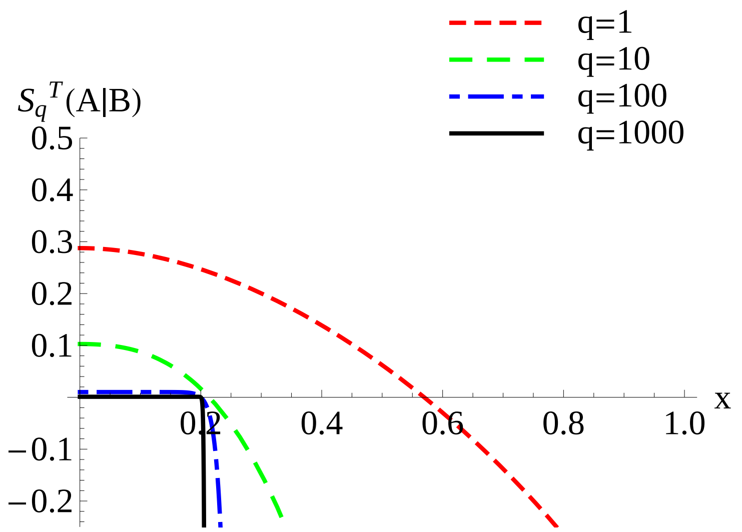

FIGURE 2.2: The AR q-conditional entropy $S_q^T(A|B)$ as a function of x for different values of q, in the $1:2$ partition of the state $\rho_3^{\mathrm{W}}(x)$.

It is to be noticed that the subsystems ρ_A, ρ_B are not maximally mixed implying that the conditional version of sandwiched relative entropy (CSTRE) will not match with the AR q-conditional entropy. In order to evaluate CSTRE (See Eq. (2.5)) for $\rho_3^{\mathrm{W}}(x)$ in its 1 : 2 partition, one has to find out

$$\tilde{Q}_q(\rho_{AB}||\rho_B) = \mathrm{Tr}\left\{\left((I_A\otimes\rho_B)^{\frac{1-q}{2q}}\rho_{AB}(I_A\otimes\rho_B)^{\frac{1-q}{2q}}\right)^q\right\}.$$

with $\rho_{AB} = \rho_3^{\mathrm{W}}(x)$ and

$$\rho_B = \mathrm{Tr}_A\,\rho_3^{\mathrm{W}}(x) = \mathrm{Tr}_1\,\rho_3^{\mathrm{W}}(x)$$

being the two-qubit marginal of $\rho_3^{\mathrm{W}}(x)$. Also, $I_A = I_2$ is a 2×2 unit matrix in the subsystem space A. To make this calculation feasible, one needs to construct the unitary operator U_B that diagonalizes ρ_B. It can be seen that

$$U_B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

and facilitates in the evaluation of $\Gamma_U = (I_2 \otimes U_B)\Gamma(I_2 \otimes U_B)^\dagger$ where

$$\Gamma = (I_2 \otimes \rho_B)^{\frac{1-q}{2q}} \rho_3^{\rm W}(x)(I_2 \otimes \rho_B)^{\frac{1-q}{2q}}$$

is the sandwiched matrix. The non-zero eigenvalues of Γ_U and Γ are found to be,

$$\begin{aligned} \gamma_1 &= \left(\frac{1-x}{4}\right)\left(\frac{1-x}{3}\right)^{\frac{1-q}{q}}, \ \gamma_2 = \left(\frac{1-x}{4}\right)\left(\frac{1}{3}\right)^{\frac{1-q}{q}} \\ \gamma_3 &= \left(\frac{1-x}{4}\right)\left(\frac{1}{3}\right)^{\frac{1}{q}}\left[(1-x)^{\frac{1-q}{q}} + 2(1+x)^{\frac{1-q}{q}}\right] \\ \gamma_4 &= \left(\frac{1+3x}{4}\right)\left(\frac{1}{3}\right)^{\frac{1}{q}}\left[1 + 2(1+x)^{\frac{1-q}{q}}\right] \end{aligned} \tag{2.33}$$

One can now readily evaluate the expression for CSTRE (See Eq.(2.16))

$$\tilde{D}_q^T(\rho_{AB}||\rho_B) = \frac{\sum_i \gamma_i^q - 1}{1-q}$$

for different values of q and obtain $\tilde{D}_q^T(\rho_3^{\rm W}(x)||\rho_B)$ as a function of x and q. The plots (Figs. 2.3 and 2.4) illustrate the variation of $\tilde{D}_q^T(\rho_3^{\rm W}(x)||\rho_B)$ with respect to x for different values of q.

It can be seen through Figs. 2.3, 2.4 and the identification of zeroes of $\tilde{D}_q^T(\rho_3^{\rm W}(x)||\rho_B)$ in the limit $q \longrightarrow \infty$ that, the $1:2$ separability range in the one-parameter family of 3-qubit W states is $0 \leq x \leq 0.1547$ through CSTRE approach [62]. It is to be recalled (See Fig. 2.4) that AR criterion yields a weaker separability range [40] $0 \leq x \leq 0.2$.

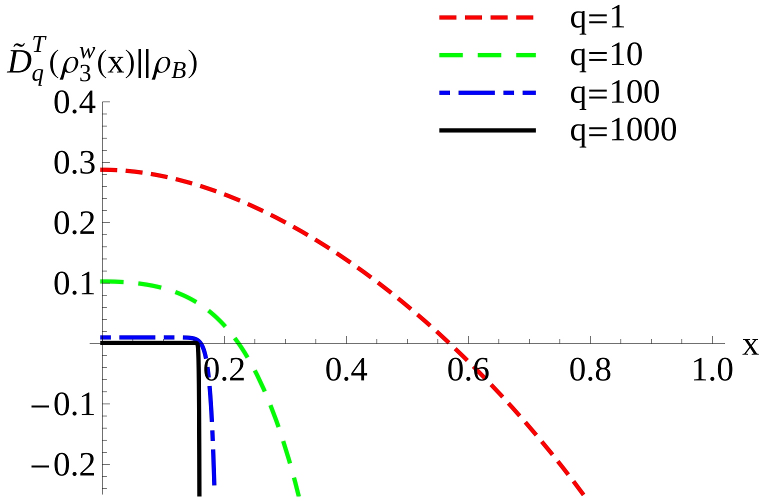

FIGURE 2.3: The CSTRE $\tilde{D}_q^T(\rho_3^{\mathrm{W}}(x)||\rho_B)$ as a function of x for different values of q, in the $1:2$ partition of the state $\rho_3^{\mathrm{W}}(x)$.

Having determined the $1:2$ separability range in $\rho_3^{\mathrm{W}}(x)$ through different entropic criteria, it would be of interest to evaluate this separability range through Peres Partial Transpose criteria. Towards this end, we evaluate the partially transposed density matrix of $\rho_3^{\mathrm{W}}(x)$ in its $1:2$ partition. The $1:2$ partially transposed density matrix of $\rho_3^{\mathrm{W}}(x)$ is explicitly given by

$$\rho^T = \begin{pmatrix} \frac{1-x}{4} & 0 & 0 & 0 & 0 & \frac{(1+3x)}{12} & \frac{(1+3x)}{12} & 0 \\ 0 & \frac{(1+3x)}{12} & \frac{(1+3x)}{12} & 0 & 0 & 0 & 0 & \frac{1-x}{12} \\ 0 & \frac{(1+3x)}{12} & \frac{(1+3x)}{12} & 0 & 0 & 0 & 0 & \frac{1-x}{12} \\ 0 & 0 & 0 & \frac{1-x}{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(1+3x)}{12} & 0 & 0 & 0 \\ \frac{(1+3x)}{12} & 0 & 0 & 0 & 0 & \frac{1-x}{12} & \frac{1-x}{12} & 0 \\ \frac{(1+3x)}{12} & 0 & 0 & 0 & 0 & \frac{1-x}{12} & \frac{1-x}{12} & 0 \\ 0 & \frac{1-x}{12} & \frac{1-x}{12} & 0 & 0 & 0 & 0 & \frac{1-x}{4} \end{pmatrix} \tag{2.34}$$

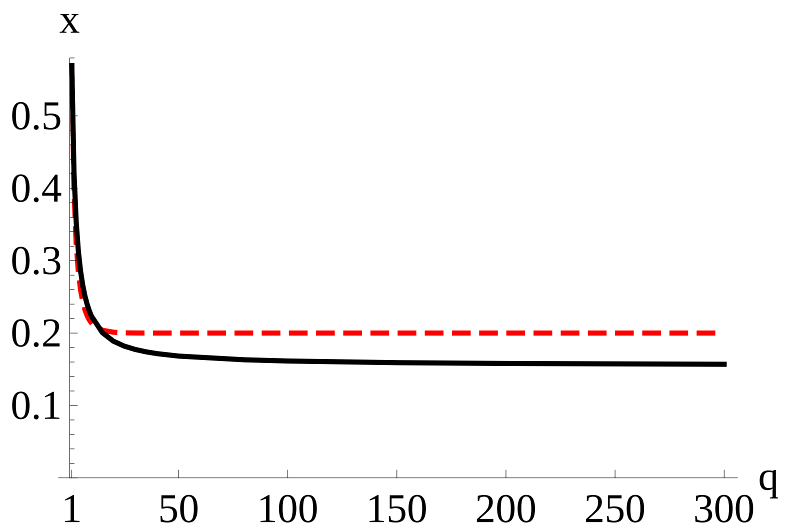

FIGURE 2.4: Implicit plot of $\tilde{D}_q^T(\rho_3^{\rm W}(x)||\rho_B) = 0$ as a function of q (solid line) indicating that $x \to 0.1547$ as $q \to \infty$ in the $1:2$ partition of the state $\rho_3^{\rm W}(x)$. In contrast, for the same partition of $\rho_3^{\rm W}(x)$, the implicit plot of Abe-Rajagopal q-conditional entropy $S_q^T(A|B) = 0$(dashed line), leads to $x \to 0.2$ as $q \to \infty$.

On evaluating the eigenvalues α_i^2 of $(\rho^T)^2$, the trace norm $||\rho^T|| = \sum_i \alpha_i$ and the negativity $N(\rho)$ is calculated using the relation $N(\rho) = \frac{||\rho^T||-1}{2}$.

$$\begin{aligned}
\alpha_1 &= \frac{1-x}{12}, \quad \alpha_2 = \frac{1+3x}{12}, \\
\alpha_{3/4} &= \frac{1}{12\sqrt{2}}\sqrt{17-2x+49x^2 \pm 5(1-x)\sqrt{9+46x+73x^2}}, \\
\alpha_{5/6} &= \frac{1}{12\sqrt{2}}\sqrt{17-2x+49x^2 \pm (5+3x)\sqrt{9-34x+89x^2}}.
\end{aligned} \tag{2.35}$$

The plot of negativity $N(\rho)$ as a function of x is shown in Fig. 2.5. One can observe through Fig. 2.5 that $N(\rho)$ remains zero till $x = 0.1547$ and its value increases monotonically with x. Thus, according to PPT criterion, in the $1:2$ partition of $\rho_3^{\rm W}(x)$, it is separable in the range $0 \leq x \leq 0.1547$ and entangled in the range $0.1547 < x \leq 1$.

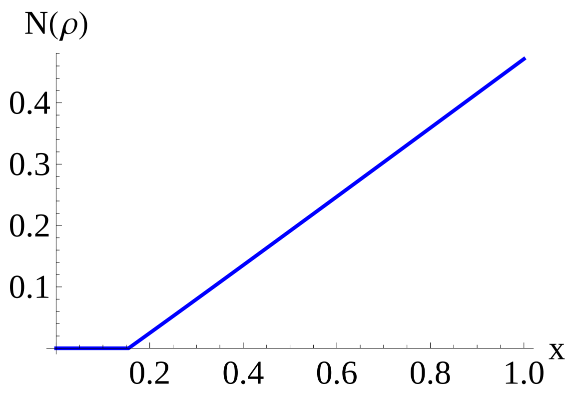

FIGURE 2.5: The plot of negativity of partial transpose as a function of x, for $\rho_3^{\mathrm{W}}(x)$ in its $1:2$ partition.

2.2.1.2 Separability of $\rho_3^{\mathrm{W}}(x)$ in its $2:1$ partition

In the $2:1$ partition, the two qubit and single qubit marginals of $\rho_3^{\mathrm{W}}(x)$ form the first and second parts A, B respectively. Owing to the symmetry of the state $\rho_3^{\mathrm{W}}(x)$ under interchange of qubits, it can be seen that

$$\rho_A = \mathrm{Tr}_1\, \rho_3^{\mathrm{W}}(\mathrm{x}) = \mathrm{Tr}_2\, \rho_3^{\mathrm{W}}(\mathrm{x}) = \mathrm{Tr}_3\, \rho_3^{\mathrm{W}}(\mathrm{x})$$

and is given in Eq. (2.29). Similarly,

$$\rho_B = \mathrm{Tr}_{12}\, \rho_3^{\mathrm{W}}(\mathrm{x}) = \mathrm{Tr}_{23}\, \rho_3^{\mathrm{W}}(\mathrm{x}) = \frac{1}{6}\,\mathrm{diag}\,(3+\mathrm{x},\, 3-\mathrm{x})$$

The eigenvalues of ρ_B being $\mu_1 = (3+x)/6$, $\mu_2 = (3-x)/6$, the entropy of the subsystem B is given by

$$S(B) = -\frac{3+x}{6}\log_2\frac{3+x}{6} - \frac{3-x}{6}\log_2\frac{3-x}{6} \tag{2.36}$$

With the entropy of the global state $\rho_3^{\mathrm{W}}(x)$ given in Eq. (2.30), and $S(B)$ given in Eq. (2.36), the von-Neumann conditional entropy $S(A|B)$ of the state $\rho_3^{\mathrm{W}}(x)$ in its $2:1$ partition can be readily evaluated as

$$\begin{aligned} S(A|B) \;\; = \;\; & -3\left(\frac{1-x}{4}\right)\log_2\frac{1-x}{4} - \frac{1+3x}{4}\log_2\frac{1+3x}{4} \\ & + \frac{3+x}{6}\log_2\frac{3+x}{6} + \frac{3-x}{6}\log_2\frac{3-x}{6} \end{aligned} \tag{2.37}$$

Through the identification of the zero of $S(A|B)$ given in Eq. (2.37), one can obtain (0, 0.7645) as the $2:1$ von-Neumann separability range of $\rho_3^{\mathrm{W}}(x)$.

The AR q-conditional entropy of $\rho_3^{\mathrm{W}}(x)$ in its $2:1$ partition can be readily obtained as

$$S_q^T(A|B) = \frac{1}{q-1}\left[1 - \frac{3\left(\frac{1-x}{4}\right)^q + \left(\frac{1+3x}{4}\right)^q}{\left(\frac{3+x}{6}\right)^q + \left(\frac{3-x}{6}\right)^q}\right] \tag{2.38}$$

The plot of $S_q^T(A|B)$ in Eq. (2.38), as a function of x, for different values of q is shown in Fig. 2.6. From Fig. 2.6, it is evident that $\lim_{q\to 1} S_q^T(A|B) = S(A|B)$ yielding the separability range $(0,\ 0.7645)$ as obtained through von-Neumann conditional entropy. But as $q \to \infty$ the AR q-conditional entropy $S_q^T(A|B)$ changes sign from positive to negative at $x = 0.4286$ yielding the $2:1$ separability range of $\rho_3^{\mathrm{W}}(x)$ as $(0,\ 0.4286)$. The $2:1$ AR-separability range is thus seen to be stricter than the $2:1$ von-Neumann separability range.

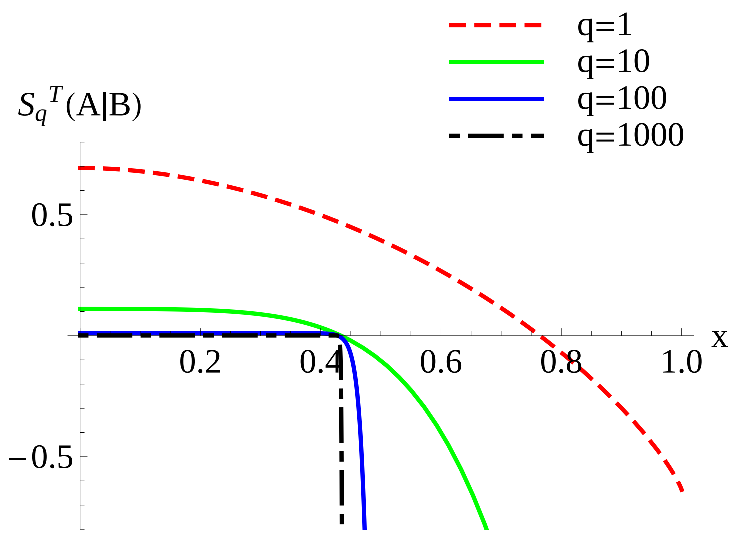

FIGURE 2.6: The AR q-conditional entropy $S_q^T(A|B)$ as a function of x for different values of q, for $\rho_3^{\mathrm{W}}(x)$, in its $2:1$ partition.

In order to employ CSTRE criterion to obtain the $2:1$ separability range in $\rho_3^{\mathrm{W}}(x)$, recall that

$$\tilde{D}_q^T(\rho_{AB}||\rho_B) = \frac{\sum_i \gamma_i^q - 1}{1-q}$$

with γ_i being the eigenvalues of

$$\Gamma = (I_4 \otimes \rho_B)^{\frac{1-q}{2q}} \rho_3^{\mathrm{W}}(x) (I_4 \otimes \rho_B)^{\frac{1-q}{2q}}$$

The single qubit marginal ρ_B being diagonal, the eigenvalues of Γ can readily be evaluated thus facilitating the evaluation of the CSTRE $\tilde{D}_q^T(\rho_{AB}||\rho_B)$ of the state $\rho_3^{\mathrm{W}}(x)$ in its $2:1$ partition. Now on evaluating $\tilde{D}_q^T(\rho_{AB}||\rho_B)$ for different values of q, one can find its variation with respect to the parameter x for each value of q. Fig. 2.7 illustrates that, in the limit $q \to 1$, the CSTRE criterion gives the $2:1$ von-Neumann separability range $(0,\ 0.7645)$ but in the limit $q \to \infty$ one can obtain $(0,\ 0.3509)$ as the $2:1$ CSTRE separability range. Though $(0,\ 0.3509)$ is stricter than von-Neumann and AR-separability ranges, it is found to be weaker than the corresponding $1:2$ separability range. Also, it is noticed that $\rho_3^{\mathrm{W}}(x)$ being a symmetric state, the $1:2$ and $2:1$ separability ranges of the state must be the same. But none of the entropic separability criteria are able to capture the symmetry of the state as they are yielding different $1:2$, $2:1$ separability ranges. But the PPT criteria accommodates the symmetry of the state and yields the equal separability ranges in the $1:2$, $2:1$ partition of the symmetric state $\rho_3^{\mathrm{W}}(x)$. This is evident from the fact that the partial transpose of the state $\rho_3^{\mathrm{W}}(x)$ in both the partitions match with each other.

2.2.2 Bipartite separability in one parameter family of four qubit noisy W-states

The symmetric one parameter family of noisy mixed states involving four-qubit W-states are given by

$$\rho_4^{\mathrm{W}}(x) = \left(\frac{1-x}{5}\right) P_4 + x|\mathrm{W}_4\rangle\langle\mathrm{W}_4|, \quad 0 \le x \le 1 \tag{2.39}$$

where

$$P_4 = \sum_M |2,\ M\rangle\langle 2,\ M|$$

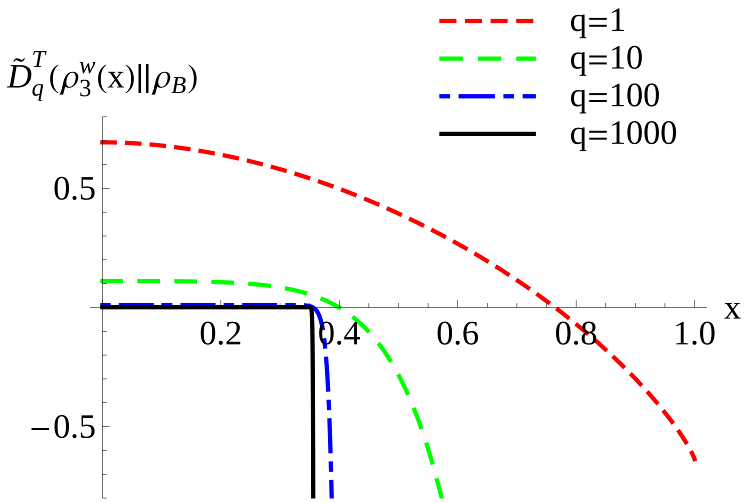

FIGURE 2.7: The CSTRE $\tilde{D}_q^T(\rho_3^{\rm W}(x)||\rho_B)$ as a function of x for different values of q, for $\rho_3^{\rm W}(x)$, in its $2:1$ partition.

is the projector onto the symmetric subspace of four-qubits spanned by the five angular momentum states $|2, M\rangle$, $M = 2, 1, 0, -1, -2$ which are basis states of the maximal multiplicity subspace with $J = 2$.

There are only two distinct non zero eigenvalues for the state $\rho_4^{\rm W}(x)$ and they are given by

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \frac{1-x}{5}, \quad \lambda_5 = \frac{1}{5}(1+4x). \tag{2.40}$$

In the following, the separability of $\rho_4^{\rm W}(x)$ in its different partitions, is investigated through various separability criteria.

2.2.2.1 Separability of $\rho_4^{\rm W}(x)$ in its $1:3$ partition

In the $1:3$ partition, the single qubit marginal of $\rho_4^{\rm W}(x)$ forms the first part A and is given by

$$\rho_A = \mathrm{Tr}_{234}\, \rho_4^{\rm W}(x) = \begin{pmatrix} \frac{2+x}{4} & 0 \\ 0 & \frac{2-x}{4} \end{pmatrix} \tag{2.41}$$

whereas the subsystem corresponding to the remaining three qubits corresponds to the second part B. It can be seen that

$$\rho_B = \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1+2x}{12} & \frac{1+2x}{12} & 0 & \frac{1+2x}{12} & 0 & 0 & 0 \\ 0 & \frac{1+2x}{12} & \frac{1+2x}{12} & 0 & \frac{1+2x}{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-x}{12} & 0 & \frac{1-x}{12} & \frac{1-x}{12} & 0 \\ 0 & \frac{1+2x}{12} & \frac{1+2x}{12} & 0 & \frac{1+2x}{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-x}{12} & 0 & \frac{1-x}{12} & \frac{1-x}{12} & 0 \\ 0 & 0 & 0 & \frac{1-x}{12} & 0 & \frac{1-x}{12} & \frac{1-x}{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1-x}{4} \end{pmatrix} \tag{2.42}$$

The nonzero eigenvalues of ρ_B are given by

$$\mu_1 = \frac{1}{4}, \quad \mu_2 = \mu_3 = \frac{1-x}{4}, \quad \mu_4 = \frac{1}{4}(1+2x). \tag{2.43}$$

The entropy $S(B)$ of subsystem ρ_B and $S(A,\, B)$ of the global state $\rho_4^{\rm W}(x)$ are obtained respectively as

$$S(B) = -\frac{1}{4}\log_2\frac{1}{4} - 2\left(\frac{1-x}{4}\right)\log_2\frac{1-x}{4} - \frac{1+2x}{4}\log_2\frac{1+2x}{4} \tag{2.44}$$

$$S(A,\, B) = \frac{-4(1-x)}{5}\log_2\frac{1-x}{5} - \frac{1+4x}{5}\log_2\frac{1+4x}{5} \tag{2.45}$$

and the conditional entropy $S(A|B) = S(A,\, B) - S(B)$ in the $1:3$ partition of the state can readily be evaluated. The plot of $S(A|B)$ as a function of x is as shown in Fig. 2.8. As $S(A|B)$ becomes negative when $x > 0.5193$, $(0,\ 0.5193)$ is the $1:3$ von-Neumann separability range of $\rho_4^{\rm W}(x)$.

The evaluation of AR q-conditional entropy in the $1:3$ partition of $\rho_4^{\rm W}(x)$ leads to [40]

$$S_q^T(A|B) = \frac{1}{q-1}\left[1 - \frac{4\left(\frac{1-x}{5}\right)^q + \left(\frac{1+4x}{5}\right)^q}{\left(\frac{1}{4}\right)^q + 2\left(\frac{1-x}{4}\right)^q + \left(\frac{1+2x}{4}\right)^q}\right] \tag{2.46}$$

The plot of $S_q^T(A|B)$ in (2.46), as a function of x, for different values of q is as shown in Fig. 2.9. It can be seen that in the limit $q \to 1$, the state $\rho_4^{\rm W}(x)$ is separable in the range $(0,\ 0.5193)$ which equals the $1:3$ von-Neumann separability range. But as $q \to \infty$ the $1:3$ separability range of $\rho_4^{\mathrm{W}}(x)$ is found to be (0, 0.1666) [40].

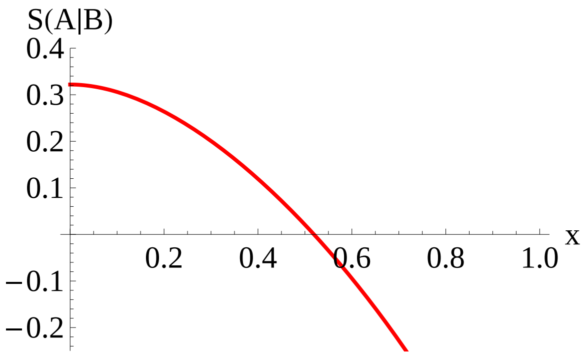

FIGURE 2.8: The von-Neumann conditional entropy $S(A|B)$ as a function of x, for the state $\rho_4^{\mathrm{W}}(x)$, in its $1:3$ partition.

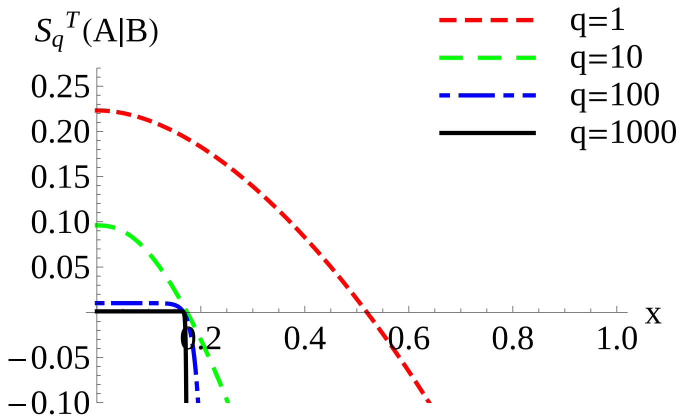

FIGURE 2.9: The AR q-conditional entropy $S_q^T(A|B)$ of $\rho_4^{\mathrm{W}}(x)$, in its $1:3$ partition, as a function of x for different values of q.

In view of the fact that ρ_B given in Eq. (2.42) is not a diagonal matrix, one needs a unitary matrix U_B which diagonalizes ρ_B so that the eigenvalues of $\Gamma_U = (I_2 \otimes U_B)\Gamma(I_2 \otimes U_B)^\dagger$ unitarily equivalent to

$$\Gamma = (I_2 \otimes \rho_B)^{\frac{1-q}{2q}} \rho_4^{\mathrm{W}}(x)(I_2 \otimes \rho_B)^{\frac{1-q}{2q}}$$

can be found out. The eigenvectors of ρ_B in Eq. (2.42) facilitate us to find out U_B and it is given by

$$U_B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{6}} & \sqrt{\frac{2}{3}} & 0 \end{pmatrix}$$

With μ_i, $i = 1, 2, 3, 4$ being the eigenvalues of ρ_B (See Eq. (2.43)), the non-zero eigenvalues γ_i of the matrix $\Gamma_U = (I_2 \otimes U_B)\Gamma(I_2 \otimes U_B)^\dagger$ unitarily equivalent to the sandwiched matrix

$$\Gamma = (I_2 \otimes \rho_B)^{\frac{1-q}{2q}} \rho_4^{\mathrm{W}}(x)(I_2 \otimes \rho_B)^{\frac{1-q}{2q}},$$

are given by,

$$\begin{aligned} \gamma_1 &= \left(\frac{1-x}{5}\right)\left(\frac{1-x}{4}\right)^{\frac{1-q}{q}}, \ \gamma_2 = \left(\frac{1-x}{5}\right)\left(\frac{1}{4}\right)^{\frac{1-q}{q}} \\ \gamma_3 &= \left(\frac{1-x}{5}\right)\left(\frac{1}{4}\right)^{\frac{1}{q}}\left[2(1-x)^{\frac{1-q}{q}} + 2(1+2x)^{\frac{1-q}{q}}\right] \\ \gamma_4 &= \left(\frac{1+4x}{5}\right)\left(\frac{1}{4}\right)^{\frac{1}{q}}\left[1 + 3(1+2x)^{\frac{1-q}{q}}\right] \end{aligned}$$

One can now readily evaluate the expression for CSTRE $\tilde{D}_q^T(\rho_4^{\mathrm{W}}(x)||\rho_B)$ in its $1:3$ partition as

$$\tilde{D}_q^T(\rho_4^{\mathrm{W}}(x)||\rho_B) = \frac{\sum_i \gamma_i^q - 1}{1-q}$$

Fig. 2.10 indicates the variation of the CSTRE $\tilde{D}_q^T(\rho_4^{\mathrm{W}}(x)||\rho_B)$ with x for increasing values of $q \geq 1$. The $1:3$ separability range of the state $\rho_4^{\mathrm{W}}(x)$ is obtained as $0 \leq x \leq 0.1123$ through identifying the zero of the function $\lim_{q\to\infty} \tilde{D}_q^T(\rho_4^{\mathrm{W}}(x)||\rho_B)$ [62].

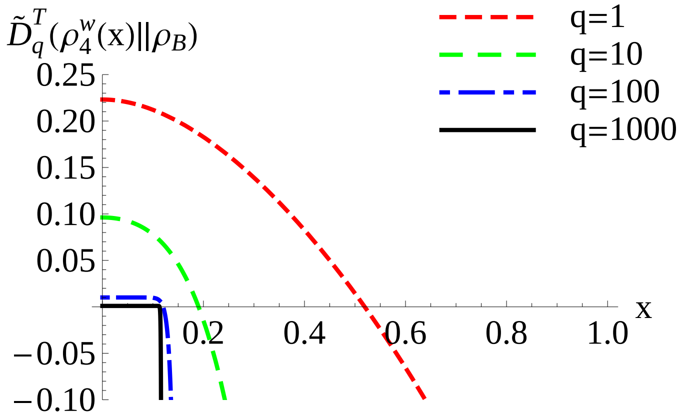

FIGURE 2.10: The CSTRE $\tilde{D}_q^T(\rho_4^{\mathrm{W}}(x)||\rho_B)$ as a function of x for different values of q, in the $1:3$ partition of the state $\rho_4^{\mathrm{W}}(x)$.

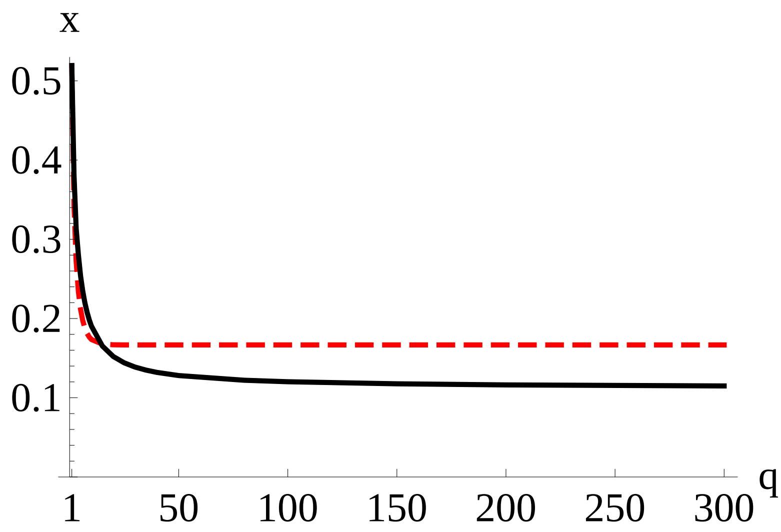

FIGURE 2.11: The implicit plots of $\tilde{D}_q^T(\rho_4^{\mathrm{W}}(x)||\rho_B) = 0$ (solid line) and $S_q^T(A|B) = 0$ (dashed line) as a function of q for the state $\rho_4^{\mathrm{W}}(x)$, in its $1:3$ partition.

Notice that $x \to 0.1124$ according to the implicit plot $\tilde{D}_q^T(\rho_4^{\mathrm{W}}(x)||\rho_B) = 0$ while $x \to 0.1666$ in the implicit plot of AR q-conditional entropy, both in the limit $q \to \infty$.

To obtain the $1:3$ separability range in $\rho_4^{\mathrm{W}}(x)$ through PPT criterion, the eigenvalues α_i^2 of $\rho^T\left(\rho^T\right)^\dagger = \left(\rho^T\right)^2$ with ρ^T being the partially transposed density matrix of $\rho_4^{\mathrm{W}}(x)$ transposed either with respect to the first subsystem (single qubit marginal ρ_A) or with respect to the second subsystem (three-qubit marginal ρ_B) are required. On an explicit evaluation of the eigenvalues α_i^2 of $\left(\rho^T\right)^2$, one gets

$$\begin{aligned} \alpha_1 &= \alpha_2 = \frac{1-x}{20}, \quad \alpha_3 = \frac{1+4x}{20}, \quad \alpha_4 = \frac{1-x}{4} \\ \alpha_{5/6} &= \frac{1}{20}\sqrt{13+4x+58x^2 \pm 6(1-x)\sqrt{4+x(22+49x)}}, \\ \alpha_{7/8} &= \frac{1}{20\sqrt{2}}\sqrt{26+x(38+161x) \pm 3(2+3x)\sqrt{16+x(241x-32)}}. \end{aligned} \tag{2.47}$$

thereby leading to the evaluation of $N(\rho) = \frac{1}{2}\left(\sum_i \alpha_i - 1\right)$. The plot of $N(\rho)$ as a function of x is shown in Fig. 2.12. The negativity remains zero till $x = 0.1123$ and increases thereon. Thus, according to partial transpose criterion the 1 : 3 separability range of the state $\rho_4^{\rm W}(x)$ is seen to be $0 \leq x \leq 0.1123$. It can be seen that 1 : 3 CSTRE separability range matches with that obtained through PPT criterion, for the noisy state $\rho_4^{\rm W}(x)$.

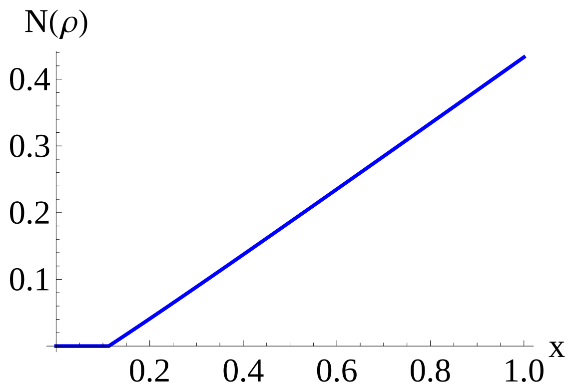

FIGURE 2.12: The plot of negativity of partial transpose as a function of x, for the state $\rho_4^{\rm W}(x)$, in its 1 : 3 partition.

2.2.2.2 Separability of $\rho_4^{\rm W}(x)$ in its $3:1$ partition

In the $3:1$ partition the three qubit marginal of $\rho_4^{\rm W}(x)$ forms the first part A and the single qubit marginal of $\rho_4^{\rm W}(x)$ forms the second part B. Due to the symmetry of the state,

$$\rho_A = {\rm Tr}_4\, \rho_4^{\rm W}({\rm x}) = {\rm Tr_i}\, \rho_4^{\rm W}({\rm x}), \quad {\rm i} = 1,\, 2,\, 3 \tag{2.48}$$

and is as given in Eq. (2.42). Also,

$$\rho_B = \mathrm{Tr}_{123}\, \rho_4^{\mathrm{W}}(\mathrm{x}) = \mathrm{Tr}_{234}\, \rho_4^{\mathrm{W}}(\mathrm{x}) = \frac{1}{4}\, \mathrm{diag}\; (2+\mathrm{x},\, 2-\mathrm{x}) \tag{2.49}$$

and the entropy of ρ_B is given by

$$S(B) = -\frac{2+x}{4}\log_2\frac{2+x}{4} - \frac{2-x}{4}\log_2\frac{2-x}{4} \tag{2.50}$$

With $S(A,\, B)$ given in Eq. (2.45), the conditional entropy $S(A|B) = S(A,\, B) - S(B)$ in the $3:1$ partition of $\rho_4^{\mathrm{W}}(x)$ can be evaluated. One can also obtain $(0,\, 0.8222)$ as the $3:1$ von-Neumann separability range of $\rho_4^{\mathrm{W}}(x)$.

With the knowledge of eigenvalues of the single qubit marginal ρ_B and that of $\rho_4^{\mathrm{W}}(x)$, the AR q-conditional entropy for $\rho_4^{\mathrm{W}}(x)$ in its $3:1$ partition is given by

$$S_q^T(A|B) = \frac{1}{q-1}\left(1 - \frac{4\left(\frac{1-x}{5}\right)^q + \left(\frac{1+4x}{5}\right)^q}{\left(\frac{2+x}{4}\right)^q + \left(\frac{2-x}{4}\right)^q}\right) \tag{2.51}$$

The plot of $S_q^T(A|B)$ as a function of x, for different values of q is shown in Fig. 2.13. From Fig. 2.13 one can observe that in the limit $q \to 1$, $S_q^T(A|B) \geq 0$ when $0 \leq x \leq 0.8222$ leading to the $3:1$ von-Neumann separability range $(0,\; 0.8222)$ for $\rho_4^{\mathrm{W}}(x)$. But as $q \to \infty$, $S_q^T(A|B) \geq 0$ when $0 \leq x \leq 0.5454$, yielding $(0,\; 0.5454)$ as the $3:1$ AR separability range for the state $\rho_4^{\mathrm{W}}(x)$.

The second subsystem ρ_B, the single qubit marginal of $\rho_4^{\mathrm{W}}(x)$ in its $3:1$ partition being diagonal, one can readily evaluate the eigenvalues γ_i of

$$\Gamma = (I_8 \otimes \rho_B)^{\frac{1-q}{2q}}\, \rho_4^{\mathrm{W}}(x)(I_8 \otimes \rho_B)^{\frac{1-q}{2q}}$$

can readily be evaluated. Using Eq.(2.16), one can obtain $\tilde{D}_q^T(\rho_4^{\mathrm{W}}(x)||\rho_B)$ as a function of x and q. The variation of $\tilde{D}_q^T(\rho_4^{\mathrm{W}}(x)||\rho_B)$ with x for different values of q is as shown in Fig. 2.14. While the zero of $\lim_{q\to 1} \tilde{D}_q^T\left(\rho_4^{\mathrm{W}}(x)||\rho_B\right)$ yields the $3:1$ von-Neumann separability range $(0,\; 0.8222)$, the CSTRE separability range is found to be $(0,\; 0.4174)$ through identifying the zero of $\tilde{D}_q^T(\rho_4^{\mathrm{W}}(x)||\rho_B)$ in the limit $q \longrightarrow \infty$. This range is weaker than the $1:3$ CSTRE separability range and the symmetry of the state is not obeyed in CSTRE approach. But PPT criteria respects the symmetry of the state $\rho_4^{\mathrm{W}}(x)$ under interchange of qubits and it is

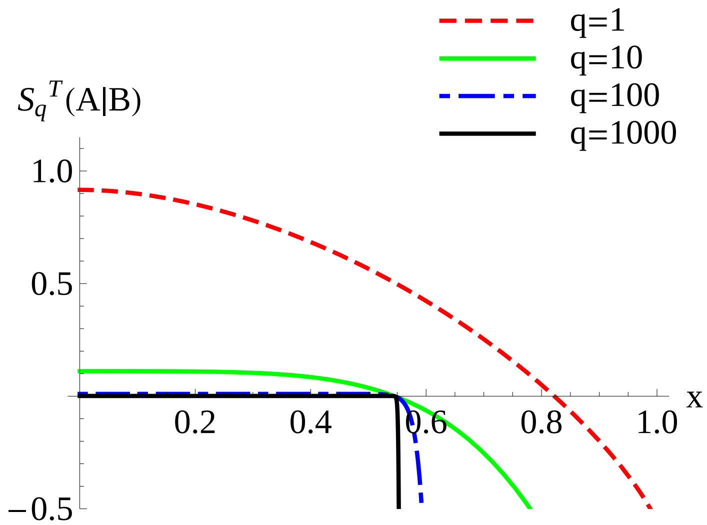

FIGURE 2.13: The AR q-conditional entropy $S_q^T(A|B)$ of the state $\rho_4^{\mathrm{W}}(x)$, in its $3:1$ partition, as a function of x for different values of q.

observed that the tracenorm of partially transposed density matrix of $\rho_4^{\mathrm{W}}(x)$ in its $1:3$ and $3:1$ partitions are same. Thus $(0, 0.1123)$ is the $3:1$ as well as $1:3$ PPT separability range for the symmetric state $\rho_4^{\mathrm{W}}(x)$.

2.2.2.3 Separability in $\rho_4^{\mathrm{W}}(x)$ in its $2:2$ partition

In the $2:2$ partition of $\rho_4^{\mathrm{W}}(x)$ both ρ_A, ρ_B are two-qubit marginals and due to the symmetry of the state $\rho_A = \rho_B$ given by

$$\rho_A = \rho_B = \begin{pmatrix} \frac{2+x}{6} & 0 & 0 & 0 \\ 0 & \frac{2+x}{12} & \frac{2+x}{12} & 0 \\ 0 & \frac{2+x}{12} & \frac{2+x}{12} & 0 \\ 0 & 0 & 0 & \frac{1-x}{3} \end{pmatrix} \tag{2.52}$$

The nonzero eigenvalues of ρ_B are

$$\mu_1 = \mu_2 = \frac{2+x}{6}, \quad \mu_3 = \frac{1-x}{3}. \tag{2.53}$$

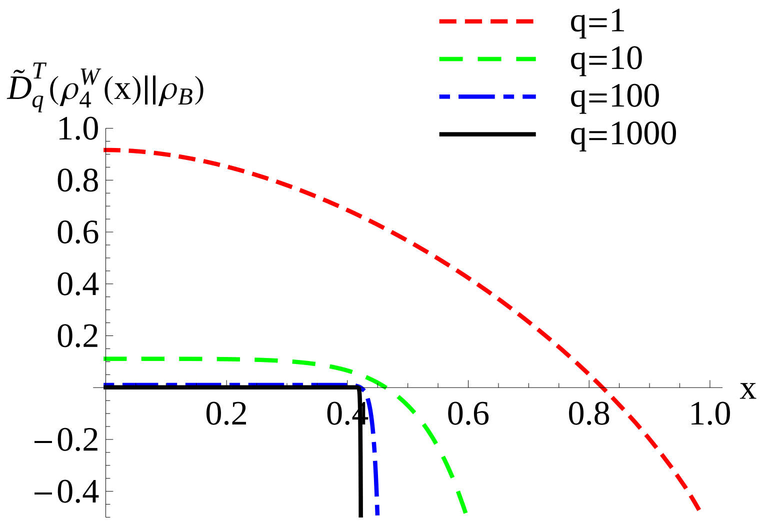

FIGURE 2.14: The CSTRE $\tilde{D}_q^T(\rho_4^{\mathrm{W}}(x)||\rho_B)$ as a function of x for different values of q, in the $3:1$ partition of $\rho_4^{\mathrm{W}}(x)$.

The von-Neumann conditional entropy $S(A|B)$ in the $2:2$ partition of $\rho_4^{\mathrm{W}}(x)$ is explicitly given by

$$\begin{aligned} S(A|B) &= -\frac{4(1-x)}{5}\log_2\frac{1-x}{5} - \frac{1+4x}{5}\log_2\frac{1+4x}{5} \\ &\quad +\frac{1-x}{3}\log_2\frac{1-x}{3} + \frac{2(2+x)}{6}\log_2\frac{2+x}{6} \end{aligned}$$

It can be identified that $(0,\ 0.6560)$ is the $2:2$ von-Neumann separability range of $\rho_4^{\mathrm{W}}(x)$.

The AR q-conditional entropy for $\rho_4^{\mathrm{W}}(x)$ in its $2:2$ partition is explicitly given by

$$S_q^T(A|B) = \frac{1}{q-1}\left(1 - \frac{4\left(\frac{1-x}{5}\right)^q + \left(\frac{1+4x}{5}\right)^q}{\left(\frac{1-x}{3}\right)^q + 2\left(\frac{2+x}{6}\right)^q}\right) \tag{2.54}$$

The plot of $S_q^T(A|B)$ with respect to x, for different values of q is shown in Fig. 2.15. It can be seen that in the limit $q\to 1$, $S_q^T(A|B)$ is non-negative in the range $[0,\ 0.6560]$ verifying that $\lim_{q\to 1} S_q^T(A|B) = S(A|B)$. In the limit $q\to\infty$, the $2:2$ AR-separability range is obtained as $(0,\ 0.2105)$ [40].

In order to employ the CSTRE approach to obtain the $2:2$ separability range

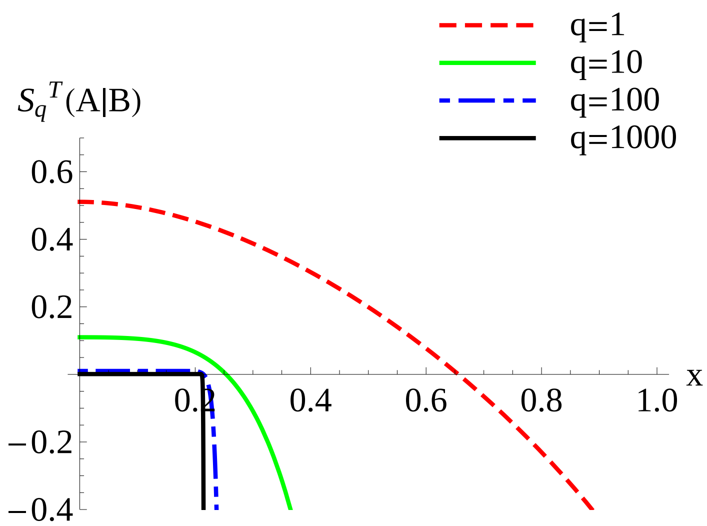

FIGURE 2.15: The AR q-conditional entropy $S_q^T(A|B)$ as a function of x for different values of q, in the $2:2$ partition of the state $\rho_4^{\rm W}(x)$.

of $\rho_4^{\rm W}(x)$, it is necessary to find out the unitary matrix U_B which diagonalizes the two qubit marginal ρ_B. With the knowledge of eigenvectors of ρ_B one can construct U_B as

$$U_B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.55}$$

The unitary matrix U_B that diagonalizes ρ_B leads to $\Gamma_U = (I_4 \otimes U_B)\Gamma(I_4 \otimes U_B)^\dagger$, the unitary equivalent of the sandwiched matrix

$$\Gamma = (I_4 \otimes \rho_B)^{\frac{1-q}{2q}} \rho_4^{\rm W}(x)(I_4 \otimes \rho_B)^{\frac{1-q}{2q}}.$$

Through the determination of eigenvalues γ_i of Γ_U, one can evaluate the expression for CSTRE $\tilde{D}_q^T(\rho_4^{\rm W}(x)||\rho_B)$ in Eq.(2.16) as a function of x and q. Fig. 2.16 illustrates that in the limit $q \to 1$, the CSTRE gives the $2:2$ von-Neumann separability range. The identification of zeroes of $\tilde{D}_q^T(\rho_4^{\rm W}(x)||\rho_B)$ in the limit $q \to \infty$, leads to $(0,\ 0.2105)$ as the $2:2$ CSTRE separability range of the state $\rho_4^{\rm W}(x)$.

On an explicit evaluation of the partially transposed density matrix ρ^T of $\rho_4^{\rm W}(x)$

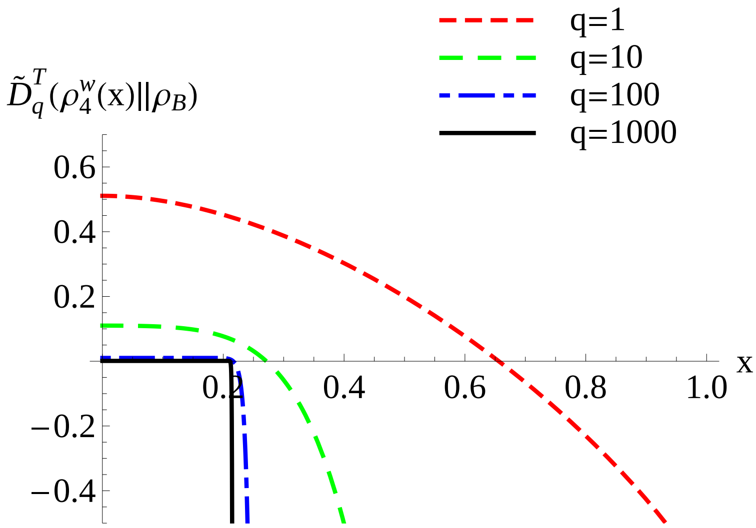

FIGURE 2.16: The CSTRE $\tilde{D}_q^T(\rho_4^{\mathrm{W}}(x)||\rho_B)$ of $\rho_4^{\mathrm{W}}(x)$, in its $2:2$ partition. as a function of x for different values of q.

in its $2:2$ partition, the tracenorm $||\rho^T||$ and the negativity $N(\rho)$ are obtained as functions of x. The variation of $N(\rho)$ with respect to x is shown in Fig. 2.17. The negativity remains zero till $x = 0.0808$. Thus, according to partial transpose criteria, the state $\rho_4^{\mathrm{W}}(x)$ is separable in the range $0 \leq x \leq 0.0808$ and entangled in the range $0.0808 < x \leq 1$, in its $2:2$ partition.

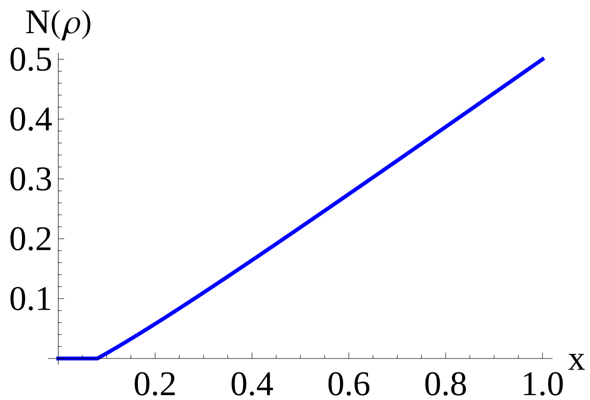

FIGURE 2.17: The plot of negativity of partial transpose as a function of x, in the $2:2$ partition of the state $\rho_4^{\mathrm{W}}(x)$.

2.3 Symmetric one-parameter family of noisy GHZ states

The symmetric one parameter family of N-qubit mixed states, involving a GHZ state is given by

$$\rho_N^{\rm GHZ}(x) = \left(\frac{1-x}{N+1}\right) P_N + x|{\rm GHZ}_N\rangle\langle {\rm GHZ}_N|, \quad 0 \leq x \leq 1. \tag{2.56}$$

$P_N = \sum_M |N/2,\, M\rangle\langle N/2,\, M|$ is the projector onto the $N+1$ dimensional maximal multiplicity subspace of the 2^N dimensional space of N-qubits. The N-qubit GHZ state is given [52, 53] by

$$|{\rm GHZ}_N\rangle = \frac{1}{\sqrt{2}}[|0_1 0_2 \cdots 0_N\rangle + |1_1 1_2 \cdots 1_N\rangle] \tag{2.57}$$

2.3.1 Bipartite separability in symmetric one parameter family of three qubit noisy GHZ states

The symmetric one parameter family of mixed states involving a three-qubit GHZ state is given by

$$\rho_3^{\rm GHZ}(x) = \left(\frac{1-x}{4}\right) P_3 + x|{\rm GHZ}_3\rangle\langle {\rm GHZ}_3| \tag{2.58}$$

and its explicit density matrix is as shown below:

$$\rho_3^{\rm GHZ}(x) = \begin{pmatrix} \frac{1+x}{4} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{x}{2} \\ 0 & \frac{1-x}{12} & \frac{1-x}{12} & 0 & \frac{1-x}{12} & 0 & 0 & 0 \\ 0 & \frac{1-x}{12} & \frac{1-x}{12} & 0 & \frac{1-x}{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-x}{12} & 0 & \frac{1-x}{12} & \frac{1-x}{12} & 0 \\ 0 & \frac{1-x}{12} & \frac{1-x}{12} & 0 & \frac{1-x}{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-x}{12} & 0 & \frac{1-x}{12} & \frac{1-x}{12} & 0 \\ 0 & 0 & 0 & \frac{1-x}{12} & 0 & \frac{1-x}{12} & \frac{1-x}{12} & 0 \\ \frac{x}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+x}{4} \end{pmatrix} \tag{2.59}$$

The non-zero eigenvalues of $\rho_3^{\rm GHZ}(x)$ are seen to be

$$\lambda_1=\lambda_2=\lambda_3=\frac{1-x}{4},\ \ \lambda_4=\frac{1}{4}(1+3x). \tag{2.60}$$

and

$$S\left(\rho_3^{\rm GHZ}(x)\right)=S(A,\,B)=-\frac{3(1-x)}{4}\,\log_2\frac{1-x}{4}-\frac{1+3x}{4}\,\log_2\frac{1+3x}{4} \tag{2.61}$$

is the entropy of the global state $\rho_3^{\rm GHZ}(x)$ with any two subsystems A, B. Quite similar to the symmetric one parameter family of states involving W states, the separability of $\rho_3^{\rm GHZ}(x)$ in its $1:2$, $2:1$ partitions are investigated here, through various separability criteria.

2.3.1.1 Separability of $\rho_3^{\rm GHZ}(x)$ in its $1:2$ partition.

The first subsystem ρ_A (the single qubit marginal) and the second subsystem ρ_B (the two qubit marginal) of $\rho_3^{\rm GHZ}(x)$ are seen to be

$$\rho_A=\mathrm{Tr}_{23}\,\rho_3^{\rm GHZ}(x)=\frac{1}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix} \tag{2.62}$$

$$\rho_B=\mathrm{Tr}_1\,\rho_3^{\rm GHZ}(x)=\begin{pmatrix}\frac{2+x}{6}&0&0&0\\0&\frac{1-x}{6}&\frac{1-x}{6}&0\\0&\frac{1-x}{6}&\frac{1-x}{6}&0\\0&0&0&\frac{2+x}{6}\end{pmatrix} \tag{2.63}$$

The nonzero eigenvalues of ρ_B and its entropy $S(B)$ are respectively given by

$$\mu_1=\frac{1-x}{3},\ \ \mu_2=\mu_3=\frac{2+x}{6} \tag{2.64}$$

$$S(B)=-\frac{1-x}{3}\,\log_2\frac{1-x}{3}-\frac{2(2+x)}{6}\,\log_2\frac{2+x}{6} \tag{2.65}$$

The von-Neumann conditional entropy $S(A|B)$ of the state $\rho_3^{\mathrm{GHZ}}(x)$ in its $1:2$ partition is given by

$$S(A|B) = -3\left(\frac{1-x}{4}\right)\log_2\left(\frac{1-x}{4}\right) - \left(\frac{1+3x}{4}\right)\log_2\left(\frac{1+3x}{4}\right) + \left(\frac{1-x}{3}\right)\log_2\left(\frac{1-x}{3}\right) + 2\left(\frac{2+x}{6}\right)\log_2\left(\frac{2+x}{6}\right)$$

The plot of $S(A|B)$ as a function of x is shown in Fig. 2.18. From this plot, one can obtain $(0,\ 0.5482)$ as the $1:2$ von-Neumann separability range of $\rho_3^{\mathrm{GHZ}}(x)$.

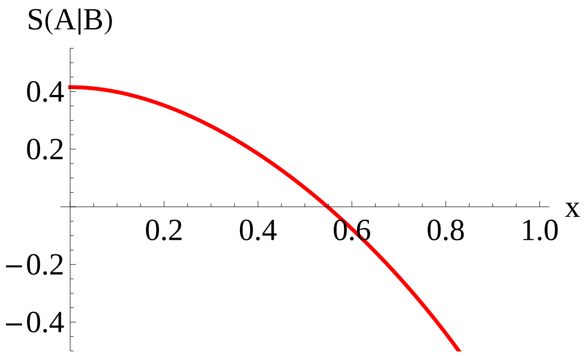

FIGURE 2.18: The von-Neumann conditional entropy $S(A|B)$ of $\rho_3^{\mathrm{GHZ}}(x)$ in its $1:2$ partition, as a function of x.

The AR q-conditional entropy for $\rho_3^{\mathrm{GHZ}}(x)$ in its $1:2$ partition is given by

$$S_q^T(A|B) = \frac{1}{q-1}\left(1 - \frac{3\left(\frac{1-x}{4}\right)^q + \left(\frac{1+3x}{4}\right)^q}{\left(\frac{1-x}{3}\right)^q + 2\left(\frac{2+x}{6}\right)^q}\right) \tag{2.66}$$

$(0,\ 0.1428)$ as the $1:2$ AR separability range [40]. From this observation it is clear that the $1:2$ AR-separability range is stricter than the $1:2$ von-Neumann separability range.

In fact, as the single qubit marginal of the state is maximally mixed thus commuting with the global density matrix $\rho_3^{\mathrm{GHZ}}(x)$, the CSTRE separability range should match with the AR-separability range. To check this, we employ the CSTRE criteria to obtain the $1:2$ separability range in $\rho_3^{\mathrm{GHZ}}(x)$ and compare the results.

The unitary operator U_B which diagonalizes ρ_B is as follows

$$U_B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{pmatrix}$$

This unitary matrix leads us to the evaluation of $\Gamma_U = (I_2 \otimes U_B)\Gamma(I_2 \otimes U_B)^\dagger$ where

$$\Gamma = (I_2 \otimes \rho_B)^{\frac{1-q}{2q}} \rho_3^{\rm GHZ}(x)(I_2 \otimes \rho_B)^{\frac{1-q}{2q}}$$

is the sandwiched matrix of $\rho_3^{\rm GHZ}(x)$ in its $1:2$ partition.

The non-zero eigenvalues of Γ_U (and of Γ) are found to be,

$$\begin{aligned} \gamma_1 &= \left(\frac{1-x}{4}\right)\left(\frac{2+x}{6}\right)^{\frac{1-q}{q}}, \ \gamma_2 = \left(\frac{1+3x}{4}\right)\left(\frac{2+x}{6}\right)^{\frac{1-q}{q}} \\ \gamma_3 &= \left(\frac{1-x}{4}\right)\left(\frac{1}{3}\right)^{\frac{1}{q}}\left[2(1-x)^{\frac{1-q}{q}} + 2\left(1+\frac{x}{2}\right)^{\frac{1-q}{q}}\right] \quad \text{(2-fold degenerate)} \end{aligned}$$

One can now readily evaluate the expression for CSTRE in Eq.(2.16) for different values of q and obtain $\tilde{D}_q^T(\rho_3^{\rm GHZ}(x)||\rho_B)$ as a function of x. Fig. 2.19 illustrates the stricter $1:2$ separability range in $\rho_3^{\rm GHZ}(x)$, for increasing values of q.

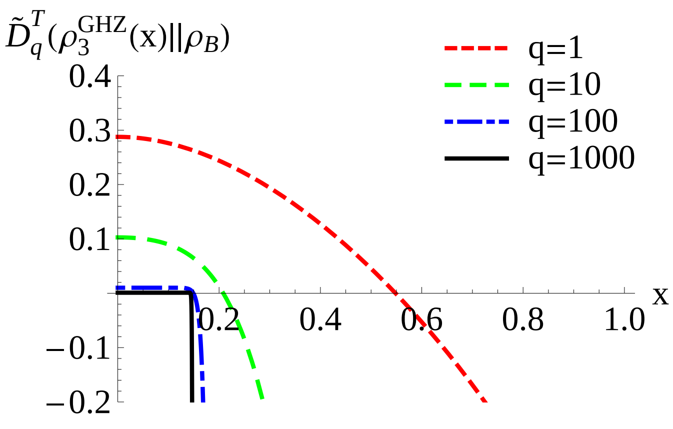

FIGURE 2.19: The CSTRE $\tilde{D}_q^T(\rho_3^{\rm GHZ}(x)||\rho_B)$ of $\rho_3^{\rm GHZ}(x)$, in its $1:2$ partition, as a function of x for different values of q.

It can be seen through Fig. 2.19 that $0 \leq x \leq 0.1428$ is the $1:2$ separability range of the one-parameter family of 3-qubit GHZ states obtained through CSTRE approach, in the limit $q \to \infty$ [62]. This separability range is in complete agreement with that obtained through AR criterion, as expected owing to the maximally mixed and hence commuting nature of ρ_A.

Now to obtain the $1:2$ separability range in $\rho_3^{\rm GHZ}(x)$ through PPT criterion, we consider the partially transposed density matrix of $\rho_3^{\rm GHZ}(x)$ in its $1:2$ partition. It is given by

$$\rho^T = \begin{pmatrix} \frac{1+x}{4} & 0 & 0 & 0 & 0 & \frac{1-x}{12} & \frac{1-x}{12} & 0 \\ 0 & \frac{1-x}{12} & \frac{1-x}{12} & 0 & 0 & 0 & 0 & \frac{1-x}{12} \\ 0 & \frac{1-x}{12} & \frac{1-x}{12} & 0 & 0 & 0 & 0 & \frac{1-x}{12} \\ 0 & 0 & 0 & \frac{1-x}{12} & \frac{x}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{x}{2} & \frac{1-x}{12} & 0 & 0 & 0 \\ \frac{1-x}{12} & 0 & 0 & 0 & 0 & \frac{1-x}{12} & \frac{1-x}{12} & 0 \\ \frac{1-x}{12} & 0 & 0 & 0 & 0 & \frac{1-x}{12} & \frac{1-x}{12} & 0 \\ 0 & \frac{1-x}{12} & \frac{1-x}{12} & 0 & 0 & 0 & 0 & \frac{1+x}{4} \end{pmatrix} \tag{2.67}$$

The square root of the eigenvalues of $(\rho^T)^2$ are seen to be

$$\alpha_1 = \frac{7x-1}{12}, \quad \alpha_2 = \frac{1+5x}{12}, \tag{2.68}$$

$$\alpha_3 = \alpha_4 = \frac{1}{\sqrt{288}}\sqrt{17+2x+17x^2+(5+x)\sqrt{9+3x(11x-2)}}$$

$$\alpha_5 = \alpha_6 = \frac{1}{\sqrt{288}}\sqrt{17+2x+17x^2-(5+x)\sqrt{9+3x(11x-2)}}$$

allowing for the evaluation of the trace norm $||\rho^T|| = \sum_i \alpha_i$. The negativity $N(\rho)$ is calculated using the relation $N(\rho) = \frac{||\rho^T||-1}{2}$.

The plot of negativity of the partially transposed density matrix ρ^T, as function of x is shown in Fig. 2.20. The negativity remains zero till $x = 0.1428$ and increases thereon. Thus, according to partial transpose criterion, the state $\rho_3^{\rm GHZ}(x)$

is separable, in its $1:2$ partition, in the range $0 \leq x \leq 0.1428$ and entangled in the range $0.1428 < x \leq 1$.

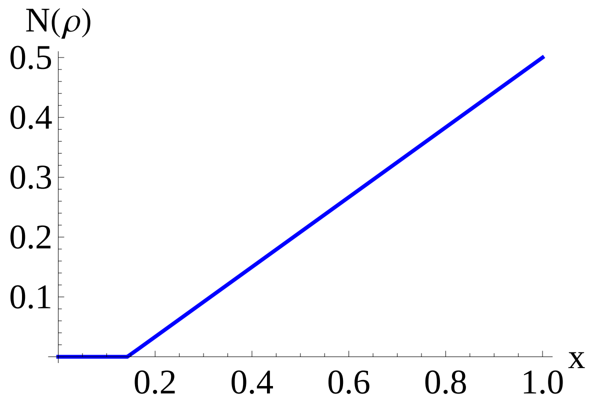

FIGURE 2.20: The plot of negativity of partial transpose as a function of x, for the $1:2$ partition of the state $\rho_3^{\mathrm{GHZ}}(x)$.

2.3.1.2 Separability of $\rho_3^{\mathrm{GHZ}}(x)$ in its $2:1$ partition.

In the $2:1$ partition, the two qubit marginal of $\rho_3^{\mathrm{GHZ}}(x)$ forms the first part A and the single qubit marginal of $\rho_3^{\mathrm{GHZ}}(x)$ forms the second part B. The two qubit marginal ρ_A of $\rho_3^{\mathrm{GHZ}}(x)$ is given by

$$\rho_A = \mathrm{Tr}_3\, \rho_3^{\mathrm{GHZ}}(x) = \begin{pmatrix} \frac{2+x}{6} & 0 & 0 & 0 \\ 0 & \frac{1-x}{6} & \frac{1-x}{6} & 0 \\ 0 & \frac{1-x}{6} & \frac{1-x}{6} & 0 \\ 0 & 0 & 0 & \frac{2+x}{6} \end{pmatrix}$$

and its nonzero eigenvalues are

$$\mu_1 = \frac{1-x}{3}, \quad \mu_2 = \mu_3 = \frac{2+x}{6}$$

The density matrix ρ_B, the single qubit marginal of $\rho_3^{\mathrm{GHZ}}(x)$ is maximally mixed with its entropy being $S(B) = 1$. That is,

$$\begin{aligned} \rho_B &= \mathrm{Tr}_{12}\, \rho_3^{\mathrm{GHZ}}(x) = \mathrm{diag}\left(\frac{1}{2}, \frac{1}{2}\right), \\ S(B) &= -\frac{1}{2}\log_2 \frac{1}{2} - \frac{1}{2}\log_2 \frac{1}{2} = 1 \end{aligned}$$

The von-Neumann conditional entropy $S(A|B) = S(A,\, B) - S(B)$ of the state $\rho_3^{\mathrm{GHZ}}(x)$ in its $2:1$ partition is given by

$$S(A|B) = -3\left(\frac{1-x}{4}\right)\log_2\left(\frac{1-x}{4}\right) - \left(\frac{1+3x}{4}\right)\log_2\left(\frac{1+3x}{4}\right) - 1$$

The identification of the zeroes of $S(A|B)$, one can obtain $(0,\ 0.7476)$ as the $2:1$ von-Neumann separability range of $\rho_3^{\mathrm{GHZ}}(x)$, which is much greater than the corresponding $1:2$ separability range.

The AR q-conditional entropy in the $2:1$ partition of $\rho_3^{\mathrm{GHZ}}(x)$ is obtained after substituting the eigenvalues corresponding to global and single qubit marginal of $\rho_3^{\mathrm{GHZ}}(x)$ in the expression for $S_q^T(A|B)$.

$$S_q^T(A|B) = \frac{1}{q-1}\left(1 - \frac{3\left(\frac{1-x}{4}\right)^q + \left(\frac{1+3x}{4}\right)^q}{2\left(\frac{1}{2}\right)^q}\right) \tag{2.69}$$

By obtaining the zero of $S_q^T(A|B) = 0$ in the limit $q \to \infty$, the $2:1$ AR separability range of $\rho_3^{\mathrm{GHZ}}(x)$ is obtained as $(0,\ 0.3333)$.

Now, the CSTRE criterion is employed to obtain the $2:1$ separability range in $\rho_3^{\mathrm{GHZ}}(x)$. As the subsystem B, the single qubit marginal of $\rho_3^{\mathrm{GHZ}}(x)$ is already in the diagonal form one can readily evaluate the sandwiched matrix

$$\Gamma \;=\; (I_4 \otimes \rho_B)^{\frac{1-q}{2q}}\, \rho_3^{\mathrm{GHZ}}(x) (I_4 \otimes \rho_B)^{\frac{1-q}{2q}}$$

and determine its eigenvalues γ_i. On evaluating the expression for CSTRE in Eq.(2.16), $\tilde{D}_q^T(\rho_3^{\mathrm{GHZ}}(x)||\rho_B)$ is obtained as a function of x and q. Fig. 2.21 illustrates that in the limit $q \to 1$, the $2:1$ von-Neumann separability range (i.e., $(0,\ 0.7476)$) results and in the limit $q \to \infty$, $(0,\ 0.3333)$ is obtained as the $2:1$ CSTRE separability range. Here too, the $2:1$ CSTRE separability range is weaker than the corresponding $1:2$ separability range. This discrepancy between $1:2$ and $2:1$ CSTRE separability range again suggests that, to obtain the better separability range, the global density matrix must be sandwiched between the biggest marginal of the bipartition under consideration.

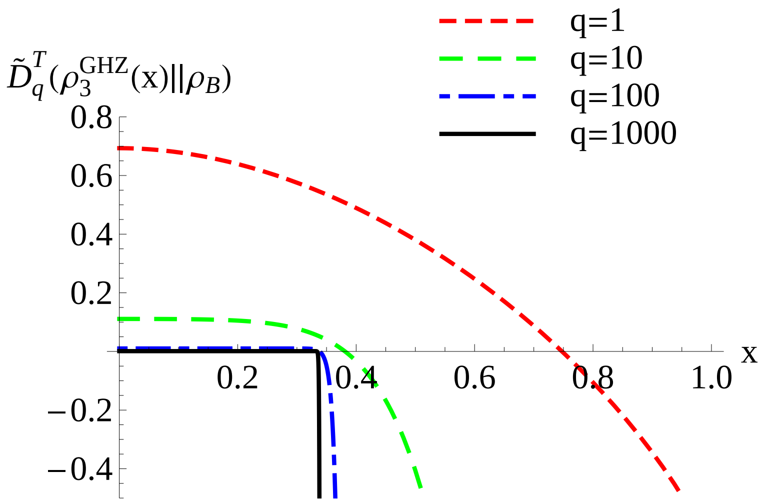

FIGURE 2.21: The CSTRE $\tilde{D}_q^T(\rho_3^{\mathrm{GHZ}}(x)||\rho_B)$ of $\rho_3^{\mathrm{GHZ}}(x)$, in its $2:1$ partition, as a function of x for different values of q.

Finally, it can be observed that the $2:1$ separability range in $\rho_3^{\mathrm{GHZ}}(x)$ obtained through Positive Partial Transpose criterion is the same as the corresponding $1:2$ separability range. This is expected because PPT criterion respects the symmetry of a state and hence the $1:2$, $2:1$ separability ranges are equal. In fact, though the $1:2$ PPT separability range of $\rho_3^{\mathrm{GHZ}}(x)$ given by $(0,\ 0.1428)$ matches with its $1:2$ CSTRE separability range, the $2:1$ CSTRE separability range given by $(0,\ 0.3333)$ is much weaker in comparison with the PPT separability range.

2.3.2 Bipartite separability in one parameter family of four qubit noisy GHZ-state

The symmetric one parameter family of 4-qubit mixed GHZ-states are given by

$$\rho_4^{\mathrm{GHZ}}(x) = \left(\frac{1-x}{5}\right) P_4 + x|\mathrm{GHZ}_4\rangle\langle\mathrm{GHZ}_4| \tag{2.70}$$

Here $0 \leq x \leq 1$ and $P_4 = \sum_M |2,\, M\rangle\langle 2,\, M|$ denotes the projector onto the symmetric subspace of 4-qubits spanned by the 5 angular momentum states $|2,\, M\rangle$,

$M = 2, 1, 0, -1, -2$ belonging to the maximum value $J = 2$ of total angular momentum.

The distinct non-zero eigenvalues of $\rho_4^{\text{GHZ}}(x)$ are,

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \frac{1-x}{5}, \qquad \lambda_5 = \frac{1}{5}(1+4x).$$

In the following, the bipartite separability of $\rho_4^{\text{GHZ}}(x)$, in its different partitions, is investigated through various separability criteria,

2.3.2.1 Separability of $\rho_4^{\text{GHZ}}(x)$ in its $1:3$ partition

In the $1:3$ partition, the single qubit marginal of $\rho_4^{\text{GHZ}}(x)$ forms the first part A and the remaining three qubit marginal of $\rho_4^{\text{GHZ}}(x)$ forms its second part B. The single qubit marginal ρ_A or the subsystem A of $\rho_4^{\text{GHZ}}(x)$, in the $1:3$ partition, is a maximally mixed state. That is,

$$\rho_A = \text{Tr}_{234}\, \rho_4^{\text{GHZ}}(x) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The density matrix ρ_B corresponding to subsystem B is given by

$$\rho_B = \text{Tr}_1\, \rho_4^{\text{GHZ}}(x) = \begin{pmatrix} \frac{1+2x}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1-x}{12} & \frac{1-x}{12} & 0 & \frac{1-x}{12} & 0 & 0 & 0 \\ 0 & \frac{1-x}{12} & \frac{1-x}{12} & 0 & \frac{1-x}{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-x}{12} & 0 & \frac{1-x}{12} & \frac{1-x}{12} & 0 \\ 0 & \frac{1-x}{12} & \frac{1-x}{12} & 0 & \frac{1-x}{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-x}{12} & 0 & \frac{1-x}{12} & \frac{1-x}{12} & 0 \\ 0 & 0 & 0 & \frac{1-x}{12} & 0 & \frac{1-x}{12} & \frac{1-x}{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+2x}{4} \end{pmatrix} \tag{2.71}$$

The distinct nonzero eigenvalues of ρ_B are

$$\mu_1 = \mu_2 = \frac{1-x}{4}, \mu_3 = \mu_4 = \frac{1+x}{4}$$

The von-Neumann conditional entropy of the state $\rho_4^{\rm GHZ}(x)$ in its $1:3$ partition is given by

$$S(A|B) = -\left(\frac{4(1-x)}{5}\right)\log_2\left(\frac{1-x}{5}\right) - \left(\frac{1+4x}{5}\right)\log_2\left(\frac{1+4x}{5}\right) \\ +2\left(\frac{1-x}{4}\right)\log_2\left(\frac{1-x}{4}\right) + 2\left(\frac{1+x}{4}\right)\log_2\left(\frac{1+x}{4}\right)$$

The plot of $S(A|B)$ as a function of x is shown in Fig. 2.22. It can be readily seen that $(0,\ 0.4676)$ is the $1:3$ von-Neumann separability range of $\rho_4^{\rm GHZ}(x)$. The AR q-conditional entropy of the state $\rho_4^{\rm GHZ}(x)$ in its $1:3$ partition can be

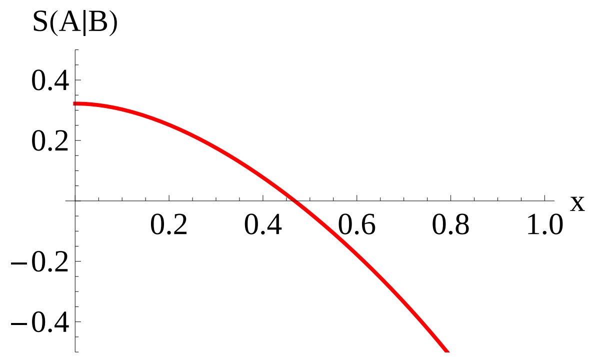

FIGURE 2.22: The von-Neumann conditional entropy $S(A|B)$ as a function of x, for $\rho_4^{\rm GHZ}(x)$ state, in its $1:3$ partitions.

evaluated as

$$S_q^T(A|B) = \frac{1}{q-1}\left(1 - \frac{4\left(\frac{1-x}{5}\right)^q + \left(\frac{1+4x}{5}\right)^q}{2\left(\frac{1-x}{4}\right)^q + 2\left(\frac{1+x}{4}\right)^q}\right) \tag{2.72}$$

Identifying the zero of $S_q^T(A|B) = 0$ in the limit $q \to \infty$, the $1:3$ AR separability range of the state $\rho_4^{\rm GHZ}(x)$ is obtained as $(0,\ 0.0909)$.

In order to evaluate the CSTRE for the state $\rho_4^{\rm GHZ}(x)$ in its $1:3$ partition, it is necessary to evaluate the unitary matrix U_B which diagonalizes the three qubit marginal ρ_B. It can be seen that

$$U_B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \\ 0 & 0 & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 & \frac{-2}{\sqrt{6}} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and one can evaluate $\Gamma_U = (I_2 \otimes U_B)\Gamma(I_2 \otimes U_B)^\dagger$ which is the unitary equivalent of

$$\Gamma = (I_2 \otimes \rho_B)^{\frac{1-q}{2q}} \rho_4^{\mathrm{GHZ}}(x)(I_2 \otimes \rho_B)^{\frac{1-q}{2q}}$$

The non-zero eigenvalues of Γ_U are found to be,

$$\begin{aligned} \gamma_1 &= \left(\frac{1-x}{5}\right)\left(\frac{1-x}{4}\right)^{\frac{1-q}{q}}, \; \gamma_2 = \left(\frac{1-x}{5}\right)\left(\frac{1+x}{4}\right)^{\frac{1-q}{q}} \\ \gamma_3 &= \left(\frac{1+4x}{5}\right)\left(\frac{1+x}{4}\right)^{\frac{1-q}{q}}, \gamma_4 = \left(\frac{1-x}{5}\right)\left(\frac{1}{4}\right)^{\frac{1}{q}}\left[3(1-x)^{\frac{1-q}{q}} + (1+x)^{\frac{1-q}{q}}\right] \end{aligned}$$

The expression for CSTRE $\tilde{D}_q^T(\rho_4^{\mathrm{GHZ}}(x)||\rho_B)$ can now be evaluated as a function of x, q using Eq.(2.16). The plot Fig. 2.23 illustrates the stricter $1:3$ separability range of $\rho_4^{\mathrm{GHZ}}(x)$, for increasing values of q. The $1:3$ separability range $0 \leq x \leq 0.0909$ of the one-parameter family of 4-qubit GHZ states, obtained in the limit $q \to \infty$ through CSTRE approach [62], is in complete agreement with that obtained through AR q-conditional entropy.

To obtain the $1:3$ separability range in $\rho_4^{\mathrm{GHZ}}(x)$ through PPT criterion, the

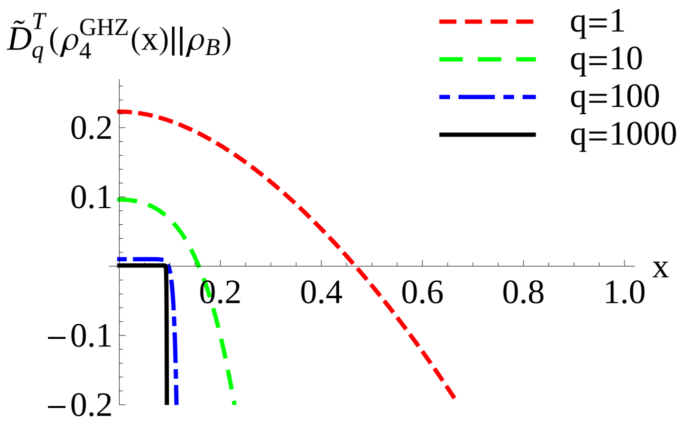

FIGURE 2.23: The CSTRE $\tilde{D}_q^T(\rho_4^{\rm GHZ}(x)||\rho_B)$ as a function of x for different values of q in the $1:3$ partition of $\rho_4^{\rm GHZ}(x)$.

partially transposed density matrix (ρ^T) of $\rho_4^{\rm GHZ}(x)$ in its $1:3$ partition is evaluated and the square root of eigenvalues of $(\rho^T)^2$ are obtained as

$$\begin{aligned}
\alpha_1 &= \frac{1-x}{20}, \qquad \alpha_2 = \frac{1+9x}{20} \\
\alpha_3 &= \frac{1-x}{4}, \qquad \alpha_4 = \frac{11x-1}{20}, \\
\alpha_5 &= \alpha_6 = \frac{1}{20}\sqrt{13+x(14+23x)+2(3+2x)\sqrt{4+x(2+19x)}}, \\
\alpha_7 &= \alpha_8 = \frac{1}{20}\sqrt{13+x(14+23x)-2(3+2x)\sqrt{(4+x(2+19x)}}
\end{aligned} \tag{2.73}$$

The plot of negativity of the partially transposed density matrix ρ^T as a function of x is shown in Fig. 2.24. One can obtain $(0,\ 0.0909)$ as the $1:3$ PPT separability range of $\rho_4^{\rm GHZ}(x)$.

2.3.2.2 Separability of $\rho_4^{\rm GHZ}(x)$ in its $3:1$ partition

In the $3:1$ partition, the three qubit marginal of $\rho_4^{\rm GHZ}(x)$ forms the first part A and the remaining single qubit marginal of $\rho_4^{\rm GHZ}(x)$ forms the second part B. The

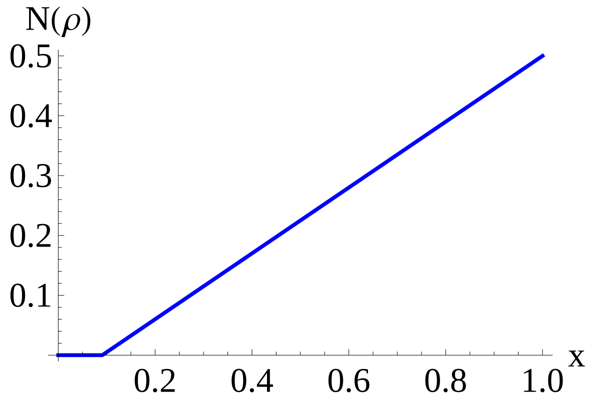

FIGURE 2.24: The plot of negativity of partial transpose of the state $\rho_4^{\mathrm{GHZ}}(x)$, in its $1:3$ partition, as a function of x.

three qubit marginal ρ_A of $\rho_4^{\mathrm{GHZ}}(x)$, in its $3:1$ partition is given by

$$\rho_A = \mathrm{Tr}_4\, \rho_4^{\mathrm{GHZ}}(x) = \begin{pmatrix} \frac{1+2x}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1-x}{12} & \frac{1-x}{12} & 0 & \frac{1-x}{12} & 0 & 0 & 0 \\ 0 & \frac{1-x}{12} & \frac{1-x}{12} & 0 & \frac{1-x}{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-x}{12} & 0 & \frac{1-x}{12} & \frac{1-x}{12} & 0 \\ 0 & \frac{1-x}{12} & \frac{1-x}{12} & 0 & \frac{1-x}{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-x}{12} & 0 & \frac{1-x}{12} & \frac{1-x}{12} & 0 \\ 0 & 0 & 0 & \frac{1-x}{12} & 0 & \frac{1-x}{12} & \frac{1-x}{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+2x}{4} \end{pmatrix}$$

The nonzero eigenvalues of ρ_A are

$$\mu_1 = \mu_2 = \frac{1-x}{4}, \qquad \mu_3 = \mu_4 = \frac{1+x}{4}.$$

The density matrix ρ_B corresponding to subsystem B is a maximally mixed state $I_2/2$ characteristic of GHZ states and noisy states involving them. Thus, $S(B) = 1$ and the von-Neumann conditional entropy of the state $\rho_4^{\mathrm{GHZ}}(x)$ in its $3:1$ partition is given by

$$S(A|B) \;=\; -\left(\frac{4(1-x)}{5}\right)\log_2\left(\frac{1-x}{5}\right) - \left(\frac{1+4x}{5}\right)\log_2\left(\frac{1+4x}{5}\right) - 1$$

From the identification of the zero of $S(A|B)$, one can obtain $(0,\ 0.7868)$ as the $3:1$ von-Neumann separability range of the state $\rho_4^{\mathrm{GHZ}}(x)$.

The AR q-conditional entropy for the state $\rho_4^{\rm GHZ}(x)$ in its $3:1$ partition is given by

$$S_q^T(A|B) = \frac{1}{q-1}\left(1 - \frac{4\left(\frac{1-x}{5}\right)^q + \left(\frac{1+4x}{5}\right)^q}{2\left(\frac{1}{2}\right)^q}\right) \tag{2.74}$$

By obtaining the zero of $S_q^T(A|B) = 0$ in the limit $q \to \infty$, one can obtain $(0,\ 0.375)$ as the $3:1$ AR separability range of the state $\rho_4^{\rm GHZ}(x)$ and it is stricter than the corresponding von-Neumann separability range.

In view of the fact that the single qubit marginal ρ_B is in the diagonal form, the matrix

$$\Gamma = (I_8 \otimes \rho_B)^{\frac{1-q}{2q}} \rho_4^{\rm GHZ}(x)(I_8 \otimes \rho_B)^{\frac{1-q}{2q}}$$

can readily be evaluated. An explicit evaluation of $\tilde{D}_q^T(\rho_4^{\rm GHZ}(x)||\rho_B)$ as a function of x and q can be done with the knowledge of eigenvalues of Γ. Fig. 2.25 illustrates that in the limit $q \to 1$, CSTRE criterion yields the $3:1$ von-Neumann separability range $(0,\ 0.7868)$ and in the limit $q \to \infty$ it gives the strictest separability range $(0,\ 0.375)$. This range matches with the $3:1$ AR-separability range and this is due to the fact that CSTRE $\tilde{D}_q^T(\rho_{AB}||\rho_B)$ reduces to AR q-conditional entropy when the single qubit marginal ρ_B is maximally mixed. Notice also that the $3:1$ CSTRE separability range is weaker than the corresponding $1:3$ separability range of the state $\rho_4^{\rm GHZ}(x)$.

It is evident that the $3:1$ PPT separability range is the same as $1:3$ PPT separability range owing to the fact that the eigenvalues of partial transpose of any density matrix in its $a:b$ partition and $b:a$ partition are same. In fact the $3:1$ PPT separability range $(0,\ 0.0909)$ though equivalent to the $1:3$ CSTRE separability range, it is much stricter than the $3:1$ CSTRE separability range $(0,\ 0.375)$.

2.3.2.3 Separability of $\rho_4^{\rm GHZ}(x)$ in its $2:2$ partition.

In the $2:2$ partition of $\rho_4^{\rm GHZ}(x)$ both the subsystems ρ_A, ρ_B contain two qubits and due to the symmetry of the state, they are identical. On explicit evaluation,

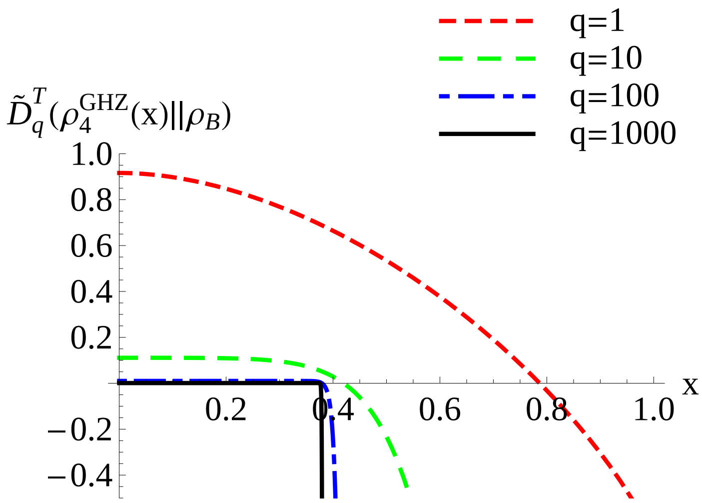

FIGURE 2.25: The CSTRE $\tilde{D}_q^T(\rho_4^{\text{GHZ}}(x)||\rho_B)$ as a function of x for different values of q in the $3:1$ partition of $\rho_4^{\text{GHZ}}(x)$.

one gets,

$$\rho_A = \rho_B = \text{Tr}_{12}\, \rho_4^{\text{GHZ}}(x) = \begin{pmatrix} \frac{2+x}{6} & 0 & 0 & 0 \\ 0 & \frac{1-x}{6} & \frac{1-x}{6} & 0 \\ 0 & \frac{1-x}{6} & \frac{1-x}{6} & 0 \\ 0 & 0 & 0 & \frac{2+x}{6} \end{pmatrix}$$

The nonzero eigenvalues of ρ_A, ρ_B are

$$\mu_1 = \mu_2 = \frac{2+x}{6}, \mu_3 = \frac{1-x}{3}$$

The von-Neumann conditional entropy of the state $\rho_4^{\text{GHZ}}(x)$ in its $2:2$ partition is given by

$$\begin{aligned} S(A|B) = &-4\left(\frac{(1-x)}{5}\right)\log_2\left(\frac{1-x}{5}\right) - \left(\frac{1+4x}{5}\right)\log_2\left(\frac{1+4x}{5}\right) \\ &+ \left(\frac{(1-x)}{3}\right)\log_2\left(\frac{1-x}{3}\right) + 2\left(\frac{(2+x)}{6}\right)\log_2\left(\frac{2+x}{6}\right) \end{aligned}$$

The $2:2$ von-Neumann separability range of $\rho_4^{\text{GHZ}}(x)$ is obtained as $(0,\ 0.6560)$.

The AR q-conditional entropy for the state $\rho_4^{\rm GHZ}(x)$ in its $2:2$ partition is given by

$$S_q^T(A|B) = \frac{1}{q-1}\left(1 - \frac{4\left(\frac{1-x}{5}\right)^q + \left(\frac{1+4x}{5}\right)^q}{\left(\frac{1-x}{3}\right)^q + 2\left(\frac{2+x}{6}\right)^q}\right) \tag{2.75}$$

In the limit $q\to\infty$, $S_q^T(A|B)$ changes sign from positive to negative and becomes zero when $x = 0.2105$. Thus the $2:2$ separability of the state $\rho_4^{\rm GHZ}(x)$ is $(0,\ 0.2105)$ [40] according to AR-criterion. It is clear that the $2:2$ AR-separability range $(0,\ 0.2105)$ is stricter than the $2:2$ von-Neumann separability range.

In order to evaluate CSTRE in the $2:2$ partition of the state $\rho_4^{\rm GHZ}(x)$, construction of the unitary U_B that diagonalizes the two qubit marginal ρ_B is necessary and it is given by

$$U_B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The unitary equivalent $\Gamma_U = (I_4\otimes U_B)\Gamma(I_4\otimes U_B)^\dagger$ of the matrix

$$\Gamma = (I_4\otimes\rho_B)^{\frac{1-q}{2q}}\rho_4^{\rm GHZ}(x)(I_4\otimes\rho_B)^{\frac{1-q}{2q}}$$

can now be evaluated and its eigenvalues γ_i help in the evaluation of CSTRE (See Eq.(2.16)) $\tilde{D}_q^T(\rho_4^{\rm GHZ}(x)||\rho_B)$ as a function of x and q. Fig. 2.26 indicates that in the limit $q\to\infty$, $(0,\ 0.2105)$ is obtained as the $2:2$ separability range of $\rho_4^{\rm GHZ}(x)$. This range is in complete agreement with that obtained through AR criterion. On an explicit evaluation of the partially transposed density matrix ρ^T of $\rho_4^{\rm GHZ}(x)$ in its $2:2$ partition, the tracenorm $||\rho^T||$ and the negativity $N(\rho)$ are obtained as functions of x. The plot of negativity of a partial transpose as a function of x is shown in Fig.2.27. $N(\rho)$ remains zero till $x = 0.0625$ and increases thereon. Thus, according to partial transpose criteria, the state $\rho_4^{\rm GHZ}(x)$ is separable in the range $0\le x\le 0.0625$ and entangled in the range $0.0625 < x \le 1$, in its $2:2$ partition.

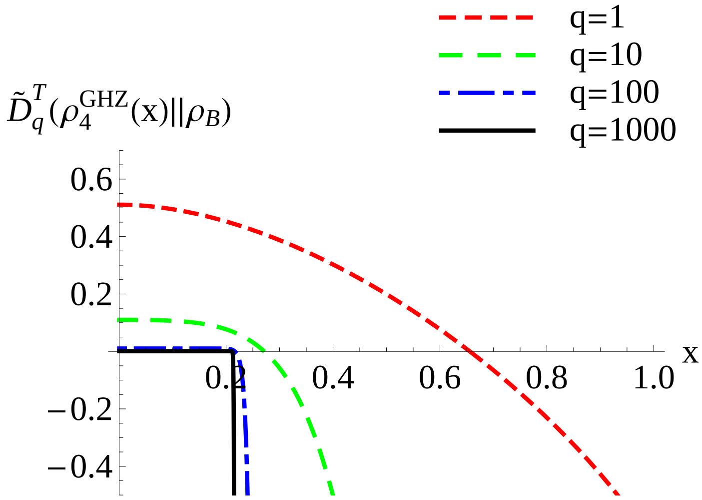

FIGURE 2.26: The CSTRE $\tilde{D}_q^T(\rho_4^{\mathrm{GHZ}}(x)||\rho_B)$ of $\rho_4^{\mathrm{GHZ}}(x)$, in its 2 : 2 partition, as a function of x for different values of q.

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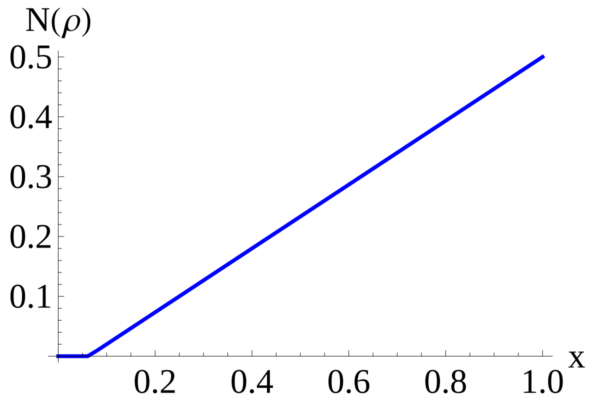

FIGURE 2.27: The plot of negativity of partial transpose in the 2 : 2 partition of the state $\rho_4^{\mathrm{GHZ}}(x)$ as a function of x.

2.4 Symmetric one parameter family of noisy states involving equal superposition of W- and obverse W states $|\mathrm{W}\bar{\mathrm{W}}_N\rangle$

The symmetric one-parameter family of noisy N-qubit W$\bar{\text{W}}$-states are defined as

$$\rho_N^{\mathrm{W}\bar{\mathrm{W}}}(x) = \left(\frac{1-x}{N+1}\right) P_N + x|\mathrm{W}\bar{\mathrm{W}}_N\rangle\langle\mathrm{W}\bar{\mathrm{W}}_N| \tag{2.76}$$

where

$$|\mathrm{W}\bar{\mathrm{W}}_N\rangle = \frac{1}{\sqrt{2}}\left(|\mathrm{W}_N\rangle + |\bar{\mathrm{W}}_N\rangle\right) \tag{2.77}$$

and

$$\left|\bar{\mathrm{W}}_N\right\rangle = \frac{1}{\sqrt{N}}[|0_1 1_2 \cdots 1_N\rangle + |1_1 0_2 1_3 \cdots 1_N\rangle + \cdots + \cdots + |1_1 1_2 1_3 \cdots 0_N\rangle] \tag{2.78}$$

is the so-called obverse W state.

2.4.1 Bipartite separability in one parameter family of noisy states involving the states $|\mathrm{W}\bar{\mathrm{W}}_3\rangle$

The three qubit symmetric one parameter family of noisy states involving the states $|\mathrm{W}\bar{\mathrm{W}}_3\rangle$ are given by

$$\rho_3^{\mathrm{W}\bar{\mathrm{W}}}(x) = \left(\frac{1-x}{4}\right) P_3 + x\left|\mathrm{W}\bar{\mathrm{W}}_3\right\rangle\left\langle\mathrm{W}\bar{\mathrm{W}}_3\right|; \quad 0 \le x \le 1 \tag{2.79}$$

The state $|\mathrm{W}\bar{\mathrm{W}}_3\rangle$ is of the form

$$\left|\mathrm{W}\bar{\mathrm{W}}_3\right\rangle = \frac{1}{\sqrt{6}}\left[|001\rangle + |010\rangle + |100\rangle + |110\rangle + |101\rangle + |011\rangle\right] \tag{2.80}$$

and its density matrix is explicitly given by

$$\rho_3^{\mathrm{W\bar{W}}}(x) = \begin{pmatrix} \frac{1-x}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1+x}{12} & \frac{1+x}{12} & \frac{x}{6} & \frac{1+x}{12} & \frac{x}{6} & \frac{x}{6} & 0 \\ 0 & \frac{1+x}{12} & \frac{1+x}{12} & \frac{x}{6} & \frac{1+x}{12} & \frac{x}{6} & \frac{x}{6} & 0 \\ 0 & \frac{x}{6} & \frac{x}{6} & \frac{1+x}{12} & \frac{x}{6} & \frac{1+x}{12} & \frac{1+x}{12} & 0 \\ 0 & \frac{1+x}{12} & \frac{1+x}{12} & \frac{x}{6} & \frac{1+x}{12} & \frac{x}{6} & \frac{x}{6} & 0 \\ 0 & \frac{x}{6} & \frac{x}{6} & \frac{1+x}{12} & \frac{x}{6} & \frac{1+x}{12} & \frac{1+x}{12} & 0 \\ 0 & \frac{x}{6} & \frac{x}{6} & \frac{1+x}{12} & \frac{x}{6} & \frac{1+x}{12} & \frac{1+x}{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1-x}{4} \end{pmatrix} \tag{2.81}$$

The distinct non-zero eigenvalues of $\rho_3^{\mathrm{W\bar{W}}}(x)$ are

$$\lambda_1 = \lambda_2 = \lambda_3 = \frac{1-x}{4}, \quad \lambda_4 = \frac{(1+3x)}{4}. \tag{2.82}$$

The entropy of the state $\rho_3^{\mathrm{W\bar{W}}}(x)$ is given by

$$S(A,B) = -3\left(\frac{1-x}{4}\right)\log_2\left(\frac{1-x}{4}\right) - \left(\frac{1+3x}{4}\right)\log_2\left(\frac{1+3x}{4}\right) \tag{2.83}$$

2.4.1.1 Separability of $\rho_3^{\mathrm{W\bar{W}}}(x)$ in its $1:2$ partition

The single qubit marginal of $\rho_3^{\mathrm{W\bar{W}}}(x)$ is given by

$$\rho_A = \mathrm{Tr}_{23}\,\rho_3^{\mathrm{W\bar{W}}}(x) = \begin{pmatrix} \frac{1}{2} & \frac{x}{3} \\ \frac{x}{3} & \frac{1}{2} \end{pmatrix} \tag{2.84}$$

The two qubit marginal of $\rho_3^{\mathrm{W\bar{W}}}(x)$ is seen to be

$$\rho_B = \begin{pmatrix} \frac{2-x}{6} & \frac{x}{6} & \frac{x}{6} & 0 \\ \frac{x}{6} & \frac{1+x}{6} & \frac{1+x}{6} & \frac{x}{6} \\ \frac{x}{6} & \frac{1+x}{6} & \frac{1+x}{6} & \frac{x}{6} \\ 0 & \frac{x}{6} & \frac{x}{6} & \frac{2-x}{6} \end{pmatrix} \tag{2.85}$$

The non-zero eigenvalues of ρ_B are

$$\eta_1 = \frac{1-x}{3}, \quad \eta_2 = \frac{2-x}{6}, \quad \eta_3 = \frac{2+3x}{6}. \tag{2.86}$$

The entropy of the subsystem ρ_B is given by

$$S(B) = -\frac{(1-x)}{3}\log_2\frac{(1-x)}{3} - \frac{(2-x)}{6}\log_2\frac{(2-x)}{6} - \frac{(2+3x)}{6}\log_2\frac{(2+3x)}{6} \tag{2.87}$$

With the entropy of the global state $\rho_3^{\mathrm{W\bar{W}}}(x)$ given in Eq. (2.83) and $S(B)$ given in Eq. (2.87), the von-Neumann conditional entropy $S(A|B)$ of the state $\rho_3^{W\bar{W}}(x)$ in its $1:2$ partition can be readily evaluated.

$$\begin{aligned} S(A|B) &= -3\left(\frac{1-x}{4}\right)\log_2\left(\frac{1-x}{4}\right) - \left(\frac{1+3x}{4}\right)\log_2\left(\frac{1+3x}{4}\right) \\ &\quad + \frac{(1-x)}{3}\log_2\frac{(1-x)}{3} + \frac{(2-x)}{6}\log_2\frac{(2-x)}{6} \\ &\quad + \frac{(2+3x)}{6}\log_2\frac{(2+3x)}{6} \end{aligned} \tag{2.88}$$

Through the identification of the zero of $S(A|B)$, one can obtain $(0,\, 0.6530)$ as the $1:2$ von-Neumann separability range of $\rho_3^{\mathrm{W\bar{W}}}(x)$.

One can now readily evaluate AR q-conditional entropy to find the $1:2$ separability range in $\rho_3^{\mathrm{W\bar{W}}}(x)$ as

$$S_q^T(A|B) = \frac{1}{q-1}\left(1 - \frac{3\left(\frac{1-x}{4}\right)^q + \left(\frac{1+3x}{4}\right)^q}{\left(\frac{2-x}{6}\right)^q + \left(\frac{1-x}{3}\right)^q + \left(\frac{2+3x}{6}\right)^q}\right)$$

The plot of $S_q^T(A|B)$ as a function of x, for different values of q is shown in Fig. 2.28. From the Fig. 2.28, in the limit $q \to \infty$, one can obtain $(0,\, 0.3333)$ as the $1:2$ AR separability range of $\rho_3^{\mathrm{W\bar{W}}}(x)$.

To employ the CSTRE criterion to obtain the $1:2$ separability range in $\rho_3^{\mathrm{W\bar{W}}}(x)$ one needs to diagonalize the two qubit marginal ρ_B. The unitary matrix which

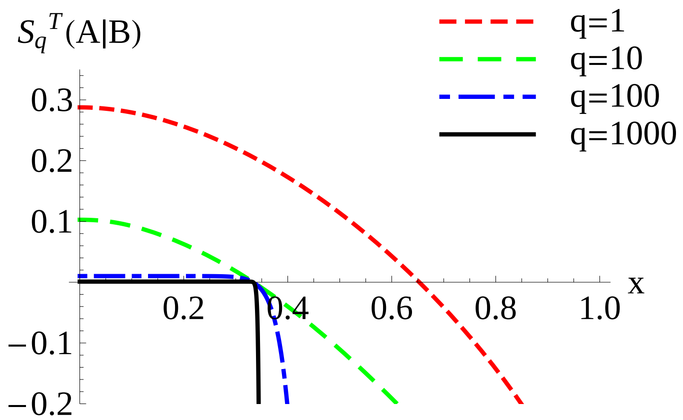

FIGURE 2.28: The AR q-conditional entropy $S_q^T(A|B)$ in the $1:2$ partition of the state $\rho_3^{\mathrm{W\bar{W}}}(x)$, as a function of x for different values of q.

diagonalizes the two qubit marginal ρ_B is given by

$$U_B = \begin{pmatrix} 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \sqrt{\frac{2}{5}} & \frac{-1}{\sqrt{10}} & \frac{-1}{\sqrt{10}} & \sqrt{\frac{2}{5}} \\ \frac{-1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{10}} & \frac{2}{\sqrt{10}} & \frac{2}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix}$$

This unitary operator U_B leads to $\Gamma_U = (I_2 \otimes U_B)\Gamma(I_2 \otimes U_B)^\dagger$ with

$$\Gamma = (I_2 \otimes \rho_B)^{\frac{1-q}{2q}} \rho_3^{\mathrm{W\bar{W}}}(x)(I_2 \otimes \rho_B)^{\frac{1-q}{2q}}.$$

From the non-zero eigenvalues γ_i's of sandwiched matrix Γ one can evaluate the expression for CSTRE of the form

$$\tilde{D}_q^T(\rho_3^{\mathrm{W\bar{W}}}(x)||\rho_B) = \frac{\sum_{i=1}^{d} \gamma_i^q - 1}{1-q}. \tag{2.89}$$

The plot of $\tilde{D}_q^T(\rho_3^{\mathrm{W\bar{W}}}(x)||\rho_B)$ as a function of x, for different values of q is shown in Fig. 2.29. It can be readily seen through Fig. 2.29 that the state $\rho_3^{\mathrm{W\bar{W}}}(x)$ is separable in the range $0 \leq x \leq 0.1896$, in its $1:2$ partition [63]. It is to be noticed (See Fig. 2.28) that AR q-conditional entropy yields a weaker separability range $0 \leq x \leq 0.3333$ for the $1:2$ partition of $\rho_3^{\mathrm{W\bar{W}}}(x)$.

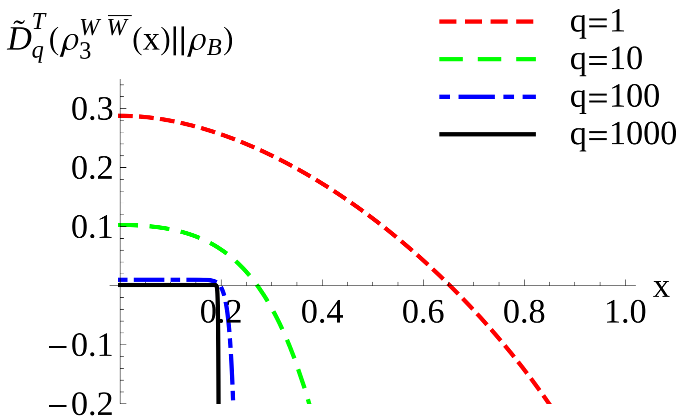

FIGURE 2.29: The CSTRE $\tilde{D}_q^T(\rho_3^{W\bar{W}}(x)||\rho_B)$ as a function of x for different values of q, in the $1:2$ partition of $\rho_3^{W\bar{W}}(x)$.

To find the $1:2$ separability range in $\rho_3^{W\bar{W}}(x)$ through PPT criterion, the partially transposed density matrix ρ^T of $\rho_3^{W\bar{W}}(x)$ in its $1:2$ partition is evaluated and it is given by

$$\rho^T = \begin{pmatrix} \frac{1-x}{4} & 0 & 0 & 0 & 0 & \frac{1+x}{12} & \frac{1+x}{12} & \frac{x}{6} \\ 0 & \frac{1+x}{12} & \frac{1+x}{12} & \frac{x}{6} & 0 & \frac{x}{6} & \frac{x}{6} & \frac{1+x}{12} \\ 0 & \frac{1+x}{12} & \frac{1+x}{12} & \frac{x}{6} & 0 & \frac{x}{6} & \frac{x}{6} & \frac{1+x}{12} \\ 0 & \frac{x}{6} & \frac{x}{6} & \frac{1+x}{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1+x}{12} & \frac{x}{6} & \frac{x}{6} & 0 \\ \frac{1+x}{12} & \frac{x}{6} & \frac{x}{6} & 0 & \frac{x}{6} & \frac{1+x}{12} & \frac{1+x}{12} & 0 \\ \frac{1+x}{12} & \frac{x}{6} & \frac{x}{6} & 0 & \frac{x}{6} & \frac{1+x}{12} & \frac{1+x}{12} & 0 \\ \frac{x}{6} & \frac{1+x}{12} & \frac{1+x}{12} & 0 & 0 & 0 & 0 & \frac{1-x}{4} \end{pmatrix} \tag{2.90}$$

On explicit evaluation of the eigenvalues α_i^2 of $(\rho^T)^2$ one can obtain the tracenorm $||\rho^T|| = \sum_i \alpha_i$ and the negativity $N(\rho) = (||\rho^T|| - 1)/2$. The plot of $N(\rho)$ as a function of x is shown in Fig. 2.30. From the Fig 2.30 one can obtain $(0,\ 0.1895)$ as the $1:2$ PPT separability range which exactly matches with the $1:2$ CSTRE separability range.

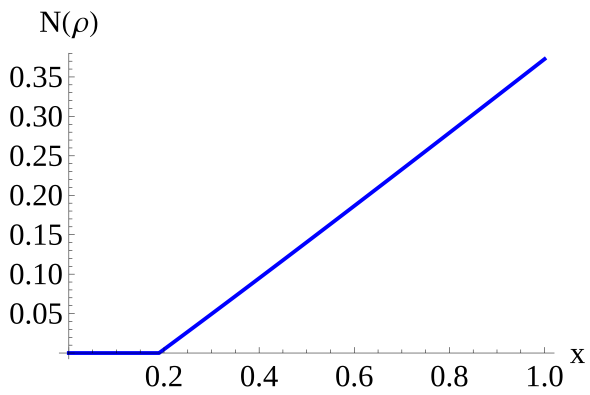

FIGURE 2.30: The plot of negativity of partial transpose in the 1 : 2 partition of the state $\rho_3^{\mathrm{W\bar{W}}}(x)$ as a function of x.

2.4.1.2 Separability of $\rho_3^{\mathrm{W\bar{W}}}(x)$ in its $2:1$ partition

In the $2:1$ partition of $\rho_3^{\mathrm{W\bar{W}}}(x)$, its two qubit marginal is considered as the first subsystem ρ_A and the single qubit marginal is the second subsystem ρ_B. It can be seen, due to the symmetry of the state $\rho_3^{\mathrm{W\bar{W}}}(x)$, that

$$\rho_A = \mathrm{Tr}_1\, \rho_3^{\mathrm{W\bar{W}}}(\mathrm{x}) = \mathrm{Tr}_2\, \rho_3^{\mathrm{W\bar{W}}}(\mathrm{x}) = \mathrm{Tr}_3\, \rho_3^{\mathrm{W\bar{W}}}(\mathrm{x})$$

and is given in Eq. (2.85). Similarly,

$$\rho_B = \mathrm{Tr}_{12}\, \rho_3^{\mathrm{W\bar{W}}}(\mathrm{x}) = \mathrm{Tr}_{23}\, \rho_3^{\mathrm{W\bar{W}}}(\mathrm{x}) = \mathrm{Tr}_{13}\, \rho_3^{\mathrm{W\bar{W}}}(\mathrm{x})$$

and is given in Eq.(2.84) The eigenvalues of ρ_B being $\mu_1 = \frac{(3-2x)}{6}$, $\mu_2 = \frac{(3+2x)}{6}$, the entropy of the subsystem ρ_B is given by

$$S(B) = -\frac{(3+2x)}{6}\log_2\frac{(3+2x)}{6} - \frac{(3-2x)}{6}\log_2\frac{(3-2x)}{6} \tag{2.91}$$

With the entropy of the global state $\rho_3^{\mathrm{W\bar{W}}}(x)$ given in Eq. (2.83) and $S(B)$ given in Eq. (2.91), the von-Neumann conditional entropy $S(A|B)$ of the state $\rho_3^{\mathrm{W\bar{W}}}(x)$ in its $2:1$ partition can be readily evaluated.

$$\begin{aligned} S(A|B) \;=\; & -3\left(\frac{1-x}{4}\right)\log_2\left(\frac{1-x}{4}\right) - \left(\frac{1+3x}{4}\right)\log_2\left(\frac{1+3x}{4}\right) \\ & + \left(\frac{3+2x}{6}\right)\log_2\left(\frac{3+2x}{6}\right) + \left(\frac{3-2x}{6}\right)\log_2\left(\frac{3-2x}{6}\right) \end{aligned} \tag{2.92}$$

Through the identification of the zero of $S(A|B)$, one can obtain $(0,\, 0.8248)$ as the $2:1$ von-Neumann separability range of $\rho_3^{\mathrm{W}\bar{\mathrm{W}}}(x)$.

The AR q-conditional entropy of $\rho_3^{\mathrm{W}\bar{\mathrm{W}}}(x)$ in its $2:1$ partition is obtained as

$$S_q^T(A|B) = \frac{1}{q-1}\left[1 - \frac{3\left(\frac{1-x}{4}\right)^q + \left(\frac{1+3x}{4}\right)^q}{\left(\frac{3+2x}{6}\right)^q + \left(\frac{3-2x}{6}\right)^q}\right] \tag{2.93}$$

It can be seen that $\lim_{q\to 1} S_q^T(A|B) = S(A|B)$ yielding the separability range $(0,\, 0.8248)$ as obtained through von-Neumann conditional entropy. But as $q \to \infty$ the $S_q^T(A|B)$ changes sign from positive to negative at $x = 0.6$ yielding the $2:1$ separability range of $\rho_3^{\mathrm{W}\bar{\mathrm{W}}}(x)$ as $(0,\, 0.6)$. The $2:1$ AR-separability range is thus seen to be stricter than the $2:1$ von-Neumann separability range.

In order to employ CSTRE to obtain the $2:1$ separability range in $\rho_3^{\mathrm{W}\bar{\mathrm{W}}}(x)$, the eigenvalues γ_i of the sandwiched matrix

$$\Gamma = (I_4 \otimes \rho_B)^{\frac{1-q}{2q}}\ \rho_3^{\mathrm{W}\bar{\mathrm{W}}}(x)\ (I_4 \otimes \rho_B)^{\frac{1-q}{2q}}$$

is to be determined and that can be done through evaluation of the eigenvalues of the unitarily equivalent matrix $\Gamma_U = (I_4 \otimes U_B)\Gamma(I_4 \otimes U_B)^\dagger$. The unitary matrix U_B that diagonalizes the single qubit marginal ρ_B is given by

$$U_B = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

and the CSTRE $\tilde{D}_q^T(\rho_3^{\mathrm{W}\bar{\mathrm{W}}}(x)||\rho_B)$ given by

$$\tilde{D}_q^T(\rho_3^{\mathrm{W}\bar{\mathrm{W}}}(x)||\rho_B) = \frac{\sum_i \gamma_i^q - 1}{1-q}$$

is readily evaluated.

Fig. 2.31 illustrates that in the limit $q \to \infty$, the zero of $\tilde{D}_q^T(\rho_3^{\mathrm{W}\bar{\mathrm{W}}}(x)||\rho_B)$ occurs at $x = 0.4104$ yielding $(0,\, 0.4104)$ as the $2:1$ CSTRE separability range. Though this is stricter than von-Neumann and AR-separability ranges, it is found to be weaker than the corresponding $1:2$ separability range. Also, it is noticed that $\rho_3^{\mathrm{W}\bar{\mathrm{W}}}(x)$ being a symmetric state, the $1:2$ and $2:1$ separability ranges of the state must be the same. But none of the entropic separability criteria are able to capture the symmetry of the state as they are yielding different 1 : 2, 2 : 1 separability ranges. But the PPT criteria accommodates the symmetry of the state and the 1 : 2, 2 : 1 PPT separability ranges of $\rho_3^{\mathrm{W\bar{W}}}(x)$ are same. This is evident from the fact that the partial transpose of the state $\rho_3^{\mathrm{W\bar{W}}}(x)$ in both the partitions match with each other.

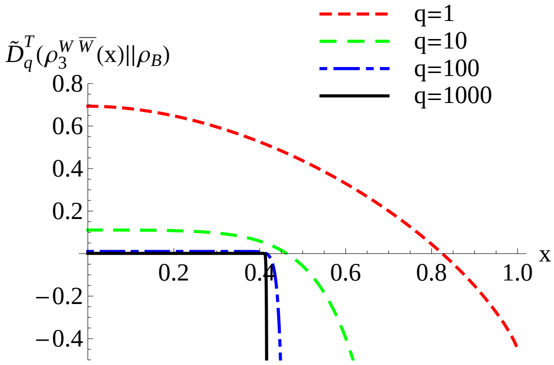

FIGURE 2.31: The CSTRE $\tilde{D}_q^T(\rho_3^{\mathrm{W\bar{W}}}(x)||\rho_B)$ as a function of x for different values of q in the $2:1$ partition of $\rho_3^{\mathrm{W\bar{W}}}(x)$.

2.4.2 Bipartite separability in one parameter family of noisy states involving the states $|\mathrm{W\bar{W}}_4\rangle$

The four qubit symmetric one parameter family of noisy states involving the state $|\mathrm{W\bar{W}}_4\rangle$ are given by

$$\rho_4^{\mathrm{W\bar{W}}}(x) = \left(\frac{1-x}{5}\right) P_4 + x\left|\mathrm{W\bar{W}}_4\right\rangle\left\langle\mathrm{W\bar{W}}_4\right|; \quad 0 \leq x \leq 1 \tag{2.94}$$

The state $|\mathrm{W\bar{W}}_4\rangle$ is explicitly seen as

$$\left|\mathrm{W\bar{W}}_4\right\rangle = \frac{1}{\sqrt{8}}\left[|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle + |1110\rangle + |1101\rangle + |1011\rangle + |0111\rangle\right]$$

The distinct non-zero eigenvalues of $\rho_4^{\mathrm{W}\bar{\mathrm{W}}}(x)$ are

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \frac{1-x}{5}, \quad \lambda_5 = \frac{1}{5}(1+4x). \tag{2.95}$$

2.4.2.1 Separability of $\rho_4^{\mathrm{W}\bar{\mathrm{W}}}(x)$ in its $1:3$ partition

To find the $1:3$ separability range in $\rho_4^{\mathrm{W}\bar{\mathrm{W}}}(x)$ one has to obtain its single qubit and three qubit marginal. The single qubit marginal of $\rho_4^{\mathrm{W}\bar{\mathrm{W}}}(x)$ is maximally mixed and is given by

$$\rho_A = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.96}$$

The three qubit marginal of $\rho_4^{\mathrm{W}\bar{\mathrm{W}}}(x)$ is given by

$$\rho_B = \begin{pmatrix}
\frac{2-x}{8} & 0 & 0 & \frac{x}{8} & 0 & \frac{x}{8} & \frac{x}{8} & 0 \\
0 & \frac{2+x}{24} & \frac{2+x}{24} & 0 & \frac{2+x}{24} & 0 & 0 & \frac{x}{8} \\
0 & \frac{2+x}{24} & \frac{2+x}{24} & 0 & \frac{2+x}{24} & 0 & 0 & \frac{x}{8} \\
\frac{x}{8} & 0 & 0 & \frac{2+x}{24} & 0 & \frac{2+x}{24} & \frac{2+x}{24} & 0 \\
0 & \frac{2+x}{24} & \frac{2+x}{24} & 0 & \frac{2+x}{24} & 0 & 0 & \frac{x}{8} \\
\frac{x}{8} & 0 & 0 & \frac{2+x}{24} & 0 & \frac{2+x}{24} & \frac{2+x}{24} & 0 \\
\frac{x}{8} & 0 & 0 & \frac{2+x}{24} & 0 & \frac{2+x}{24} & \frac{2+x}{24} & 0 \\
0 & \frac{x}{8} & \frac{x}{8} & 0 & \frac{x}{8} & 0 & 0 & \frac{2-x}{8}
\end{pmatrix} \tag{2.97}$$

The eigenvalues of ρ_B are

$$\eta_1 = \eta_2 = \frac{1-x}{4}, \quad \eta_3 = \eta_4 = \frac{1+x}{4}. \tag{2.98}$$

In order to find the $1:3$ separability range in $\rho_4^{\mathrm{W}\bar{\mathrm{W}}}(x)$ through AR criterion, the AR q-conditional entropy for $\rho_4^{\mathrm{W}\bar{\mathrm{W}}}(x)$ in its $1:3$ partition is evaluated and it is given by

$$S_q^T(A|B) = \frac{1}{q-1}\left(1 - \frac{4\left(\frac{1-x}{5}\right)^q + \left(\frac{1+4x}{5}\right)^q}{2\left(\frac{1-x}{4}\right)^q + 2\left(\frac{1+x}{4}\right)^q}\right) \tag{2.99}$$

Identifying the zero of the AR q-conditional entropy $S_q^T(A|B)$, in the limit $q \to \infty$, one can obtain $(0,\ 0.0909)$ as the $1:3$ AR separability range in $\rho_4^{\mathrm{W}\bar{\mathrm{W}}}(x)$.

In order to obtain the separability range through CSTRE criterion one needs to diagonalize the three qubit marginal ρ_B. The unitary matrix which diagonalizes the three qubit marginal ρ_B is given by

$$U_B = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & 0 & \frac{-1}{2\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & 0 \\ 0 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & 0 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{6}} & \sqrt{\frac{2}{3}} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

Using U_B and with the knowledge of eigenvalues of ρ_B, one can evaluate the diagonal matrix $\Gamma_U = (I_2 \otimes U_B)\Gamma(I_2 \otimes U_B)^\dagger$ unitarily equivalent to

$$\Gamma = (I_2 \otimes \rho_B)^{\frac{1-q}{2q}} \rho_4^{\mathrm{W\bar{W}}}(x)(I_2 \otimes \rho_B)^{\frac{1-q}{2q}}.$$

Now the expression for CSTRE $\tilde{D}_q^T(\rho_4^{\mathrm{W\bar{W}}}(x)||\rho_B)$ in its $1:3$ partition can be evaluated as

$$\tilde{D}_q^T(\rho_4^{\mathrm{W\bar{W}}}(x)||\rho_B) = \frac{\sum_i \gamma_i^q - 1}{1-q}$$

The Fig. 2.32 illustrates the behavior of $\tilde{D}_q^T(\rho_4^{\mathrm{W\bar{W}}}(x)||\rho_B)$ as a function of x, for different values of q. From Fig. 2.32 one can get $(0,\ 0.0909)$ as the separability range for $\rho_4^{\mathrm{W\bar{W}}}(x)$, in its $1:3$ partition [63]. It is to be noticed that $1:3$ AR separability range is also same as that of the one obtained through the CSTRE. This again supports the fact that whenever a marginal is maximally mixed, the CSTRE reduces to AR q-conditional entropy and hence both the separability ranges match with each other.

The $1:3$ separability range in $\rho_4^{\mathrm{W\bar{W}}}(x)$ through PPT criterion is evaluated through explicit evaluation of the partially transposed density matrix of $\rho_4^{\mathrm{W\bar{W}}}(x)$ and determination of its trace norm. The plot of negativity of partial transpose of $\rho_4^{\mathrm{W\bar{W}}}(x)$, in its $1:3$ partition, as a function of x is shown in Fig. 2.33. From the Fig. 2.33, one obtain $(0,\ 0.0909)$ as the $1:3$ PPT separability range of $\rho_4^{\mathrm{W\bar{W}}}(x)$.

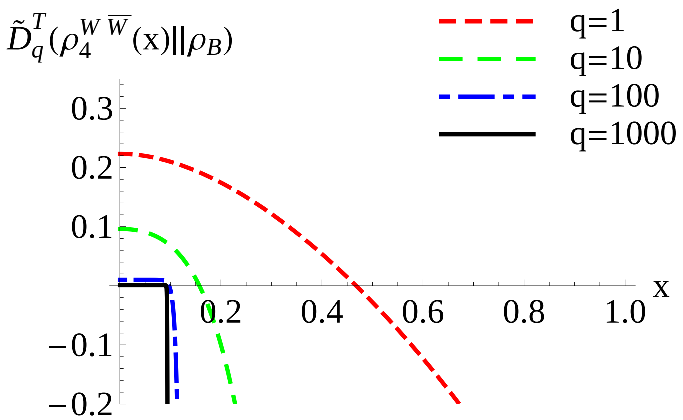

FIGURE 2.32: The CSTRE $\tilde{D}_q^T(\rho_4^{\mathrm{W\bar{W}}}(x)||\rho_B)$ as a function of x for different values of q in the $1:3$ partition of $\rho_4^{\mathrm{W\bar{W}}}(x)$.

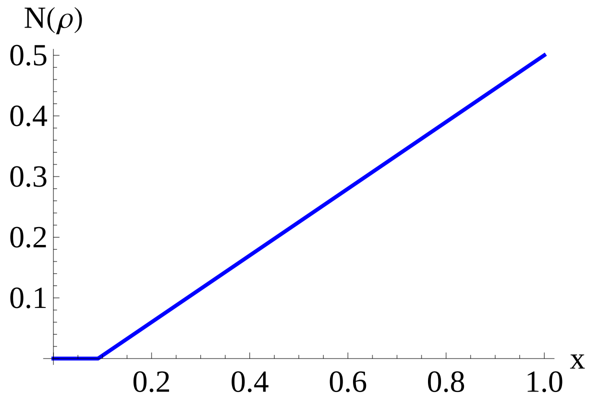

FIGURE 2.33: The plot of negativity of partial transpose in the $1:3$ partition of the state $\rho_4^{\mathrm{W\bar{W}}}(x)$ as a function of x.

It is exactly seen to match with the $1:3$ CSTRE separability range for the family of states $\rho_4^{\mathrm{W\bar{W}}}(x)$.

2.4.2.2 Separability of $\rho_4^{\mathrm{W\bar{W}}}(x)$ in its $3:1$ partition

In the $3:1$ partition, the three qubit marginal is considered as the first subsystem ρ_A and the single qubit marginal the second subsystem ρ_B. It can be seen that, due to the symmetry of the state $\rho_4^{\mathrm{W\bar{W}}}(x)$,

$$\rho_A = \mathrm{Tr}_4\, \rho_4^{\mathrm{W\bar{W}}}(\mathrm{x}) = \mathrm{Tr}_1\, \rho_4^{\mathrm{W\bar{W}}}(\mathrm{x})$$

and is given in Eq. (2.97). Similarly,

$$\rho_B = \mathrm{Tr}_{123}\,\rho_4^{\mathrm{W}\bar{\mathrm{W}}}(\mathrm{x}) = \mathrm{Tr}_{234}\,\rho_4^{\mathrm{W}\bar{\mathrm{W}}}(\mathrm{x}) = \mathrm{diag}\left(\frac{1}{2},\frac{1}{2}\right)$$

The entropy of the subsystem ρ_B is given by

$$S(B) = -2\left(\frac{1}{2}\right)\log_2\left(\frac{1}{2}\right) \tag{2.100}$$

Through the identification of the zero of $S(A|B) = S(A,\,B) - S(B)$, (0, 0.7868) is obtained as the 3 : 1 von-Neumann separability range of $\rho_4^{\mathrm{W}\bar{\mathrm{W}}}(x)$. The AR q-conditional entropy of $\rho_4^{\mathrm{W}\bar{\mathrm{W}}}(x)$ in its 3 : 1 partition is given by

$$S_q^T(A|B) = \frac{1}{q-1}\left[1 - \frac{4\left(\frac{1-x}{5}\right)^q + \left(\frac{1+4x}{5}\right)^q}{2\left(\frac{1}{2}\right)^q}\right] \tag{2.101}$$

It is seen that in the limit $q \to \infty$, the monotonically decreases function $S_q^T(A|B)$ becomes zero at $x = 0.375$ yielding the 3 : 1 separability range of $\rho_4^{\mathrm{W}\bar{\mathrm{W}}}(x)$ as $(0,\ 0.375)$.

In view of the fact that the single qubit marginal of $\rho_4^{\mathrm{W}\bar{\mathrm{W}}}(x)$ being diagonal, one can readily evaluate the eigenvalues γ_i of the sandwiched matrix

$$\Gamma = \left(I_8 \otimes \mathrm{diag}\left[\left(\frac{1}{2}\right)^{\frac{1-q}{2q}},\left(\frac{1}{2}\right)^{\frac{1-q}{2q}}\right]\right)\rho_4^{\mathrm{W}\bar{\mathrm{W}}}(x)\left(I_8 \otimes \mathrm{diag}\left[\left(\frac{1}{2}\right)^{\frac{1-q}{2q}},\left(\frac{1}{2}\right)^{\frac{1-q}{2q}}\right]\right).$$

On evaluating the expression for the CSTRE $\tilde{D}_q^T(\rho_4^{\mathrm{W}\bar{\mathrm{W}}}(x)||\rho_B) = \frac{\sum_i \gamma_i^q - 1}{1-q}$ its variation with respect to the parameter x is plotted for different values of q in Fig. 2.34. It is seen that in the limit $q \to \infty$, $(0,\ 0.375)$ is obtained as the 3 : 1 CSTRE separability range. Though this is stricter than von-Neumann separability ranges, it is found to be weaker than the corresponding 1 : 3 PPT separability range.

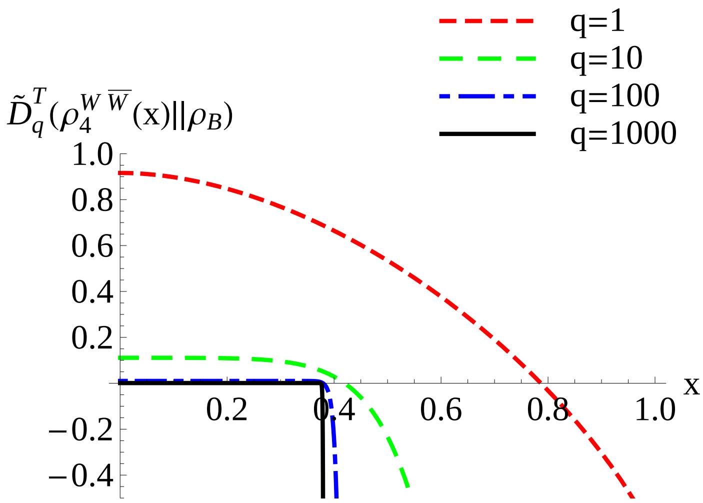

FIGURE 2.34: The CSTRE $\tilde{D}_q^T(\rho_4^{\mathrm{W\bar{W}}}(x)||\rho_B)$ as a function of x for different values of q in the $3:1$ partition of $\rho_4^{\mathrm{W\bar{W}}}(x)$.

2.4.2.3 Separability of $\rho_4^{\mathrm{W\bar{W}}}(x)$ in its $2:2$ partition

In the $2:2$ partition of $\rho_4^{\mathrm{W\bar{W}}}(x)$ both ρ_A, ρ_B are two-qubit marginals and due to the symmetry of the state $\rho_A = \rho_B$ given by

$$\rho_A = \rho_B = \begin{pmatrix} \frac{4-x}{12} & 0 & 0 & 0 \\ 0 & \frac{2+x}{12} & \frac{2+x}{12} & 0 \\ 0 & \frac{2+x}{12} & \frac{2+x}{12} & 0 \\ 0 & 0 & 0 & \frac{4-x}{12} \end{pmatrix} \tag{2.102}$$

The non-zero eigenvalues of ρ_B and its entropy $S(B)$ are respectively given by

$$\mu_1 = \mu_2 = \frac{2+x}{6}, \quad \mu_3 = \frac{1-x}{3}.$$

$$S(B) = -2\left(\frac{2+x}{6}\right)\log_2\left(\frac{2+x}{6}\right) - \left(\frac{1-x}{3}\right)\log_2\left(\frac{1-x}{3}\right)$$

The von-Neumann conditional entropy $S(A|B)$ in the $2:2$ partition of $\rho_4^{\mathrm{W\bar{W}}}(x)$ can be explicitly evaluated and it is seen that the $2:2$ von-Neumann separability range of $\rho_4^{\mathrm{W\bar{W}}}(x)$ is $(0,\ 0.6560)$.

The AR q-conditional entropy for $\rho_4^{\mathrm{W\bar{W}}}(x)$ in its $2:2$ partition is given by

$$S_q^T(A|B) = \frac{1}{q-1}\left(1 - \frac{4\left(\frac{1-x}{5}\right)^q + \left(\frac{1+4x}{5}\right)^q}{2\left(\frac{2+x}{6}\right)^q + \left(\frac{1-x}{3}\right)^q}\right)$$

By obtaining the zero of of $S_q^T(A|B) = 0$ in the limit $q \to \infty$ one can obtain $(0,\ 0.2105)$ as the $2:2$ AR separability range in $\rho_4^{\mathrm{W\bar{W}}}(x)$.

To employ the CSTRE criterion to obtain the $2:2$ separability range in $\rho_4^{\mathrm{W\bar{W}}}(x)$ one needs to diagonalize the two qubit marginal ρ_B. The unitary matrix which diagonalizes the two qubit marginal ρ_B is given by

$$U_B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \tag{2.103}$$

The unitarily equivalent matrix $\Gamma_U = (I_4 \otimes U_B)\Gamma(I_4 \otimes U_B)\dagger$ of the sandwiched matrix

$$\Gamma = (I_4 \otimes \rho_B)^{\frac{1-q}{2q}} \rho_4^{\mathrm{W\bar{W}}}(x)(I_4 \otimes \rho_B)^{\frac{1-q}{2q}}$$

allows for the evaluation of the eigenvalues γ_i of Γ and thereby the evaluation of the CSTRE $\tilde{D}_q^T(\rho_4^{\mathrm{W\bar{W}}}(x)||\rho_B)$ in its $2:2$ partition through

$$\tilde{D}_q^T(\rho_4^{\mathrm{W\bar{W}}}(x)||\rho_B) = \frac{\sum_i \gamma_i^q - 1}{1-q}.$$

Fig. 2.35 illustrates the behavior of $\tilde{D}_q^T(\rho_4^{\mathrm{W\bar{W}}}(x)||\rho_B)$ as a function of x, for different values of q. The $2:2$ CSTRE separability range of $\rho_4^{\mathrm{W\bar{W}}}(x)$ is obtained to be $(0,\ 0.2105)$. On an explicit evaluation of the partially transposed density matrix ρ^T of $\rho_4^{\mathrm{W\bar{W}}}(x)$ in its $2:2$ partition, the tracenorm $||\rho^T||$ and the negativity $N(\rho)$ are obtained as functions of x. The variation of $N(\rho)$ with respect to x is shown in Fig. 2.36. The negativity remains zero till $x = 0.0625$ and increases thereafter. Thus, according to partial transpose criteria, the state $\rho_4^{\mathrm{W\bar{W}}}(x)$ is separable in the range $0 \leq x \leq 0.0625$ and entangled in the range $0.0625 < x \leq 1$, in its $2:2$ partition.

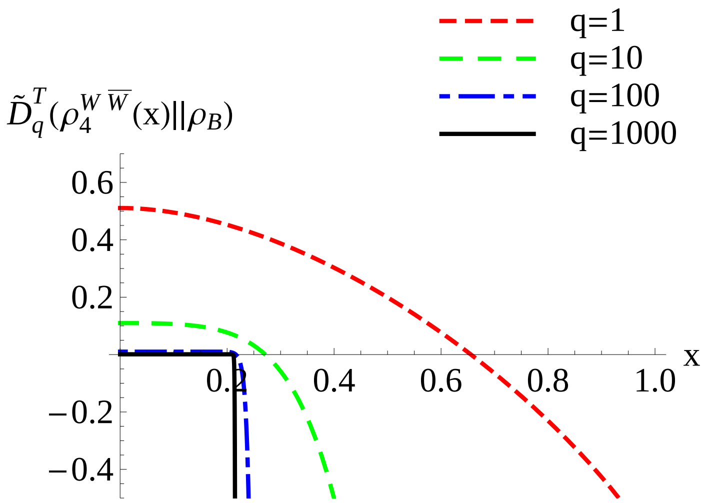

FIGURE 2.35: The CSTRE $\tilde{D}_q^T(\rho_4^{\mathrm{W}\bar{\mathrm{W}}}(x)||\rho_B)$ as a function of x in the $2:2$ partition of $\rho_4^{\mathrm{W}\bar{\mathrm{W}}}(x)$

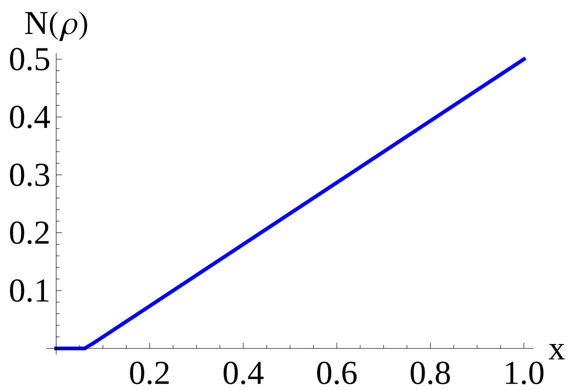

FIGURE 2.36: The plot of negativity of partial transpose of $\rho_4^{\mathrm{W}\bar{\mathrm{W}}}(x)$, in its $2:2$ partition as a function of x

Table 2.1 summarizes the results of the analysis carried out on the one parameter family of W-, GHZ-, $\mathrm{W}\bar{\mathrm{W}}$ states.

TABLE 2.1: Comparison of separability range of one parameter families of states through positivity of conditional entropies $S(A|B)$, $\lim_{q\to\infty} S_q^T(A|B)$, $\lim_{q\to\infty} \tilde{D}_q^T(\rho_{AB}||\rho_B)$ and Peres' PPT criterion

Quantum State	Von-Neumann conditional entropy	AR q-conditional entropy	CSTRE	PPT
$\rho_3^{\mathrm{W}}(x)$				
1 : 2 partition	{0, 0.5695}	{0, 0.2}	{0, 0.1547}	{0, 0.1547}
2 : 1 partition	{0, 0.7645}	{0, 0.4286}	{0, 0.3509}	{0, 0.1547}
$\rho_3^{\mathrm{GHZ}}(x)$				
1 : 2 partition	{0, 0.5482}	{0, 1/7}	{0, 1/7}	{0, 1/7}
2 : 1 partition	{0, 0.7476}	{0, 1/3}	{0, 1/3}	{0, 1/7}
$\rho_3^{\mathrm{WW}}(x)$				
1 : 2 partition	{0, 0.6530}	{0, 0.3333}	{0, 0.1895}	{0, 0.1895}
2 : 1 partition	{0, 0.8248}	{0, 0.6}	{0, 0.4104}	{0, 0.1895}
$\rho_4^{\mathrm{W}}(x)$				
1 : 3 partition	{0, 0.5193}	{0. 0.1666}	{0, 0.1123}	{0, 0.1123}
2 : 2 partition	{0, 0.6560}	{0, 0.2105}	{0, 0.2105}	{0, 0.0808}
3 : 1 partition	{0, 0.8222}	{0, 0.5454}	{0, 0.4174}	{0, 0.1123}
$\rho_4^{\mathrm{GHZ}}(x)$				
1 : 3 partition	{0, 0.4676}	{0, 0.0909}	{0, 0.0909}	{0, 0.0909}
2 : 2 partition	{0, 0.6560}	{0, 0.2105}	{0, 0.2105}	{0, 0.0625}
3 : 1 partition	{0, 0.7868}	{0, 0.375}	{0, 0.375}	{0, 0.0909}
$\rho_4^{\mathrm{WW}}(x)$				
1 : 3 partition	{0, 0.4675}	{0. 0.0909}	{0, 0.0909}	{0, 0.0909}
2 : 2 partition	{0, 0.6560}	{0, 0.2105}	{0, 0.2105}	{0, 0.0625}
3 : 1 partition	{0, 0.7868}	{0, 0.375}	{0, 0.375}	{0, 0.0909}

It is to be recalled here that the conditional entropies used hitherto, in the analysis of symmetric one parameter families of states, are

$$S(A|B) = S(A,\, B) - S(B)$$

$$S_q^T(A|B) = \frac{1}{q-1}\left(1 - \frac{\mathrm{Tr}(\rho_{AB})^q}{\mathrm{Tr}(\rho_B)^q}\right)$$

$$\tilde{D}_q^T(\rho_{AB}||\rho_B) = \frac{\tilde{Q}_q(\rho_{AB}||\rho_B) - 1}{1-q},$$

and the conditional entropies

$$S(B|A) = S(A,\, B) - S(A)$$

$$S_q^T(B|A) = \frac{1}{q-1}\left(1 - \frac{\mathrm{Tr}(\rho_{AB})^q}{\mathrm{Tr}(\rho_A)^q}\right)$$

$$\tilde{D}_q^T(\rho_{AB}||\rho_A) = \frac{\tilde{Q}_q(\rho_{AB}||\rho_A) - 1}{1-q},$$

could as well have been used for the present analysis. In fact, Table 2.2 gives the results of the analysis carried out using the positivity of conditional entropies $S(B|A)$, $\lim_{q\to\infty} S_q^T(B|A)$, $\lim_{q\to\infty} \tilde{D}_q^T(\rho_{AB}||\rho_A)$ on the one parameter family of W-, GHZ-, W$\bar{\text{W}}$ states.

It can be readily seen that the separability ranges in the $1:2$, $1:3$ partitions given in Table 2.1 are equivalent to the corresponding $2:1$, $3:1$ separability ranges given in Table 2.2. As $1:2$ separability of a state must be equivalent to its $2:1$ state for a symmetric state, one can conclude that suitable conditional entropies (either $A|B$ or $B|A$) are to be chosen in discerning the $A:B$, $B:A$ separability ranges using entropic criteria.

Through Table 2.1 (and Table 2.2) it can be concluded that the CSTRE approach yields a separability range that is either equal to or *stricter* than the range obtained through AR criterion and it matches with PPT criterion in $1:2$, $1:3$ bipartitions.

2.5 Summary

In this chapter, motivated by the recently introduced sandwiched Rényi relative entropy [56, 57], the corresponding version of Quantum Tsallis relative entropy for a pair of non-commuting density matrices is defined. The CSTRE is seen to reduce to the traditional form of Tsallis conditional entropy (AR q-conditional entropy) developed by Abe and Rajagopal [33] when the subsystem density matrix under consideration is a maximally mixed state. A theorem which states that the negative values of CSTRE necessarily imply entanglement is proved. Using the result of this theorem, the CSTRE is employed to investigate bipartite separability of symmetric noisy one parameter family involving 3 and 4 qubit W, GHZ, $\mathrm{W\bar{W}}$ states when $q \to \infty$. For the case of one parameter family of 3-, 4-qubit W states, CSTRE criterion yields the $1:2$, $1:3$ separability ranges stricter than that obtained using AR criterion and matching with the PPT separability range. For the case of one parameter family of 3-, 4- qubit GHZ states, the $1:2$, $1:3$ separability ranges obtained using CSTRE-, AR- and PPT criteria match with one another. While the $1:2$ CSTRE separability range of one parameter family of 3-qubit $\mathrm{W\bar{W}}$ state is stricter than the corresponding AR-separability range and matches with the PPT separability range, the $1:3$ CSTRE separability range of the one parameter family of 4-qubit $\mathrm{W\bar{W}}$ states is seen to be equivalent to the corresponding AR-, PPT separability range. It is also observed that for the one parameter family of 4-qubit $\mathrm{W\bar{W}}$ states, separability ranges in all bipartitions match with the corresponding separability ranges of the one parameter family of 4-qubit GHZ states. The results of this chapter clearly identify that the new entropic separability criterion using conditional version of sandwiched Tsallis relative entropy is superior to the AR-criterion. The separability ranges obtained using Peres' PPT criterion in some bipartitions of the symmetric one parameter families of states considered here are seen to be either equivalent or stricter than the ones obtained using CSTRE criterion thus implying the superiority of PPT criterion over entropic criteria.

* * * * * * * * * * * * *

TABLE 2.2: Comparison of separability ranges of one parameter families of states through positivity of conditional entropies $S(B|A)$, $\lim_{q\to\infty} S_q^T(B|A)$, $\lim_{q\to\infty} \tilde{D}_q^T(\rho_{AB}||\rho_A)$ and Peres' PPT criterion

Quantum State	Von-Neumann conditional entropy	AR q-conditional entropy	CSTRE	PPT
$\rho_3^W(x)$				
$1:2$ partition	{0, 0.7645}	{0, 0.4286}	{0, 0.3509}	{0, 0.1547}
$2:1$ partition	{0, 0.5695}	{0, 0.2}	{0, 0.1547}	{0, 0.1547}
$\rho_3^{GHZ}(x)$				
$1:2$ partition	{0, 0.7476}	{0, 1/3}	{0, 1/3}	{0, 1/7}
$2:1$ partition	{0, 0.5482}	{0, 1/7}	{0, 1/7}	{0, 1/7}
$\rho_3^{W\bar{W}}(x)$				
$1:2$ partition	{0, 0.8248}	{0, 0.6}	{0, 0.4104}	{0, 0.1895}
$2:1$ partition	{0, 0.6530}	{0, 0.3333}	{0, 0.1895}	{0, 0.1895}
$\rho_4^W(x)$				
$1:3$ partition	{0, 0.8222}	{0, 0.5454}	{0, 0.4174}	{0, 0.1123}
$2:2$ partition	{0, 0.6560}	{0, 0.2105}	{0, 0.2105}	{0, 0.0808}
$3:1$ partition	{0, 0.5193}	{0. 0.1666}	{0, 0.1123}	{0, 0.1123}
$\rho_4^{GHZ}(x)$				
$1:3$ partition	{0, 0.7868}	{0, 0.375}	{0, 0.375}	{0, 0.0909}
$2:2$ partition	{0, 0.6560}	{0, 0.2105}	{0, 0.2105}	{0, 0.0625}
$3:1$ partition	{0, 0.4676}	{0, 0.0909}	{0, 0.0909}	{0, 0.0909}
$\rho_4^{W\bar{W}}(x)$				
$1:3$ partition	{0, 0.7868}	{0, 0.375}	{0, 0.375}	{0, 0.0909}
$2:2$ partition	{0, 0.6560}	{0, 0.2105}	{0, 0.2105}	{0, 0.0625}
$3:1$ partition	{0, 0.4675}	{0. 0.0909}	{0, 0.0909}	{0, 0.0909}

Chapter 3

Bipartite separability of symmetric N-qubit noisy states using conditional quantum relative Tsallis entropy

The present chapter (Chapter 3) is an extension of the analysis carried out on the bipartite separability of one-parameter families of symmetric noisy states involving 3-, 4-qubit W-, GHZ, W$\bar{\rm W}$ states to their N-qubit counterparts. In particular, the $1 : N-1$ separability ranges of the symmetric one parameter families of states involving N-qubit W, GHZ and W$\bar{\rm W}$ states are obtained using CSTRE criterion. A comparison of the result obtained using CSTRE criterion is compared with that due to other entropic and Peres' PPT criteria.

Chapter 3 is organized as under. In Sec. 3.1, the symmetric one parameter family of states involving a symmetric N-qubit pure state is defined. In Sec. 3.2, the eigenvalues of the sandwiched matrix of the one parameter family of states involving 5-, 6- qubit W states in their respective $1 : 4$, $1 : 5$ partitions are evaluated (Secs. 3.2.1, 3.2.2). Sec. 3.2.3 details the generalization of these eigenvalues for arbitrary $N \geq 3$ and obtaining the $1 : N-1$ separability range for the symmetric one parameter family of states containing N-qubit W states. Secs. 3.3, 3.4 outline similar procedure carried out for the one parameter family of states involving N-qubit GHZ, W$\bar{\text{W}}$ states respectively. In Sec.3.5, the CSTRE criterion is made use of, to find out the separability range in higher dimensional systems such as qubit-qutrit and qutrit-qutrit systems. Sec. 3.6 gives a summary of the results in this Chapter.

3.1 Symmetric one parameter families of noisy N-qubit mixed states

The symmetric one parameter family of noisy N-qubit mixed states are given by

$$\rho_N(x) = \left(\frac{1-x}{N+1}\right) P_N + x|\Phi_N\rangle\langle\Phi_N| \tag{3.1}$$

with $P_N = \sum_{M=-\frac{N}{2}}^{\frac{N}{2}} \left|\frac{N}{2}, M\right\rangle \left\langle\frac{N}{2}, M\right|$ being the projector onto the $N+1$ dimensional maximal multiplicity subspace of the collective angular momentum of N-qubits spanned by the basis states $|\frac{N}{2}, M\rangle$, $M = \frac{N}{2}$ to $-\frac{N}{2}$ in unit steps. $|\Phi_N\rangle$ is any pure state belonging to this symmetric subspace. Notice that x is a parameter lying in the range $[0, 1]$ and when $x = 0$ the state $\rho_N(x)$ is maximally mixed (in the symmetric subspace) whereas it is a pure state $|\Phi_N\rangle$ when $x = 1$. **The separability range of one parameter family of mixed states refers to the range of values of the parameter x in which the state $\rho_N(x)$ is separable, in a chosen bipartition of the state.** The separability ranges differ for each bipartition and the $1 : N-1$ bipartition of the state $\rho_N(x)$ is analyzed in this section.

3.2 $1:N{-}1$ separability in symmetric one-parameter family of noisy W-states

It can be recalled here that the symmetric one-parameter family of noisy W states are defined in Sec. 2.2 (See Eq. (2.22)) and are of the form

$$\rho_N^{\rm W}(x) = \left(\frac{1-x}{N+1}\right) P_N + x|{\rm W}_N\rangle\langle {\rm W}_N|, \quad 0\leq x\leq 1$$
$$P_N = \sum_M \left|\frac{N}{2},\, M\right\rangle\left\langle\frac{N}{2},\, M\right|, \quad M=\frac{N}{2},\, \frac{N}{2}-1,\cdots,-\frac{N}{2}$$

where $|{\rm W}_N\rangle \equiv \left|\frac{N}{2},\, \frac{N}{2}-1\right\rangle$ is one among the basis states of the $N+1$ dimensional symmetric subspace of collective angular momentum of N-qubits. In Sec. 2.2.1 it has been shown that using the Abe-Rajagopal q-conditional entropy [33], the $1:2$ separability range of the 3-qubit state $\rho_3^{\rm W}(x)$, is found to be [0, 0.2] while the PPT criterion gives the stricter separability range [0, 0.1547] [40, 62]. In the $1:3$ partition of the 4-qubit state $\rho_4^{\rm W}(x)$ also, the AR-criterion leads to the weaker separability range [0, 0.1666] compared to the range [0, 0.1123] obtained through PPT criterion. An observation of the fact that the single qubit density matrix of $\rho_N^{\rm W}(x)$ is not maximally mixed led Rajagopal et.al., [62] to make use of the non-commuting version of the Tsallis relative entropy [56, 57] to obtain a better separability range for the case under examination. They proposed the conditional version of the sandwiched relative entropy [62] (CSTRE) $\tilde{D}_q^T(\rho_{AB}||\rho_B)$ and showed that the negative values of $\tilde{D}_q^T(\rho_{AB}||\rho_B)$ indicate entanglement in the state ρ_{AB}. Quite in accordance with the expectations, the CSTRE criterion resulted in a better separability range [62] than that through AR-criterion and it even matched with the $1:2$, $1:3$ separability ranges of $\rho_3^{\rm W}(x)$, $\rho_4^{\rm W}(x)$ obtained through PPT criterion. Continuing further with the use of AR criterion, the $1:N{-}1$ separability range of the one-parameter family of noisy W states has been obtained in Ref. [40] and it is found to be

$$0 \;\leq\; x \leq \frac{1}{N+2} \tag{3.2}$$

for any $N\geq 3$. This has been a generalization of their result for $\rho_3^{\rm W}(x), \rho_4^{\rm W}(x)$, in their respective $1:N-1$ partitions, to $\rho_N^{\rm W}(x)$. As it is seen that [62] for

3- and 4-qubit noisy W-states the CSTRE criterion yields a stricter $1:2$, $1:3$ separability range than that obtained through AR criterion, the immediate interest is to generalize the $1:N-1$ CSTRE separability range to N-qubit states $\rho_N^{\rm W}(x)$, for any $N \geq 3$. This is carried out in the following.

In Sec. 2.1, the conditional version of Tsallis relative entropy (CSTRE) of a bipartite state ρ_{AB}, corresponding to its $A:B$ partition is given by (See Eqs. (2.7), (2.8))

$$\tilde{D}_q^T(\rho_{AB}||\rho_B) = \frac{\tilde{Q}_q(\rho_{AB}||\rho_B) - 1}{1-q}$$

where

$$\tilde{Q}_q(\rho_{AB}||\rho_B) = \ \mathrm{Tr}\left\{\left((I\otimes\rho_B)^{\frac{1-q}{2q}}\rho_{AB}(I\otimes\rho_B)^{\frac{1-q}{2q}}\right)^q\right\}.$$

In order to find the $1:N-1$ separability range of the state $\rho_N^{\rm W}(x)$, one need to evaluate the eigenvalues λ_i of the sandwiched matrix,

$$\Gamma = (I_A\otimes\rho_B)^{\frac{1-q}{2q}}\rho_N^{\rm W}(x)(I_A\otimes\rho_B)^{\frac{1-q}{2q}},$$

with ρ_B being the subsystem density matrix of $\rho_N^{\rm W}(x)$ corresponding to $N-1$ qubits, so that (See Eqs. (2.12), (2.15))

$$\tilde{Q}_q(\rho_N^{\rm W}(x)||\rho_B) = \sum_{i=1}^{N+1}\lambda_i^q \tag{3.3}$$

and

$$\tilde{D}_q^T(\rho_N^{\rm W}(x)||\rho_B) = \frac{\sum_{i=1}^{N+1}\lambda_i^q - 1}{1-q}. \tag{3.4}$$

Here, as the interest is to find out the $1:N-1$ separability range, it is considered that the subsystems A, B to correspond respectively to a *single qubit* and the remaining $N-1$ qubits (as the state $\rho_N^{\rm W}(x)$ is symmetric, it does not matter which qubit is taken as subsystem A).

According to CSTRE criterion, the $1:N-1$ separability range of $\rho_N^{\rm W}(x)$ is the range in which the parameter x gives non-negative values for $\tilde{D}_q^T(\rho_N^{\rm W}(x)||\rho_B)$, in the limit $q\to\infty$.

The non-zero eigenvalues λ_i, $i = 1, 2, \ldots, N+1$ being crucial in the evaluation of $\tilde{D}_q^T(\rho_N^{\rm W}(x)||\rho_B)$, the form of these eigenvalues is examined when $N = 3, 4, 5, 6$

to analyze whether a generalization to the case of any N is possible. An explicit evaluation of the eigenvalues λ_i of the sandwiched matrix $(I_A\otimes\rho_B)^{\frac{1-q}{2q}}\,\rho_N^{\rm W}(x)(I_A\otimes\rho_B)^{\frac{1-q}{2q}}$ is carried out when $N = 3, 4, 5, 6$. In fact, as the detailed evaluation of the eigenvalues λ_i of the corresponding sandwiched matrices in the $1:2, 1:3$ partitions of 3 and 4 qubits has been carried out, the eigenvalues of the sandwiched matrices in the $1:4$, $1:5$ partition of $\rho_5^{\rm W}(x)$, $\rho_6^{\rm W}(x)$ are evaluated here.

3.2.1 $1:4$ separability in symmetric one-parameter family of noisy W-states with 5 qubits

The 5 qubit symmetric one parameter family of noisy W-states is given by

$$\rho_5^{\rm W}(x)=\left(\frac{1-x}{6}\right)P_5+x\,|{\rm W}_5\rangle\,\langle{\rm W}_5|\,;\quad 0\leq x\leq 1$$

Here,

$$P_5=\sum_{M=-\frac{5}{2}}^{\frac{5}{2}}\left|\frac{5}{2},M\right\rangle\left\langle\frac{5}{2},M\right|$$

is the projector onto the symmetric subspace of 5-qubits spanned by the six angular momentum states $|\frac{5}{2},\,M\rangle$, which are the basis states of the maximal multiplicity subspace with $J=5/2$.

The distinct non-zero eigenvalues of the state $\rho_5^{\rm W}(x)$ are given by

$$\lambda_1=\lambda_2=\lambda_3=\lambda_4=\lambda_5=\frac{1-x}{6},\;\;\lambda_6=\frac{(1+5x)}{6}. \tag{3.5}$$

In the $1:4$ partition, the single qubit marginal of $\rho_5^{\rm W}(x)$ forms the first part A and is given by

$$\rho_A={\rm Tr}_{2345}\,\rho_5^{\rm W}(x)=\begin{pmatrix}\frac{5+3x}{10} & 0\\ 0 & \frac{5-3x}{10}\end{pmatrix} \tag{3.6}$$

The four qubit marginal of $\rho_5^{\rm W}(x)$ forms the second part B.

The non-zero eigenvalues of ρ_B are given by

$$\eta_1=\frac{1}{5},\quad \eta_2=\eta_3=\eta_4=\frac{1-x}{5},\quad \eta_5=\frac{1+3x}{5}.$$

With the knowledge of the eigenvalues of ρ_B and that of $\rho_5^{\mathrm{W}}(x)$ both the von-Neumann conditional entropy $S(A|B)$ and the AR q-conditional entropy of $\rho_5^{\mathrm{W}}(x)$ in its $1:4$ partition are evaluated. The AR q-conditional entropy is explicitly given by

$$\begin{aligned} S_q^T(A|B) &= \frac{1}{q-1}\left[1-\frac{\mathrm{Tr}\left[(\rho_{AB})^q\right]}{\mathrm{Tr}\left[(\rho_B)^q\right]}\right] \\ &= \frac{1}{q-1}\left(1-\frac{5\left(\frac{1-x}{6}\right)^q+\left(\frac{1+5x}{6}\right)^q}{\left(\frac{1}{5}\right)^q+3\left(\frac{1-x}{5}\right)^q+\left(\frac{1+3x}{5}\right)^q}\right) \end{aligned}$$

The $1:5$ AR separability range of $\rho_5^{\mathrm{W}}(x)$ is obtained as $(0,\ 0.1428)$ whereas the von-Neumann conditional entropy yields the $1:5$ separability range as $(0,\ 0.4854)$.

It is observed that ρ_B is not a diagonal matrix but it can be diagonalized by a unitary matrix U_B, which is constructed using the orthonormal eigenvectors of ρ_B. The unitary matrix U_B is given by

$$U_B = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{-1}{2} & 0 & 0 & 0 & \frac{-1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{-1}{2} & \frac{-1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{-1}{2} & 0 & \frac{-1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{-1}{\sqrt{12}} & 0 & \frac{-1}{\sqrt{12}} & \frac{2}{\sqrt{12}} & 0 & 0 & \frac{2}{\sqrt{12}} & \frac{-1}{\sqrt{12}} & 0 & \frac{-1}{\sqrt{12}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

The unitary equivalent matrix of

$$\Gamma = (I_2 \otimes \rho_B)^{\frac{1-q}{2q}} \rho_5^{\mathrm{W}}(x)(I_2 \otimes \rho_B)^{\frac{1-q}{2q}}$$

is given by $\Gamma_U = (I_2 \otimes U_B)\Gamma(I_2 \otimes U_B)^\dagger$ and the non-zero eigenvalues γ_i of Γ_U (hence of Γ) are given by

$$\begin{aligned}
\gamma_1 &= \left(\frac{1-x}{6}\right)\left(\frac{1-x}{5}\right)^{\frac{1-q}{q}} \qquad \text{(3-fold degenerate)},\\
\gamma_2 &= \left(\frac{1-x}{6}\right)\left(\frac{1}{5}\right)^{\frac{1-q}{q}} \\
\gamma_3 &= \left(\frac{1-x}{6}\right)\left(\frac{1}{5}\right)^{\frac{1}{q}}\left[3(1-x)^{\frac{1-q}{q}} + 2(1+3x)^{\frac{1-q}{q}}\right] \\
\gamma_4 &= \left(\frac{1+5x}{6}\right)\left(\frac{1}{5}\right)^{\frac{1}{q}}\left[1 + 4(1+3x)^{\frac{1-q}{q}}\right]
\end{aligned} \tag{3.7}$$

One can now readily evaluate the expression for CSTRE $\tilde{D}_q^T(\rho_5^{\mathrm{W}}(x)||\rho_B)$ in its $1:4$ partition as

$$\tilde{D}_q^T(\rho_5^{\mathrm{W}}(x)||\rho_B) = \frac{\sum_i \gamma_i^q - 1}{1-q}$$

The zero of the function $\lim_{q\to\infty} \tilde{D}_q^T(\rho_5^{\mathrm{W}}(x)||\rho_B) = 0$ gives $0 \leq x \leq 0.0883$ as the $1:4$ separability range of the state $\rho_5^{\mathrm{W}}(x)$ [63]. Fig. 3.1 illustrates the $1:4$ separability range of $\rho_5^{\mathrm{W}}(x)$ when different conditional entropies are used to discern its separability. To obtain the $1:4$ separability range in $\rho_5^{\mathrm{W}}(x)$ through PPT criterion, the eigenvalues α_i^2 of $\left(\rho^T\right)^2$ with ρ^T being the partially transposed density

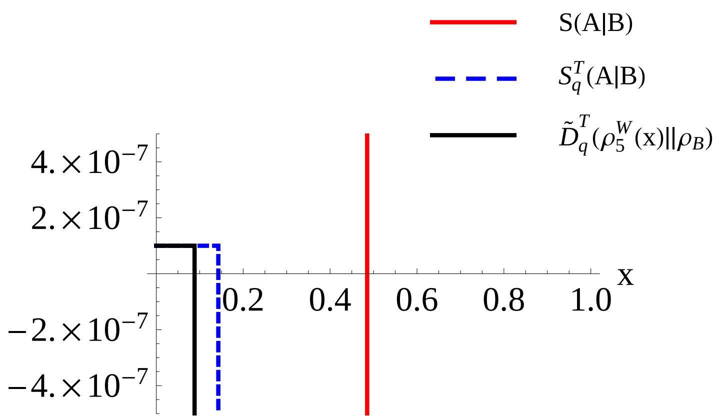

FIGURE 3.1: Plot of conditional entropies, $\lim_{q\to\infty} \tilde{D}_q^T(\rho_5^{\mathrm{W}}(x)||\rho_B)$, $\lim_{q\to\infty} S_q^T(A|B)$, $S(A|B)$ of $\rho_5^{\mathrm{W}}(x)$, in its 1 : 4 partition, as a function of x.

matrix of $\rho_5^{\mathrm{W}}(x)$ is evaluated. From the eigenvalues α_i^2 of $\left(\rho^T\right)^2$,

$$\begin{aligned}
\alpha_1 &= \alpha_2 = \alpha_3 = \frac{(1-x)}{30}, \\
\alpha_4 &= \frac{(1+5x)}{30}, \\
\alpha_5 &= \alpha_6 = \frac{(1-x)}{5}, \\
\alpha_{7/8} &= \left(\frac{37+x(22+229x)\pm 7(x-1)\sqrt{25+x(142+409x)}}{30\sqrt{2}}\right), \\
\alpha_{9/10} &= \left(\frac{37+x(118+421x)\pm (7+17x)\sqrt{25+x(553x-2)}}{30\sqrt{2}}\right).
\end{aligned} \tag{3.8}$$

the trace norm $||\rho^T|| = \sum_i \alpha_i$ and the negativity $N(\rho)$ is evaluated using the relation $N(\rho) = \frac{||\rho^T||-1}{2}$. The plot of $N(\rho)$ as a function of x, is shown in Fig. 3.2. It can be seen that, according to PPT criterion, the state $\rho_5^{\mathrm{W}}(x)$ is separable in the range $0 \le x \le 0.0883$ and entangled in the range $0.0883 < x \le 1$, in its 1 : 4 partition. Here it can be noted that the 1 : 5 CSTRE separability range (0, 0.0883) matches with the PPT separability range, whereas the 1 : 5 AR separability range

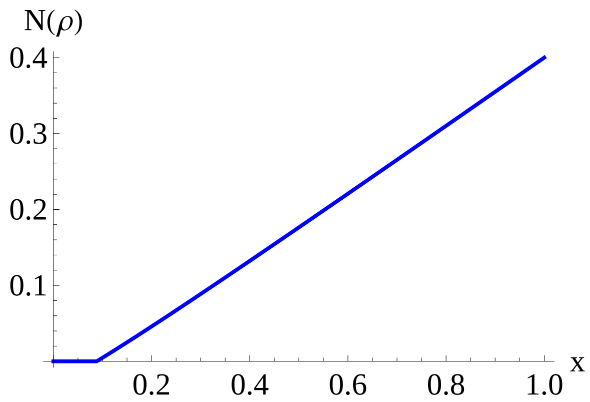

FIGURE 3.2: The variation of negativity of partial transpose of the state $\rho_5^{\rm W}(x)$, as a function of x.

(0, 0.1428) is weaker than both PPT and CSTRE separability range.

3.2.2 $1:5$ separability in symmetric one-parameter family of noisy W-states with 6 qubits

The symmetric one parameter family of noisy W-states with 6 qubits is given by

$$\rho_6^{\rm W}(x) = \left(\frac{1-x}{7}\right) P_6 + x\,|{\rm W}_6\rangle\,\langle {\rm W}_6|\,; \quad 0 \leq x \leq 1 \tag{3.9}$$

where

$$P_6 = \sum_{M=-3}^{3} |3,\, M\rangle\,\langle 3,\, M|$$

is the projector onto the symmetric subspace of 6 qubits spanned by the angular momentum states $|3,\, M\rangle$ which are basis states of the maximal multiplicity subspace with $J = 3$. There are only two distinct non zero eigenvalues for the state $\rho_6^{\rm W}(x)$ and they are given by

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \frac{1-x}{7}, \quad \lambda_7 = \frac{(1+6x)}{7}.$$

The single qubit marginal of $\rho_6^{\rm W}(x)$ is given by

$$\rho_A = {\rm Tr}_{23456}\, \rho_6^{\rm W}(x) = \frac{1}{6}\begin{pmatrix} 3+2x & 0 \\ 0 & 3-2x \end{pmatrix} \tag{3.10}$$

The non-zero eigenvalues of ρ_B, the five-qubit marginal of $\rho_6^{\rm W}(x)$, are given by

$$\eta_1 = \frac{1}{6}, \qquad \eta_2 = \eta_3 = \eta_4 = \eta_5 = \frac{1-x}{6}, \qquad \eta_6 = \frac{1+4x}{6}.$$

The AR q-conditional entropy for the state $\rho_6^{\rm W}(x)$, in its $1:5$ partition is given by

$$S_q^T(A|B) = \frac{1}{q-1}\left(1 - \frac{6\left(\frac{1-x}{7}\right)^q + \left(\frac{1+6x}{7}\right)^q}{\left(\frac{1}{6}\right)^q + 4\left(\frac{1-x}{6}\right)^q + \left(\frac{1+4x}{6}\right)^q}\right)$$

The identification of the zero of the monotonically decreasing function $S_q^T(A|B)$, in the limit $q \to \infty$ occurs at $x = 0.125$ yielding $(0,\ 0.125)$ as the $1:5$ AR separability range. In order to evaluate the conditional sandwiched Tsallis relative entropy of $\rho_6^{\rm W}(x)$ in its $1:5$ partition, the eigenvalues γ_i of the matrix

$$\Gamma = (I_2 \otimes \rho_B)^{\frac{1-q}{2q}}\, \rho_6^{\rm W}(x)\, (I_2 \otimes \rho_B)^{\frac{1-q}{2q}}$$

are to be evaluated. Towards this, it is necessary to construct the unitary matrix U_B which diagonalizes ρ_B. On constructing U_B using the orthonormal eigenvectors of ρ_B the eigenvalues γ_i of the sandwiched matrix Γ can be evaluated as the eigenvalues of the unitary equivalent matrix $\Gamma_U = (I_2 \otimes U_B)\Gamma(I_2 \otimes U_B)^\dagger$. Explicitly,

$$\begin{aligned}
\gamma_1 &= \left(\frac{1-x}{7}\right)\left(\frac{1-x}{6}\right)^{\frac{1-q}{q}}, \quad 4-\text{fold degenerate} \\
\gamma_2 &= \left(\frac{1-x}{7}\right)\left(\frac{1}{6}\right)^{\frac{1-q}{q}} \\
\gamma_3 &= \left(\frac{1-x}{7}\right)\left(\frac{1}{6}\right)^{\frac{1}{q}}\left[4(1-x)^{\frac{1-q}{q}} + 2(1+4x)^{\frac{1-q}{q}}\right] \\
\gamma_4 &= \left(\frac{1+6x}{7}\right)\left(\frac{1}{6}\right)^{\frac{1}{q}}\left[1 + 5(1+4x)^{\frac{1-q}{q}}\right]
\end{aligned} \tag{3.11}$$

The expression for CSTRE $\tilde{D}_q^T(\rho_6^{\rm W}(x)||\rho_B)$ in its $1:5$ partition is evaluated using the expression

$$\tilde{D}_q^T(\rho_6^{\rm W}(x)||\rho_B) = \frac{\sum_i \gamma_i^q - 1}{1-q}$$

Fig. 3.3 shows the variation of von-Neumann conditional entropy $S(A|B)$ and the q-conditional entropies $\lim_{q\to\infty} S_q^T(A|B)$, $\lim_{q\to\infty} \tilde{D}_q^T(\rho_6^{\rm W}(x)||\rho_B)$ of $\rho_6^{\rm W}(x)$, in its $1:5$ partition. The zero of the monotonically decreasing function $\tilde{D}_q^T(\rho_6^{\rm W}(x)||\rho_B)$

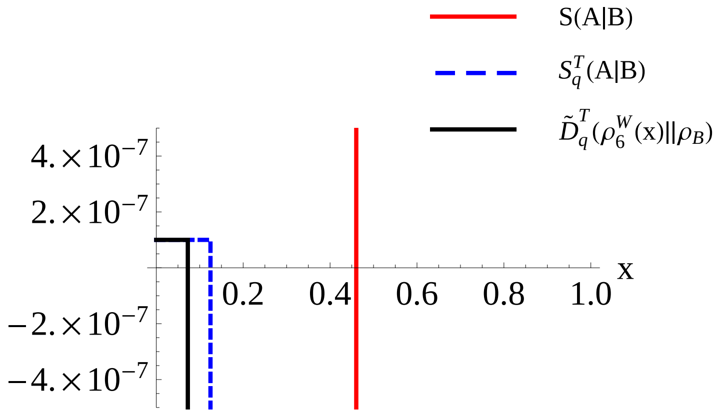

FIGURE 3.3: Plot of conditional entropies, $\lim_{q\to\infty} \tilde{D}_q^T(\rho_6^{\rm W}(x)||\rho_B)$, $\lim_{q\to\infty} S_q^T(A|B)$, $S(A|B)$ of $\rho_6^{\rm W}(x)$, in its $1:5$ partition.

is identified at $x = 0.0727$, in the limit $q\to\infty$. The $1:5$ separability range of $\rho_6^{\rm W}(x)$ is thus obtained as $(0,\ 0.0727)$. It is seen that the $1:5$ PPT separability range of the state $\rho_6^{\rm W}(x)$ is also $(0,\ 0.0727)$ while the corresponding AR-separability range $(0,\ 0.125)$ is weaker. The plot of negativity of partial transpose of $\rho_6^{\rm W}(x)$ as a function of x is shown in Fig. 3.4.

3.2.3 Eigenvalues of the sandwiched matrix in the $1:N-1$ partition of $\rho_N^{\rm W}(x)$ and its $1:N-1$ separability

The Table 3.1 provides the non-zero eigenvalues of the sandwiched matrix

$$\Gamma = (I_A\otimes\rho_B)^{\frac{1-q}{2q}}\rho_N^{\rm W}(x)(I_A\otimes\rho_B)^{\frac{1-q}{2q}}$$

when $N = 3,\ 4,\ 5,\ 6$ [63] .

TABLE 3.1: The non-zero eigenvalues γ_i of the sandwiched matrix $(I_A \otimes \rho_B)^{\frac{1-q}{2q}} \rho_N^{\mathrm{W}}(x)(I_A \otimes \rho_B)^{\frac{1-q}{2q}}$ for $N = 3$ *to* 6

Number of qubits (N)	γ_1 ($N-2$) fold degenerate	γ_2	γ_3	γ_4
$N=3$	$\left(\frac{1-x}{4}\right)\left(\frac{1-x}{3}\right)^{\frac{1-q}{q}}$	$\left(\frac{1-x}{4}\right)\left(\frac{1}{3}\right)^{\frac{1-q}{q}}$	$\left(\frac{1-x}{4}\right)\left(\frac{1}{3}\right)^{\frac{1}{q}}\left[(1-x)^{\frac{1-q}{q}}+2(1+x)^{\frac{1-q}{q}}\right]$	$\left(\frac{1+3x}{4}\right)\left(\frac{1}{3}\right)^{\frac{1}{q}}\left[1+2(1+x)^{\frac{1-q}{q}}\right]$
$N=4$	$\left(\frac{1-x}{5}\right)\left(\frac{1-x}{4}\right)^{\frac{1-q}{q}}$	$\left(\frac{1-x}{5}\right)\left(\frac{1}{4}\right)^{\frac{1-q}{q}}$	$\left(\frac{1-x}{5}\right)\left(\frac{1}{4}\right)^{\frac{1}{q}}\left[2(1-x)^{\frac{1-q}{q}}+2(1+2x)^{\frac{1-q}{q}}\right]$	$\left(\frac{1+4x}{5}\right)\left(\frac{1}{4}\right)^{\frac{1}{q}}\left[1+3(1+2x)^{\frac{1-q}{q}}\right]$
$N=5$	$\left(\frac{1-x}{6}\right)\left(\frac{1-x}{5}\right)^{\frac{1-q}{q}}$	$\left(\frac{1-x}{6}\right)\left(\frac{1}{5}\right)^{\frac{1-q}{q}}$	$\left(\frac{1-x}{6}\right)\left(\frac{1}{5}\right)^{\frac{1}{q}}\left[3(1-x)^{\frac{1-q}{q}}+2(1+3x)^{\frac{1-q}{q}}\right]$	$\left(\frac{1+5x}{6}\right)\left(\frac{1}{5}\right)^{\frac{1}{q}}\left[1+4(1+3x)^{\frac{1-q}{q}}\right]$
$N=6$	$\left(\frac{1-x}{7}\right)\left(\frac{1-x}{6}\right)^{\frac{1-q}{q}}$	$\left(\frac{1-x}{7}\right)\left(\frac{1}{6}\right)^{\frac{1-q}{q}}$	$\left(\frac{1-x}{7}\right)\left(\frac{1}{6}\right)^{\frac{1}{q}}\left[4(1-x)^{\frac{1-q}{q}}+2(1+4x)^{\frac{1-q}{q}}\right]$	$\left(\frac{1+6x}{7}\right)\left(\frac{1}{6}\right)^{\frac{1}{q}}\left[1+5(1+4x)^{\frac{1-q}{q}}\right]$

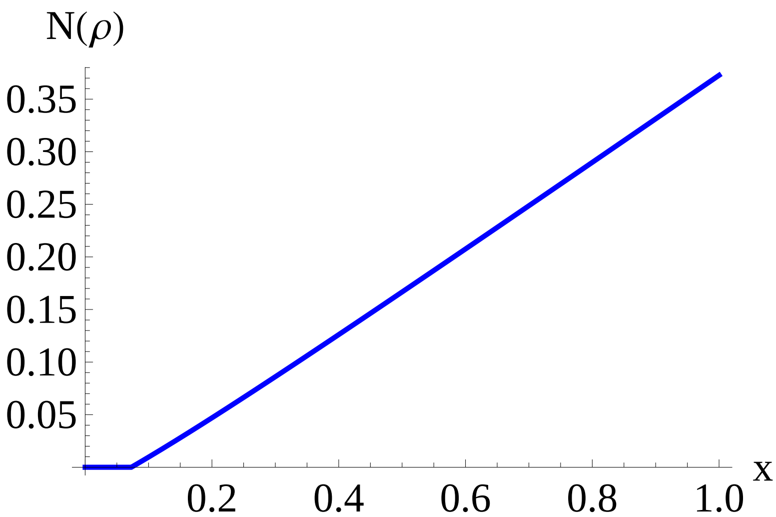

FIGURE 3.4: The plot of negativity of partial transpose of the state $\rho_6^{\rm W}(x)$ in its $1:5$ partition as a function of x.

On observing the nature of each eigenvalue presented in Table 3.1 one can obtain the eigenvalues γ_i of the sandwiched matrix Γ for arbitrary N and they are explicitly given below.

$$\begin{aligned}
\gamma_1 &= \left(\frac{1-x}{N+1}\right)\left(\frac{1-x}{N}\right)^{\frac{1-q}{q}}, \qquad (N-2) \text{ fold degenerate;} \\
\gamma_2 &= \left(\frac{1-x}{N+1}\right)\left(\frac{1}{N}\right)^{\frac{1-q}{q}}, \\
\gamma_3 &= \left(\frac{1-x}{N+1}\right)\left(\frac{1}{N}\right)^{\frac{1}{q}}\left[(N-2)(1-x)^{\frac{1-q}{q}}+2(1+(N-2)x)^{\frac{1-q}{q}}\right], \\
\gamma_4 &= \left(\frac{1+Nx}{N+1}\right)\left(\frac{1}{N}\right)^{\frac{1}{q}}\left[1+(N-1)\left(1+(N-2)x\right)^{\frac{1-q}{q}}\right].
\end{aligned} \tag{3.12}$$

On making use of the values of γ_i, $i = 1,\ 2,\ 3,\ 4$, the zero of CSTRE $\tilde{D}_q^T(\rho_N^{\rm W}(x)||\rho_B)$ (See Eq.(2.7)) can be identified for any N in the limit $q \to \infty$. It can be seen that $\tilde{D}_q^T(\rho_N^{\rm W}(x)||\rho_B)$ is a monotonically decreasing function for all values of N when $q > 1$. In the limit $q \to \infty$, $\tilde{D}_q^T(\rho_N^{\rm W}(x)||\rho_B)$ changes sign from positive to negative at its only zero which occurs at

$$x = \frac{-N + \sqrt{2N(N-1)}}{N(N-2)}. \tag{3.13}$$

Thus, the $1 : N-1$ separability range of the state $\rho_N^{\mathrm{W}}(x)$ $(N \geq 3)$, obtained using CSTRE criterion is found to be [63]

$$0 \leq x \leq \frac{-N + \sqrt{2N(N-1)}}{N(N-2)}. \tag{3.14}$$

Note that Eq.(3.14) is different from the AR result in Eq.(3.2). One can assert here that it is the non-commutativity of the single qubit marginal given by

$$\rho_1 = \rho_2 = \cdots = \rho_N = \mathrm{diag}\left(\frac{N+(N-2)x}{2N},\ \frac{N-(N-2)x}{2N}\right)$$

with $\rho_N^{\mathrm{W}}(x)$ that results in a stricter separability range through CSTRE criterion compared to its commuting version, the AR criterion. Notice also that one can immediately recover the range $(0,\ 0.1547), (0,\ 0.1123)$ respectively for the states $\rho_3^{\mathrm{W}}(x), \rho_4^{\mathrm{W}}(x)$ using the relation Eq.(3.14) and this is in accordance with the range obtained using the CSTRE criterion directly for the 3-, 4- qubit states $\rho_3^{\mathrm{W}}(x)$, $\rho_4^{\mathrm{W}}(x)$ in Ref. [62]. One can also obtain the separability ranges $(0,\ 0.0883)$, $(0,\ 0.07275)$ in the $1:4$, $1:5$ partitions respectively for the 5- and 6- qubit states of the family of noisy W-states. It is verified that these separability ranges (for $N = 3,\ 4,\ 5,\ 6$) match with those obtained through PPT criterion. One can thus conjecture that the CSTRE separability range in Eq. (3.14) for the $1 : N-1$ partition of the states $\rho_N^{\mathrm{W}}(x)$ is also the PPT separability range.

Fig. 3.5 illustrates the relatively rapid convergence of the value of x to 0.1 for AR q-conditional entropy in comparison with the convergence of x to 0.0538 for the CSTRE $\tilde{D}_q^T(\rho_8^{\mathrm{W}}(x)||\rho_B)$. From Eq. (3.14) and the discussion following it, it can be readily seen that the $1 : N-1$ separability range of $\rho_N^{\mathrm{W}}(x)$ reduces considerably with the increase in N. Thus, for large N (macroscopic limit), one can expect that a single qubit and its remaining $N-1$ qubits are entangled for the whole range $0 \leq x \leq 1$ in the state $\rho_N^{\mathrm{W}}(x)$ (See Fig. 3.6).

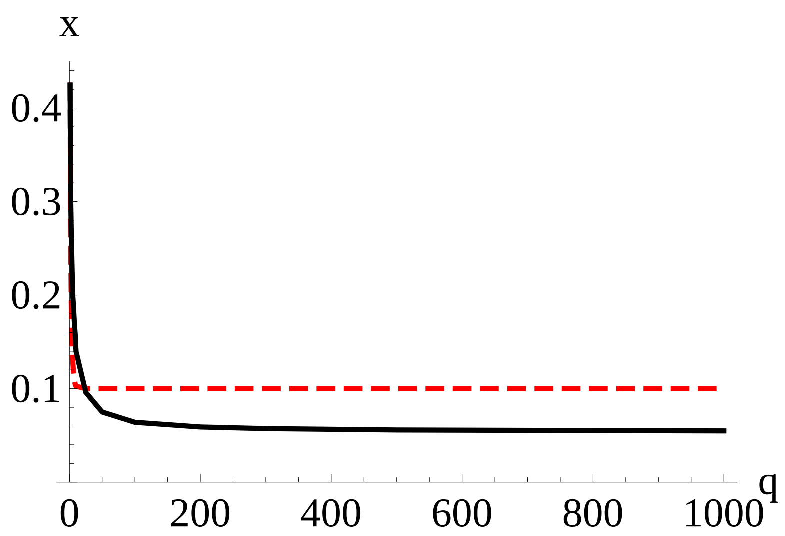

FIGURE 3.5: Implicit plot of $\tilde{D}_q^T(\rho_8^{\mathrm{W}}(x)||\rho_B) = 0$ as a function of q (solid line) indicating that $x \to 0.0538$ as $q \to \infty$. In contrast, the implicit plot of Abe-Rajagopal q-conditional entropy $S_q^T(A|B) = 0$ of the state $\rho_8^{\mathrm{W}}(x)$, in its $1:7$ partition(dashed line), leads to $x \to 0.1$ as $q \to \infty$.

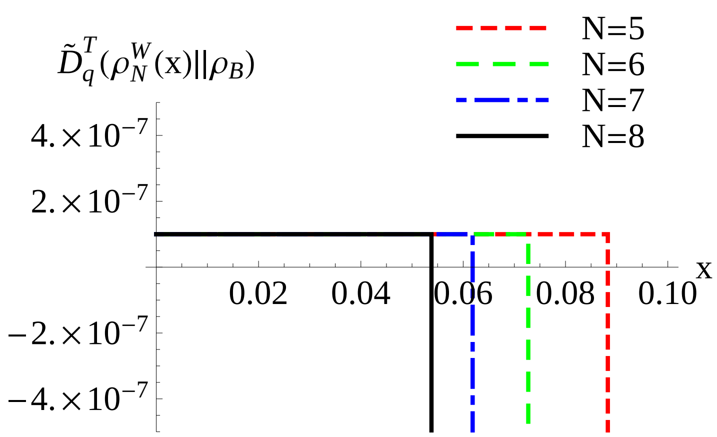

FIGURE 3.6: Illustration of the reduction of $1 : N-1$ CSTRE separability range in $\rho_N^{\mathrm{W}}(x)$ with increase in the number of qubits N from $N = 5$ to $N = 8$

3.3 $1:N-1$ separability in symmetric one-parameter family of noisy GHZ-states

The symmetric one-parameter family of noisy GHZ states with N-qubits is given by (See Eq. (2.56))

$$\rho_N^{\mathrm{GHZ}}(x) = \left(\frac{1-x}{N+1}\right) P_N + x|\mathrm{GHZ}\rangle_N\langle\mathrm{GHZ}|, \quad 0 \leq x \leq 1.$$

In chapter 2 it has been shown that for the 3-, 4-qubit states $\rho_3^{\mathrm{GHZ}}(x)$, $\rho_4^{\mathrm{GHZ}}(x)$, their respective $1:N-1$ separability ranges obtained using CSTRE criterion matched exactly with that through AR- and PPT-criteria [62]. Here, this result is generalized to N-qubit states $\rho_N^{\mathrm{GHZ}}(x)$ and the $1:N-1$ separability range of $\rho_N^{\mathrm{GHZ}}(x)$ is explicitly obtained.

3.3.1 $1:4$ separability in symmetric one-parameter family of noisy GHZ states with 5 qubits

The symmetric one parameter family of noisy GHZ-state with 5 qubits is given by

$$\rho_5^{\mathrm{GHZ}}(x) = \left(\frac{1-x}{6}\right) P_5 + x\,|\mathrm{GHZ}_5\rangle\,\langle\mathrm{GHZ}_5|\,; \quad 0 \leq x \leq 1 \tag{3.15}$$

There are only two distinct non zero eigenvalues for the state $\rho_5^{\mathrm{GHZ}}(x)$ and they are given by

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \frac{1-x}{6}, \quad \lambda_6 = \frac{(1+5x)}{6}. \tag{3.16}$$

The single qubit marginal ρ_A of $\rho_5^{\mathrm{GHZ}}(x)$ is a maximally mixed state

$$\rho_A = \mathrm{Tr}_{2345}\,\rho_5^{\mathrm{GHZ}}(x) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2/2$$

The remaining four qubit marginal of ρ_B has only two non-zero distinct eigenvalues and they are

$$\eta_1 = \eta_2 = \eta_3 = \frac{1-x}{5}, \qquad \eta_4 = \eta_5 = \frac{2+3x}{10}$$

The AR q-conditional entropy for the state $\rho_5^{\mathrm{GHZ}}(x)$ is seen to be

$$S_q^T(A|B) = \frac{1}{q-1}\left[1 - \frac{5\left(\frac{1-x}{6}\right)^q + \left(\frac{1+5x}{6}\right)^q}{3\left(\frac{1-x}{5}\right)^q + 2\left(\frac{2+3x}{10}\right)^q}\right] \tag{3.17}$$

As the 4-qubit marginal ρ_B of $\rho_5^{\mathrm{GHZ}}(x)$ is not in the diagonal form, one needs the unitary matrix which diagonalizes ρ_B so that the eigenvalues of $\Gamma_U = (I_2 \otimes U_B)\Gamma(I_2 \otimes U_B)^\dagger$ which is unitarily equivalent to the sandwiched matrix

$$\Gamma = (I_2 \otimes \rho_B)^{\frac{1-q}{2q}}\, \rho_5^{\mathrm{GHZ}}(x)\, (I_2 \otimes \rho_B)^{\frac{1-q}{2q}}$$

can be readily evaluated. In order to employ the CSTRE criterion to obtain the $1:4$ separability range in $\rho_5^{\mathrm{GHZ}}(x)$. The explicit form of the unitary matrix is given by

$$U_B = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{-1}{2} & 0 & 0 & 0 & \frac{-1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{-1}{2} & \frac{-1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{-1}{2} & 0 & \frac{-1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{-1}{\sqrt{12}} & 0 & \frac{-1}{\sqrt{12}} & \frac{2}{\sqrt{12}} & 0 & 0 & \frac{2}{\sqrt{12}} & \frac{-1}{\sqrt{12}} & 0 & \frac{-1}{\sqrt{12}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

The non-zero eigenvalues of Γ_U and hence of Γ are found to be,

$$\begin{aligned}
\gamma_1 &= \left(\frac{1-x}{6}\right)\left(\frac{1-x}{5}\right)^{\frac{1-q}{q}} \quad \text{(2-fold degenerate)}, \\
\gamma_2 &= \left(\frac{1-x}{6}\right)\left(\frac{2+3x}{10}\right)^{\frac{1-q}{q}} \\
\gamma_3 &= \left(\frac{1+5x}{6}\right)\left(\frac{2+3x}{10}\right)^{\frac{1-q}{q}} \\
\gamma_4 &= \left(\frac{1-x}{6}\right)\left(\frac{1}{5}\right)^{\frac{1}{q}}\left[4(1-x)^{\frac{1-q}{q}} + \left(1+\frac{3}{2}x\right)^{\frac{1-q}{q}}\right] \text{(2-fold degenerate)}
\end{aligned} \tag{3.18}$$

One can now evaluate the expression for CSTRE $\tilde{D}_q^T(\rho_5^{\mathrm{GHZ}}(x)||\rho_B)$ in its $1:4$ partition as

$$\tilde{D}_q^T(\rho_5^{\mathrm{GHZ}}(x)||\rho_B) = \frac{\sum_i \gamma_i^q - 1}{1-q}$$

Fig. 3.7 indicates the decreasing $1:4$ separability range of $\rho_5^{\mathrm{GHZ}}(x)$ when the von-Neumann conditional entropy, CSTRE $\tilde{D}_q^T(\rho_5^{\mathrm{GHZ}}(x)||\rho_B)$ are respectively employed to discern the separability. One can obtain $(0,\ 0.0625)$ as the $1:4$ separability range of $\rho_5^{\mathrm{GHZ}}(x)$ which matches with the AR separability range. In order to obtain

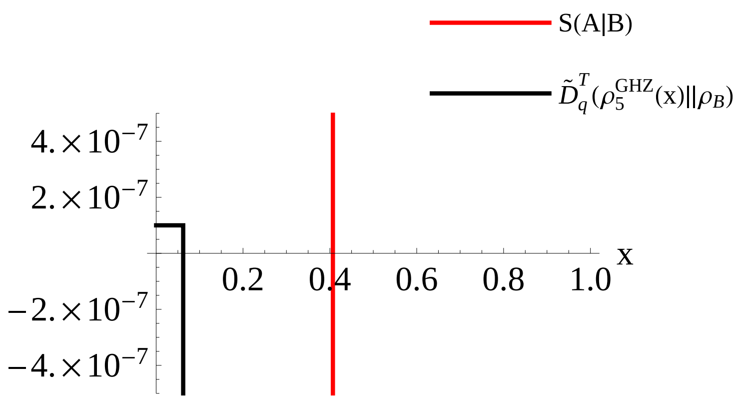

FIGURE 3.7: The variation of von-Neumann conditional entropy $S(A|B)$ and $\lim_{q\to\infty} \tilde{D}_q^T(\rho_5^{\mathrm{GHZ}}(x)||\rho_B)$ as a function of x for the state $\rho_5^{\mathrm{GHZ}}(x)$, in its $1:4$ partition.

the PPT separability range of $\rho_5^{\rm GHZ}(x)$, in its $1:4$ partition, the negativity of the $1:4$ partially transposed density matrix of $\rho_5^{\rm GHZ}(x)$ is evaluated. The graph of negativity as a function of x is as shown in Fig. 3.8. One can obtain $(0,\ 0.0625)$ as the $1:4$ PPT separability range and it matches with both AR and CSTRE separability range of $\rho_5^{\rm GHZ}(x)$, in its $1:4$ partition.

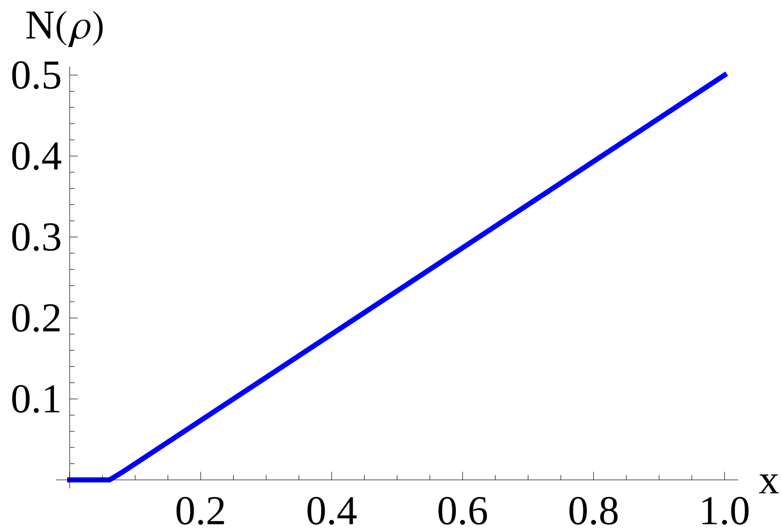

FIGURE 3.8: The graph of negativity of partial transpose of the state $\rho_5^{\rm GHZ}(x)$ as a function of x

3.3.2 $1:5$ separability in symmetric one-parameter family of noisy GHZ-states with 6 qubits

The symmetric one parameter family of noisy GHZ states with 6-qubits are given by

$$\rho_6^{\rm GHZ}(x) = \left(\frac{1-x}{7}\right) P_6 + x \left|{\rm GHZ}_6\right\rangle \left\langle{\rm GHZ}_6\right|; \tag{3.19}$$

where $0 \leq x \leq 1$ and $|{\rm GHZ}_6\rangle$ is the six qubit GHZ state.

There are only two distinct non-zero eigenvalues for $\rho_6^{\rm GHZ}(x)$ and they are given by

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \frac{1-x}{7}, \qquad \lambda_7 = \frac{(1+6x)}{7}.$$

The single qubit marginal ρ_A of $\rho_6^{\rm GHZ}(x)$ is a maximally mixed state $I_2/2$ and the non-zero eigenvalues of ρ_B, the 5- qubit marginal of $\rho_6^{\rm GHZ}(x)$ are

$$\eta_1 = \eta_2 = \eta_3 = \eta_4 = \frac{1-x}{6}, \quad \eta_5 = \eta_6 = \frac{1+2x}{6}.$$

With the knowledge of λ_i, η_j, The AR q-conditional entropy of the state $\rho_6^{\rm GHZ}(x)$, in its $1:5$ partition is evaluated as

$$S_q^T(A|B) = \frac{1}{q-1}\left(1 - \frac{6\left(\frac{1-x}{7}\right)^q + \left(\frac{1+6x}{7}\right)^q}{4\left(\frac{1-x}{6}\right)^q + 2\left(\frac{1+2x}{6}\right)^q}\right) \tag{3.20}$$

One can obtain $(0,\ 0.0454)$ as the $1:5$ AR separability range, through the identification of the zero of $S_q^T(A|B)$ in the limit $q \longrightarrow \infty$.

In order to make use of CSTRE criterion to obtain the $1:5$ separability range in $\rho_6^{\rm GHZ}(x)$ the 5-qubit marginal ρ_B is to be diagonalized by the unitary matrix U_B. On constructing the unitary matrix U_B through the orthonormal eigenvectors of the 5-qubit marginal ρ_B, one can obtain $\Gamma_U = (I_2 \otimes U_B)\,\Gamma\,(I_2 \otimes U_B)^\dagger$ where

$$\Gamma = (I_2 \otimes \rho_B)^{\frac{1-q}{2q}}\,(I_2 \otimes U_B)\rho_6^{\rm GHZ}(x)(I_2 \otimes U_B)^\dagger\,(I_2 \otimes \rho_B)^{\frac{1-q}{2q}}\,.$$

The non-zero eigenvalues of Γ_U which are the eigenvalues of the unitarily equivalent sandwiched matrix Γ are found to be,

$$\begin{aligned}
\gamma_1 &= \left(\frac{1-x}{7}\right)\left(\frac{1-x}{6}\right)^{\frac{1-q}{q}} \quad \text{(3-fold degenerate)},\\
\gamma_2 &= \left(\frac{1-x}{7}\right)\left(\frac{1+2x}{6}\right)^{\frac{1-q}{q}}\\
\gamma_3 &= \left(\frac{1+6x}{7}\right)\left(\frac{1+2x}{6}\right)^{\frac{1-q}{q}}\\
\gamma_4 &= \left(\frac{1-x}{7}\right)\left(\frac{1}{6}\right)^{\frac{1}{q}}\left[5(1-x)^{\frac{1-q}{q}} + (1+2x)^{\frac{1-q}{q}}\right] \quad \text{(2-fold degenerate)}
\end{aligned} \tag{3.21}$$

The expression for CSTRE $\tilde{D}_q^T(\rho_6^{\mathrm{GHZ}}(x)||\rho_B)$ in its $1:5$ partition can be explicitly evaluated through

$$\tilde{D}_q^T(\rho_6^{\mathrm{GHZ}}(x)||\rho_B) = \frac{\sum_i \gamma_i^q - 1}{1-q}.$$

It can be seen through Fig. 3.9 that the state $\rho_6^{\mathrm{GHZ}}(x)$ is separable in the range $0 \leq x \leq 0.0454$, in its $1:5$ partition and this range is much stricter than $0 \leq 0.3601$, the separability range obtained using von-Neumann conditional entropy. Note that the CSTRE separability range matches with the AR separability range [63] here and this is expected because whenever ρ_A is maximally mixed, $\tilde{D}_q^T(\rho_{AB}||\rho_B)$ reduces to the AR q-conditional entropy $S_q^T(A|B)$. The $1:5$ separability range

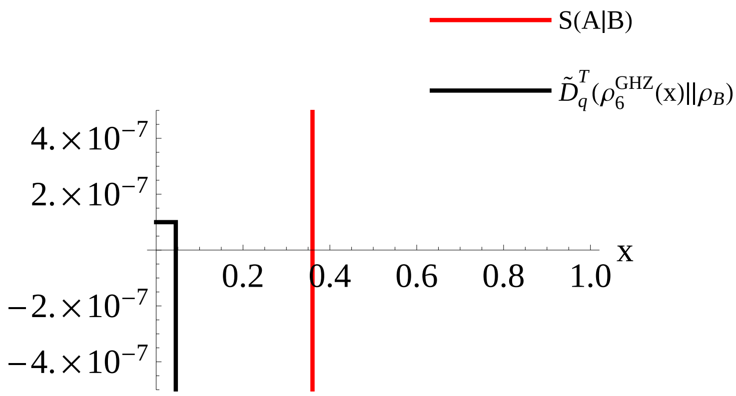

FIGURE 3.9: The variation of von-Neumann conditional entropy $S(A|B)$ and CSTRE $\lim_{q\to\infty}\tilde{D}_q^T(\rho_6^{\mathrm{GHZ}}(x)||\rho_B)$ as a function of x for $\rho_6^{\mathrm{GHZ}}(x)$, in its $1:5$ partition.

of $\rho_6^{\mathrm{GHZ}}(x)$ through PPT criterion is also evaluated through determination of the negativity of its partial transpose. The graph of negativity of partial transpose of $\rho_6^{\mathrm{GHZ}}(x)$, in its $1:5$ partition, as a function of x is shown in Fig. 3.10. It can be seen that the $1:5$ PPT separability range of $\rho_6^{\mathrm{GHZ}}(x)$ is $(0,\ 0.0454)$ and it matches with that obtained through AR- and CSTRE criteria. Here, one can observe that all the three separability criteria that is PPT, AR and CSTRE yields the same separability range $(0,\ 0.0454)$.

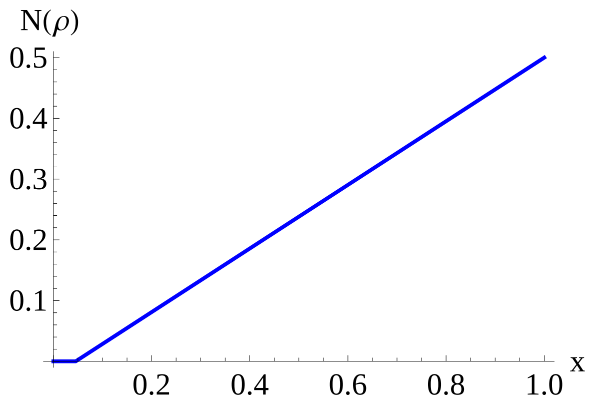

FIGURE 3.10: The graph of negativity of partial transpose of $\rho_6^{\mathrm{GHZ}}(x)$, in its $1:5$ partition, versus x.

3.3.3 Eigenvalues of the sandwiched matrix in the $1:N-1$ partition of $\rho_N^{\mathrm{GHZ}}(x)$ and its $1:N-1$ separability

The Table 3.2 provides the non-zero eigenvalues of the sandwiched matrix $(I_A \otimes \rho_B)^{\frac{1-q}{2q}} \rho_N^{\mathrm{GHZ}}(x) (I_A \otimes \rho_B)^{\frac{1-q}{2q}}$ when $N = 3, 4, 5, 6$ [63].

The eigenvalues γ_i of the sandwiched matrix $(I_A \otimes \rho_B)^{\frac{1-q}{2q}} \rho_N^{\mathrm{GHZ}}(x) (I_A \otimes \rho_B)^{\frac{1-q}{2q}}$ for $N = 3,\ 4,\ 5,\ 6$ are given in Table 3.2. Here too, there are only four distinct non-zero eigenvalues of the sandwiched matrix, two of which have $N-3$ and 2-fold degeneracies respectively. As in the case of $\rho_N^{\mathrm{W}}(x)$, here too the calculation of the eigenvalues for general N is obtained by observing the trends of each column in Table 3.2 for $N = 3, 4, 5, 6$. This leads to the four non-zero eigenvalues for all N and they are given below;

$$
\begin{aligned}
\gamma_1 &= \left(\frac{1-x}{N+1}\right)\left(\frac{1-x}{N}\right)^{\frac{1-q}{q}}, \quad (N-3)\text{-fold degenerate;} \\
\gamma_2 &= \left(\frac{1-x}{N+1}\right)\left(\frac{2+x(N-2)}{2N}\right)^{\frac{1-q}{q}}, \\
\gamma_3 &= \left(\frac{1+Nx}{N+1}\right)\left[\frac{2+x(N-2)}{2N}\right]^{\frac{1-q}{q}}, \\
\gamma_4 &= \gamma_5 = \left(\frac{1-x}{N+1}\right)\left(\frac{1}{N}\right)^{\frac{1}{q}}\left[(N-1)\,(1-x)^{\frac{1-q}{q}} + \left(1+\left(\frac{N}{2}-1\right)x\right)^{\frac{1-q}{q}}\right];
\end{aligned}
\tag{3.22}
$$

TABLE 3.2: The eigenvalues γ_i of the sandwiched matrix $(I_A \otimes \rho_B)^{\frac{1-q}{2q}} \rho_N^{\rm GHZ}(x)(I_A \otimes \rho_B)^{\frac{1-q}{2q}}$ for $N = 3,\ 4,\ 5,\ 6$

Number of qubits (N)	γ_1 ($N-3$)-fold degenerate	γ_2	γ_3	γ_4 2-fold degenerate
$N=3$	–	$\left(\frac{1-x}{4}\right)\left(\frac{2+x}{6}\right)^{\frac{1-q}{q}}$	$\left(\frac{1+3x}{4}\right)\left(\frac{2+x}{6}\right)^{\frac{1-q}{q}}$	$\left(\frac{1-x}{4}\right)\left(\frac{1}{3}\right)^{\frac{1}{q}}\left[2(1-x)^{\frac{1-q}{q}}+(1+x/2)^{\frac{1-q}{q}}\right]$
$N=4$	$\left(\frac{1-x}{5}\right)\left(\frac{1-x}{4}\right)^{\frac{1-q}{q}}$	$\left(\frac{1-x}{5}\right)\left(\frac{1+x}{4}\right)^{\frac{1-q}{q}}$	$\left(\frac{1+4x}{5}\right)\left(\frac{1+x}{4}\right)^{\frac{1-q}{q}}$	$\left(\frac{1-x}{5}\right)\left(\frac{1}{4}\right)^{\frac{1}{q}}\left[3(1-x)^{\frac{1-q}{q}}+(1+x)^{\frac{1-q}{q}}\right]$
$N=5$	$\left(\frac{1-x}{6}\right)\left(\frac{1-x}{5}\right)^{\frac{1-q}{q}}$	$\left(\frac{1-x}{6}\right)\left(\frac{2+3x}{10}\right)^{\frac{1-q}{q}}$	$\left(\frac{1+5x}{6}\right)\left(\frac{2+3x}{10}\right)^{\frac{1-q}{q}}$	$\left(\frac{1-x}{6}\right)\left(\frac{1}{5}\right)^{\frac{1}{q}}\left[4(1-x)^{\frac{1-q}{q}}+(1+3x/2)^{\frac{1-q}{q}}\right]$
$N=6$	$\left(\frac{1-x}{7}\right)\left(\frac{1-x}{6}\right)^{\frac{1-q}{q}}$	$\left(\frac{1-x}{7}\right)\left(\frac{1+2x}{6}\right)^{\frac{1-q}{q}}$	$\left(\frac{1+6x}{7}\right)\left(\frac{1+2x}{6}\right)^{\frac{1-q}{q}}$	$\left(\frac{1-x}{7}\right)\left(\frac{1}{6}\right)^{\frac{1}{q}}\left[5(1-x)^{\frac{1-q}{q}}+(1+2x)^{\frac{1-q}{q}}\right]$

The eigenvalues γ_i in Eq.(3.22) allow for the evaluation of $\tilde{D}_q^T(\rho_N^{\rm GHZ}(x)||\rho_B) = \frac{\sum_{i=1}^{N+1}\gamma_i^q-1}{1-q}$ and the zero of the monotonically decreasing function $\tilde{D}_q^T(\rho_N^{\rm GHZ}(x)||\rho_B)$ is found to be at $x=\frac{2}{N^2+N+2}$ when $q\to\infty$. One can thus obtain the $1:N-1$ separability range of the state $\rho_N^{\rm GHZ}(x)$ using CSTRE criterion as [63]

$$0 \leq x \leq \frac{2}{N^2+N+2} \tag{3.23}$$

for any $N \geq 3$. One can recall here that, in Ref. [40], the separability range in the $1:N-1$ partition of the one parameter family of GHZ states was found using AR q-conditional entropy criterion and it matches exactly with Eq. (3.23). This is to be expected as, the CSTRE criterion and AR criterion give the same results when the single qubit reduced density matrix turns out be a maximally mixed state thus commuting with its original density matrix [62]. Such a situation occurs in the case of one parameter family of noisy GHZ states [62] as the single qubit density matrix turns out to be $\frac{I_2}{2}$[1]. Thus the results of CSTRE criterion match exactly with that of AR criterion in the case of one parameter family of noisy GHZ states. But the difference between the CSTRE and AR criteria even in this case lies in the different modes of convergence of the parameter x with the increase of q. In fact, x converges slowly to the limit $\frac{2}{N^2+N+2}$ when CSTRE criterion is used whereas the convergence of x is relatively fast for AR criterion. This feature for $\rho_6^{\rm GHZ}(x)$ is illustrated in Fig. 3.11.

Quite similar to the case of $\rho_N^{\rm W}(x)$ (See Fig. 3.6), the $1:N-1$ separability range in the family of states $\rho_N^{\rm GHZ}(x)$ decreases with increasing N and this fact is illustrated through Fig. 3.12.

It is to be noticed that PPT criterion also gives the same $1:N-1$ separability range for $N=3,4,5,6$ for one-parameter family of GHZ states. Therefore, one can conjecture that Eq. (3.23) gives the PPT separability range in the $1:N-1$ partition of the one parameter family of noisy GHZ states $\rho_N^{\rm GHZ}(x)$. It is also

[1]It is to be noticed here that if the CSTRE $\tilde{D}_q^T(\rho_N^{\rm GHZ}(x)||\rho_A)$ with respect to the subsystem A is evaluated in the $1:N-1$ partition of the state $\rho_N^{\rm GHZ}(x)$, then $\rho_A=\frac{I_2}{2}$ is the single qubit marginal. Thus, the sandwiched matrix is given by

$$(\rho_A\otimes I_B)^{\frac{1-q}{2q}}\rho_N^{\rm GHZ}(x)(\rho_A\otimes I_B)^{\frac{1-q}{2q}} = (\frac{I_2}{2}\otimes I_B)^{\frac{1-q}{2q}}\rho_N^{\rm GHZ}(x)(\frac{I_2}{2}\otimes I_B)^{\frac{1-q}{2q}}$$

and it can be seen that $\tilde{D}_q^T(\rho_N^{\rm GHZ}(x)||\rho_A) = S_q^T(B|A)$

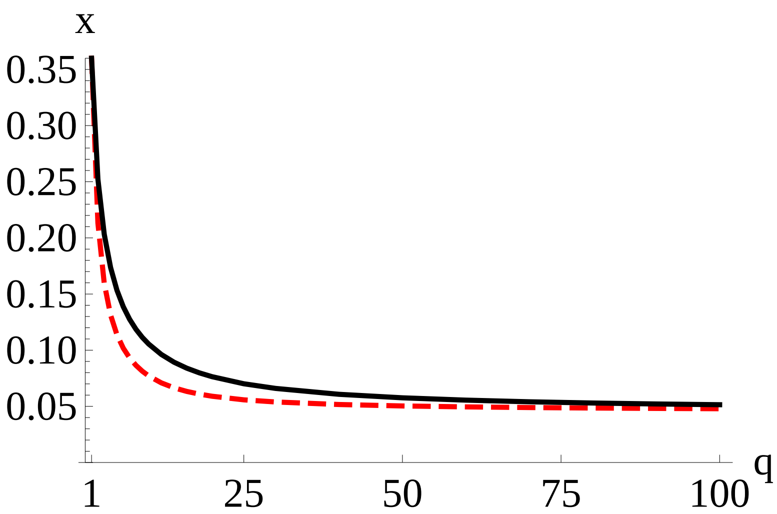

FIGURE 3.11: Implicit plots of $\tilde{D}_q^T(\rho_6^{\mathrm{GHZ}}(x)||\rho_B) = 0$ as a function of q (solid line) and Abe-Rajagopal q-conditional entropy $S_q^T(A|B) = 0$ (dashed line) of the state $\rho_6^{\mathrm{GHZ}}(x)$, in its $1:5$ partition. The relatively slow convergence of the parameter x to 0.04545 with the increase of q in the case of CSTRE criterion is readily seen.

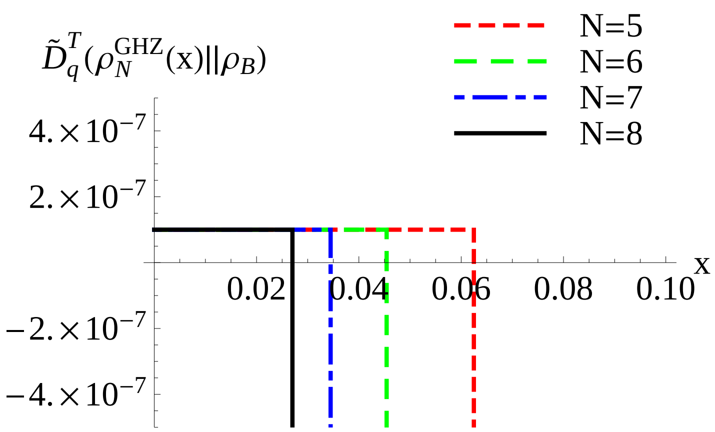

FIGURE 3.12: Illustration of the reduction of $1:N-1$ CSTRE separability range in $\rho_N^{\mathrm{GHZ}}(x)$ with increase in the number of qubits N from $N=5$ to $N=8$

observed that for large N (macroscopic limit), $x \approx \frac{2}{N^2}$ for $\rho_N^{\mathrm{GHZ}}(x)$ and $x \approx \frac{\sqrt{2}-1}{N}$ for $\rho_N^{\mathrm{W}}(x)$. Thus with the increase of N, the $1:N-1$ separability range decreases much faster for one parameter family of GHZ states than for one parameter family of W states.

3.4 $1:N-1$ separability in symmetric one-parameter family involving the states $|\mathrm{W\bar{W}}_N\rangle$

Having examined the $1:N-1$ separability of the state $\rho_N(x)$ (See Eq. (3.1)) when $|\Phi_N\rangle$ corresponds to the N-qubit W-, or GHZ- state, it would be of interest to evaluate $1:N-1$ separability range of the state $\rho_N(x)$ when $|\Phi_N\rangle$ corresponds to another symmetric N-qubit state $|\mathrm{W\bar{W}}_N\rangle$, the equal superposition of the states $|\mathrm{W}\rangle_N$ and its obverse counterpart $|\mathrm{\bar{W}}\rangle$.

3.4.1 $1:4$ separability in symmetric one-parameter family of noisy $\mathrm{W\bar{W}}$-states with 5 qubits

The five qubit symmetric one parameter family of mixed $\mathrm{W\bar{W}}$-state are defined as

$$\rho_5^{\mathrm{W\bar{W}}}(x) = \left(\frac{1-x}{6}\right) P_5 + x\left|\mathrm{W\bar{W}}_5\right\rangle\left\langle\mathrm{W\bar{W}}_5\right|; \quad 0 \leq x \leq 1 \tag{3.24}$$

The distinct non-zero eigenvalues of the state $\rho_5^{\mathrm{W\bar{W}}}(x)$ are

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \frac{1-x}{6}, \ \ \lambda_6 = \frac{(1+5x)}{6}. \tag{3.25}$$

The single qubit marginal of $\rho_5^{\mathrm{W\bar{W}}}(x)$ is a maximally mixed state $I_2/2$. The non-zero eigenvalues of the remaining four qubit marginal ρ_B are

$$\eta_1 = \eta_2 = \eta_3 = \frac{1-x}{5}, \ \eta_4 = \eta_5 = \frac{2+3x}{10}. \tag{3.26}$$

The AR q-conditional entropy for the state $\rho_5^{\mathrm{W\bar{W}}}(x)$in its $1:4$ partition is given by

$$S_q^T(A|B) = \frac{1}{q-1}\left(1 - \frac{5\left(\frac{1-x}{6}\right)^q + \left(\frac{1+5x}{6}\right)^q}{3\left(\frac{1-x}{5}\right)^q + 2\left(\frac{2+3x}{10}\right)^q}\right) \tag{3.27}$$

Identifying the zero of $S_q^T(A|B)$ in the limit $q \to \infty$, one can obtain $(0,\ 0.0625)$ as the $1:4$ AR separability range.

The unitary matrix

$$U_B = \begin{pmatrix}
0 & 0 & 0 & \sqrt{\frac{5}{6}} & 0 & \frac{-1}{\sqrt{30}} & \frac{-1}{\sqrt{30}} & 0 & 0 & \frac{-1}{\sqrt{30}} & \frac{-1}{\sqrt{30}} & 0 & \frac{-1}{\sqrt{30}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{2}{\sqrt{5}} & \frac{-1}{2\sqrt{5}} & 0 & 0 & \frac{-1}{2\sqrt{5}} & \frac{-1}{2\sqrt{5}} & 0 & \frac{-1}{2\sqrt{5}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & \frac{-1}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{6}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & \frac{\sqrt{3}}{2} & \frac{-1}{2\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & 0 & 0 & 0 & \frac{-1}{2\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{\frac{2}{3}} & 0 & \frac{-1}{\sqrt{6}} & 0 & 0 & 0 & \frac{-1}{\sqrt{6}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & \frac{-1}{2\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\
0 & \frac{-1}{2\sqrt{5}} & \frac{-1}{2\sqrt{5}} & 0 & \frac{-1}{2\sqrt{5}} & 0 & 0 & 0 & \frac{-1}{2\sqrt{5}} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{\sqrt{5}} \\
\frac{2}{\sqrt{5}} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{2\sqrt{5}} & 0 & 0 & 0 & \frac{-1}{2\sqrt{5}} & 0 & \frac{-1}{2\sqrt{5}} & \frac{-1}{2\sqrt{5}} & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} & 0 & 0 & 0 & \frac{1}{\sqrt{5}} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{5}} \\
\frac{1}{\sqrt{5}} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{5}} & 0 & 0 & 0 & \frac{1}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0
\end{pmatrix}$$

which diagonalizes the four qubit marginal ρ_B leads to the evaluation of $\Gamma_U = (I_2 \otimes U_B)\,\Gamma\,(I_2 \otimes U_B)^\dagger$ where the sandwiched matrix Γ is given by

$$\Gamma = (I_2 \otimes \rho_B)^{\frac{1-q}{2q}}\,(I_2 \otimes U_B)\rho_5^{\mathrm{W}\bar{\mathrm{W}}}(x)(I_2 \otimes U_B)^\dagger\,(I_2 \otimes \rho_B)^{\frac{1-q}{2q}}.$$

One can now readily evaluate the expression for CSTRE $\tilde{D}_q^T(\rho_5^{\mathrm{W}\bar{\mathrm{W}}}(x)||\rho_B)$ in its $1:4$ partition as

$$\tilde{D}_q^T(\rho_5^{\mathrm{W}\bar{\mathrm{W}}}(x)||\rho_B) = \frac{\sum_i \gamma_i^q - 1}{1-q}$$

Fig. 3.13 indicates the variation of the von-Neumann conditional entropy $S(A|B)$ and CSTRE $\lim_{q\to\infty} \tilde{D}_q^T(\rho_5^{\mathrm{W}\bar{\mathrm{W}}}(x)||\rho_B)$ with x. The $1:4$ separability range of the state $\rho_5^{\mathrm{W}\bar{\mathrm{W}}}(x)$ is obtained as $0 \leq x \leq 0.0625$ through identifying the zero of the function $\lim_{q\to\infty} \tilde{D}_q^T(\rho_5^{\mathrm{W}\bar{\mathrm{W}}}(x)||\rho_B)$ [63]. To find the PPT separability range of $\rho_5^{\mathrm{W}\bar{\mathrm{W}}}(x)$, in its $1:4$ partition, the negativity of partially transposed density matrix of $\rho_5^{\mathrm{W}\bar{\mathrm{W}}}(x)$ is explicitly evaluated. The plot of negativity of partially transposed matrix ρ^T as a function of x is as shown in Fig. 3.14. It can be seen that $(0,\ 0.0625)$ is the $1:4$ PPT separability range of $\rho_5^{\mathrm{W}\bar{\mathrm{W}}}(x)$ which matches with that obtained using CSTRE criterion.

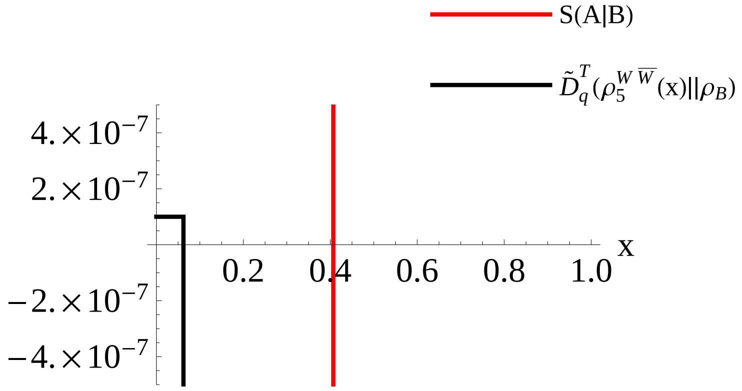

FIGURE 3.13: Both von-Neumann conditional entropy $S(A|B)$ and CSTRE $\lim_{q\to\infty}\tilde{D}_q^T(\rho_5^{\mathrm{W\bar{W}}}(x)||\rho_B)$ varying as a function of x for $\rho_5^{\mathrm{W\bar{W}}}(x)$ in its 1 : 4 partition. It can be readily seen that the 1 : 4 separability range obtained through von-Neumann conditional entropy is weaker than the corresponding CSTRE separability range.

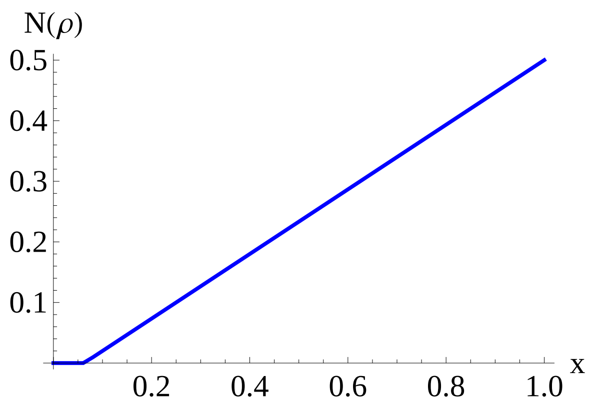

FIGURE 3.14: The plot of negativity as a function of x for the state $\rho_5^{\mathrm{W\bar{W}}}(x)$.

3.4.2 1 : 5 separability in symmetric one-parameter family of noisy $\mathrm{W\bar{W}}$-states with 6 qubits

The six qubit symmetric one parameter family of mixed $\mathrm{W\bar{W}}$ states are given by

$$\rho_6^{\mathrm{W\bar{W}}}(x) = \left(\frac{1-x}{7}\right) P_6 + x\left|\mathrm{W\bar{W}}_6\right\rangle\left\langle\mathrm{W\bar{W}}_6\right|;\quad 0\le x\le 1 \tag{3.28}$$

The distinct non-zero eigenvalues of the state $\rho_6^{\mathrm{W}\bar{\mathrm{W}}}(x)$ are

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \frac{1-x}{7}, \qquad \lambda_7 = \frac{(1+6x)}{7}.$$

The single qubit marginal of $\rho_6^{\mathrm{W}\bar{\mathrm{W}}}(x)$ is a maximally mixed state $I_2/2$. The non-zero eigenvalues of the 5-qubit marginal of $\rho_6^{\mathrm{W}\bar{\mathrm{W}}}(x)$ are

$$\eta_1 = \eta_2 = \eta_3 = \eta_4 = \frac{1-x}{6}, \qquad \eta_5 = \eta_6 = \frac{1+2x}{6}.$$

The AR q-conditional entropy for the state $\rho_6^{\mathrm{W}\bar{\mathrm{W}}}(x)$, in its $1:5$ partition can be evaluated readily as

$$S_q^T(A|B) = \frac{1}{q-1}\left(1 - \frac{6\left(\frac{1-x}{7}\right)^q + \left(\frac{1+6x}{7}\right)^q}{4\left(\frac{1-x}{6}\right)^q + 2\left(\frac{1+2x}{6}\right)^q}\right) \tag{3.29}$$

In order to employ the CSTRE approach to obtain the $1:5$ separability range in $\rho_6^{\mathrm{W}\bar{\mathrm{W}}}(x)$, it is necessary to find out the unitary matrix U_B which diagonalizes the five qubit marginal ρ_B. The unitary matrix U_B that diagonalizes ρ_B leads to the evaluation of the eigenvalues γ_i of the sandwiched matrix

$$\Gamma = (I_2 \otimes \rho_B)^{\frac{1-q}{2q}} \rho_6^{\mathrm{W}\bar{\mathrm{W}}}(x) (I_2 \otimes \rho_B)^{\frac{1-q}{2q}}$$

through its unitarily equivalent matrix $\Gamma_U = (I_2 \otimes U_B)\Gamma(I_2 \otimes U_B)^\dagger$. One can now evaluate the expression for CSTRE $\tilde{D}_q^T(\rho_6^{\mathrm{W}\bar{\mathrm{W}}}(x)||\rho_B)$ in its $1:5$ partition as

$$\tilde{D}_q^T(\rho_6^{\mathrm{W}\bar{\mathrm{W}}}(x)||\rho_B) = \frac{\sum_i \gamma_i^q - 1}{1-q}$$

Fig. 3.15 indicates the values of the parameter x at which the von-Neumann conditional entropy $S(A|B)$ and the q-conditional entropy $\lim_{q\to\infty} \tilde{D}_q^T(\rho_6^{\mathrm{W}\bar{\mathrm{W}}}(x)||\rho_B)$ change values from positive to negative. Whereas the von-Neumann separability range is obtained as $(0,\ 0.3601)$, the much stricter $(0,\ 0.0454)$ is the $1:5$ separability range of $\rho_6^{\mathrm{W}\bar{\mathrm{W}}}(x)$ obtained through both the q-entropic criteria [63]. The matching of the AR- and CSTRE separability ranges can be attributed to the maximally mixed nature of the single qubit marginal of $\rho_6^{\mathrm{W}\bar{\mathrm{W}}}(x)$. The $1:5$ separability range in $\rho_6^{\mathrm{W}\bar{\mathrm{W}}}(x)$ through PPT criterion is also evaluated explicitly through

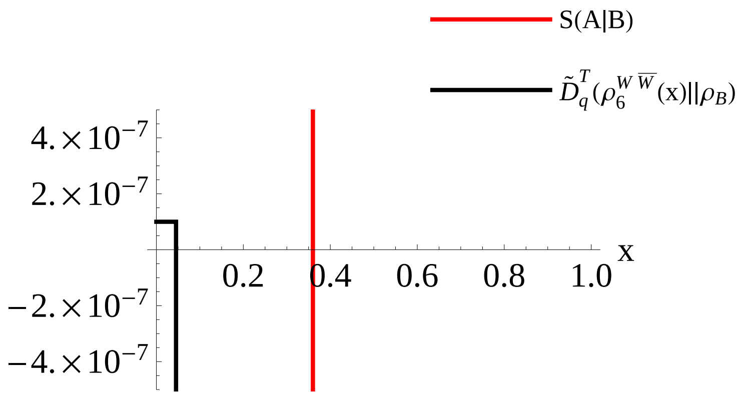

FIGURE 3.15: A plot illustrating the variation of conditional entropies $S(A|B)$, $\lim_{q\to\infty} \tilde{D}_q^T(\rho_6^{\mathrm{W\bar{W}}}(x)||\rho_B)$ of $\rho_6^{\mathrm{W\bar{W}}}(x)$, in its $1:5$ partition as a function of x.

the evaluation of the trace norm, negativity of the partially transposed density matrix of $\rho_6^{\mathrm{W\bar{W}}}(x)$, in its $1:5$ partition. The plot of negativity of partial transpose as a function of x is as shown in Fig. 3.16. One can readily obtain $(0,\ 0.0454)$ as the $1:5$ PPT separability range of $\rho_6^{\mathrm{W\bar{W}}}(x)$.

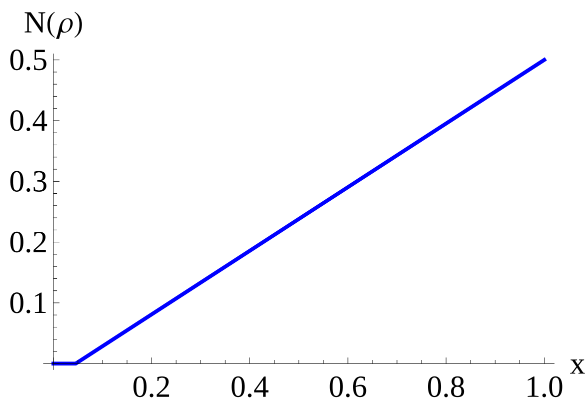

FIGURE 3.16: The plot of negativity of partial transpose of the state $\rho_6^{\mathrm{W\bar{W}}}(x)$ in its $1:5$ partition, as a function of x.

At this juncture some important points are to be noticed. One of them is that the 3-qubit GHZ state $|\mathrm{GHZ}_3\rangle$ and $|\mathrm{W\bar{W}}_3\rangle$ are convertible into one another through Stochastic Local Operations and Classical Communications (SLOCC). Both these states belong to the family of three distinct Majorana spinors whereas the W-state $|\mathrm{W}_3\rangle$ belongs to the family of two distinct Majorana spinors [66]. In spite of belonging to the same SLOCC family, the entanglement features of $|\mathrm{GHZ}_3\rangle$ and $|\mathrm{W}\bar{\mathrm{W}}_3\rangle$ are shown to be quite different in Ref. [66]. Such a feature is also reflected in the symmetric noisy states containing these states. While the single qubit marginal of $\rho_3^{\mathrm{GHZ}}(x)$ is maximally mixed thereby yielding the strictest $1:2$ separability range through AR-criterion itself, the corresponding $\rho_A \otimes I_4$ does not commute with $\rho_3^{\mathrm{W}\bar{\mathrm{W}}}(x)$ hence requiring CSTRE criterion for proper identification of its $1:2$ separability range. In fact,

$$\rho_A = \frac{1}{6}\begin{pmatrix} 3 & 2x \\ 2x & 3 \end{pmatrix} \Rightarrow \rho_1 \otimes I_4 \ \text{ does not commute with } \ \rho_3^{\mathrm{W}\bar{\mathrm{W}}}(x).$$

The $1:2$ separability range $(0,\ 0.3333)$ of the state $\rho_3^{\mathrm{W}\bar{\mathrm{W}}}(x)$ identified through AR-criterion is evidently weaker compared to $(0,\ 0.1896)$ the separability range obtained through CSTRE- as well as PPT criteria [63]. But the $1:N-1$ separability range of the N-qubit state $\rho_N^{\mathrm{W}\bar{\mathrm{W}}}(x)$ where $N \geq 4$, carried out through a similar analysis as that for $\rho_N^{\mathrm{W}}(x), \rho_N^{\mathrm{GHZ}}(x)$ is found to be $0 \leq x \leq \frac{2}{N^2+N+2}$ for $N \geq 4$. It can be readily seen that this is identical to the $1:N-1$ separability range for the state $\rho_N^{\mathrm{GHZ}}(x)$ (See Eq.(3.23)). The AR-criterion is also found to give the same $1:N-1$ separability range $0 \leq x \leq \frac{2}{N^2+N+2}$ $(N \geq 4)$ for $\rho_N^{\mathrm{W}\bar{\mathrm{W}}}(x)$. It is verified that the equivalence of the $1:N-1$ separability ranges for $\rho_N^{\mathrm{W}\bar{\mathrm{W}}}(x)$ through CSTRE- and AR-criteria when $N \geq 4$ is due to the maximally mixed (hence commuting) nature of single qubit density matrix of $\rho_N^{\mathrm{W}\bar{\mathrm{W}}}(x)$ for $N \geq 4$ [63]. Thus one can conclude that the 3-qubit symmetric noisy state $\rho_3^{\mathrm{W}\bar{\mathrm{W}}}(x)$ stands out in showing different entanglement features than its higher qubit counterparts $\rho_N^{\mathrm{W}\bar{\mathrm{W}}}(x)$, $N \geq 4$.

Having used the conditional version of sandwiched relative Tsallis entropy to find the separability range in symmetric one parameter family of noisy W-, GHZ and $\mathrm{W}\bar{\mathrm{W}}$ states, the utility of conditional version of sandwiched Rényi relative entropy in finding whether a bipartite state is entangled or not is also examined. It is found that both Tsallis and Rényi entropies play the same role in the detection of bipartite entanglement in a quantum state.

The conditional version of sandwiched Rényi relative entropy is given by

$$\tilde{D}_q^R(\rho_{AB}||\rho_B) = \frac{\log\left[\tilde{Q}_q(\rho_{AB}||\rho_B)\right]}{1-q} \tag{3.30}$$

where $\tilde{Q}_q(\rho_{AB}||\rho_B)$ is as shown in Eq.(2.8). The range of the parameter x is evaluated where $\tilde{D}_q^R(\rho_{AB}||\rho_B)$ is greater than zero and observed that the same result as obtained through CSTRE is obtained for $\rho_N^{\rm W}(x)$, $\rho_N^{\rm GHZ}(x)$ and $\rho_N^{\rm W\bar{W}}(x)$. This implies that Rényi entropy which is additive plays the same role as the non-additive Tsallis entropy in the identification of entanglement in the symmetric one-parameter families of N-qubit states. One can expect that this feature remains true for all bipartite states and one can either choose $\tilde{D}_q^R(\rho_{AB}||\rho_B)$ (Eq. (3.30)) or $\tilde{D}_q^T(\rho_{AB}||\rho_B)$ (Eqs.(2.7), (2.8)) for detecting bipartite entanglement.

3.5 Separability of qutrit-qutrit and qubit-qutrit states using CSTRE criterion

In this section, an effort is made to illustrate that the applicability of CSTRE is not restricted to symmetric one-parameter family of noisy states involving W, GHZ and W$\bar{\rm W}$ states. In fact, the CSTRE criterion is applicable for identifying any bipartite entangled state and one can use it for obtaining the separability ranges in chosen bipartitions of several one-parameter, two-parameter families of states including X states, cluster/graph states. An example of the use of CSTRE in identifying entanglement in an isospectral family of 2-qubit X states is illustrated in Ref. [62]. Also the applicability of CSTRE is not restricted to composite quantum states with two level systems (qubits) alone and it encompasses mixed composite states with qudits also. For instance, let us consider the one parameter family of 3×3 isotropic state given by [67]

$$\rho_{ab}(x) = \left(\frac{1-x}{8}\right)\, I_9 + \left(\frac{9x-1}{8}\right)|\Phi\rangle\langle\Phi|\,, \quad |\Phi\rangle = \frac{1}{\sqrt{3}}\left(|00\rangle + |11\rangle + |22\rangle\right), \tag{3.31}$$

with $0\leq x\leq 1$, I_9 is $3^2\times 3^2$ identity matrix,

$$|0\rangle = (1,0,0)\,, \quad |1\rangle = (0,1,0)\,, \quad |2\rangle = (0,0,1) \tag{3.32}$$

are the basis states in the qutrit space. The density matrix of $\rho_{ab}(x)$ is given as follows

$$\rho_{ab}(x) = \begin{pmatrix} \frac{1+3x}{12} & 0 & 0 & 0 & \frac{-1+9x}{24} & 0 & 0 & 0 & \frac{-1+9x}{24} \\ 0 & \frac{1-x}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-x}{8} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-x}{8} & 0 & 0 & 0 & 0 & 0 \\ \frac{-1+9x}{24} & 0 & 0 & 0 & \frac{1+3x}{12} & 0 & 0 & 0 & \frac{-1+9x}{24} \\ 0 & 0 & 0 & 0 & 0 & \frac{1-x}{8} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1-x}{8} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1-x}{8} & 0 \\ \frac{-1+9x}{24} & 0 & 0 & 0 & \frac{-1+9x}{24} & 0 & 0 & 0 & \frac{1+3x}{12} \end{pmatrix}$$

The distinct eigenvalue of $\rho_{ab}(x)$ are

$$\lambda_1 = \lambda_2 = \cdots \lambda_8 = \frac{1-x}{8}, \quad \lambda_9 = x.$$

The single qutrit reduced subsystems ρ_a, ρ_b turn out to be $I_3/3$ thereby commuting with ρ_{ab}. The CSTRE criterion thus reduces to AR-criterion.

The AR q-conditional entropy for the state $\rho_{ab}(x)$ is given as follows

$$S_q^T(a|b) = \frac{1}{q-1}\left(1 - \frac{8\left(\frac{1-x}{8}\right)^q + (x)^q}{3\left(\frac{1}{3}\right)^q}\right)$$

The plot of $S_q^T(a|b)$ as a function of x, for different values of q is shown in Fig. 3.17. From this plot one can obtain $(0,\ 0.3333)$ as the AR separability range, in the limit $q \to \infty$. As the single qutrit density matrix of $\rho_{ab}(x)$ is already in the diagonal form, one can directly evaluate the sandwiched matrix $\Gamma = \sigma\ \rho_{ab}(x)\ \sigma$, where

$$\sigma = I_3 \otimes \ \mathrm{diag}\left(\left(\frac{1}{3}\right)^{\frac{1-q}{2q}}, \left(\frac{1}{3}\right)^{\frac{1-q}{2q}}, \left(\frac{1}{3}\right)^{\frac{1-q}{2q}}\right)$$

The non-zero eigenvalues of Γ are found to be,

$$\gamma_1 = \left(\frac{1-x}{8}\right)\left(\frac{1}{3}\right)^{\frac{1-q}{q}} \text{ (8-fold degenerate)}, \quad \gamma_2 = \left(\frac{1}{3}\right)^{\frac{1-q}{q}}$$

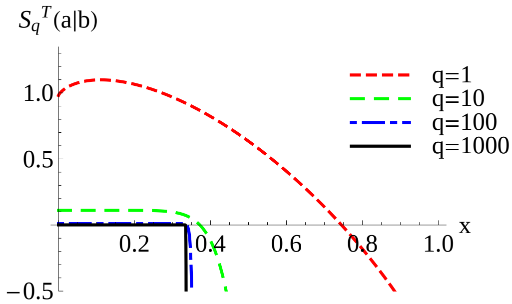

FIGURE 3.17: The AR q-conditional entropy $S_q^T(a|b)$ as a function of x for different values of q.

One can now readily evaluate the expression for CSTRE $\tilde{D}_q^T(\rho_{ab}||\rho_a)$ $\left(\equiv \tilde{D}_q^T(\rho_{ab}||\rho_b)\right)$ for different values of q, as

$$\tilde{D}_q^T(\rho_{ab}(x)||\rho_a) = \frac{\sum_i \gamma_i^q - 1}{1-q}$$

a function of x, as shown in the Fig. 3.18. From Fig. 3.18 one can obtain $(0,\ 0.3333)$ as the CSTRE separability range [63]. Clearly the AR separability range matches with the CSTRE separability range.

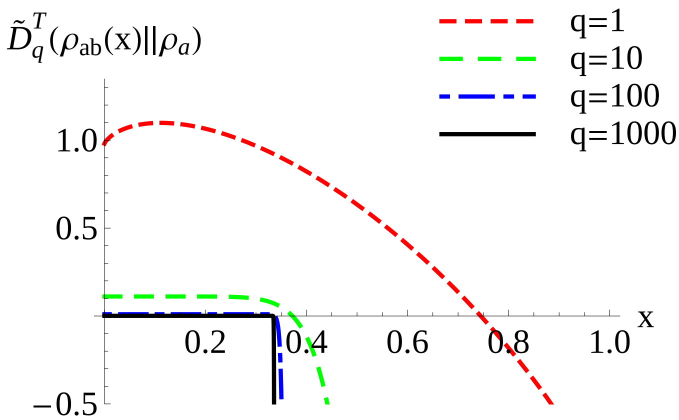

FIGURE 3.18: The CSTRE $\tilde{D}_q^T(\rho_{ab}(x)||\rho_a)$ as a function of x for different values of q.

The partial transposed density matrix of $\rho_{ab}(x)$ is given by

$$\rho^T(x) = \begin{pmatrix} \frac{1+3x}{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1-x}{8} & 0 & \frac{-1+9x}{24} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-x}{8} & 0 & 0 & 0 & \frac{-1+9x}{24} & 0 & 0 \\ 0 & \frac{-1+9x}{24} & 0 & \frac{1-x}{8} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1+3x}{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-x}{8} & 0 & \frac{-1+9x}{24} & 0 \\ 0 & 0 & \frac{-1+9x}{24} & 0 & 0 & 0 & \frac{1-x}{8} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-1+9x}{24} & 0 & \frac{1-x}{8} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+3x}{12} \end{pmatrix}$$

The distinct eigenvalues of $\left(\rho^T(x)^2\right)$ being α_i^2, one has,

$$\alpha_1 = \frac{(1+3x)}{12} \quad \text{(6-fold degenerate)}, \quad \alpha_2 = \frac{(1-3x)}{6} \quad \text{(3-fold degenerate)}.$$

On evaluating the negativity of partial transpose using α_i and identifying the value of x at which $N(\rho) > 0$, one can obtain $(0,\ 0.3333)$ as the PPT separability range of $\rho_{ab}(x)$.

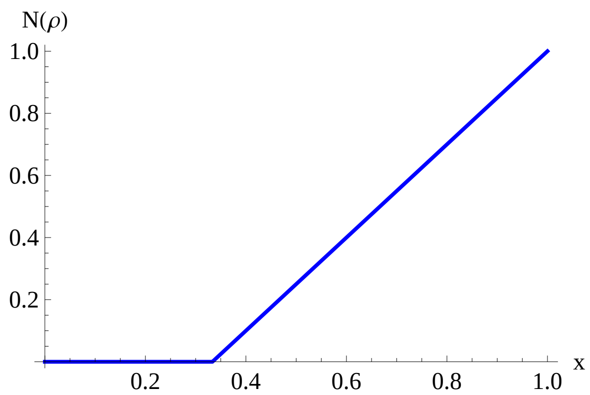

FIGURE 3.19: The plot of negativity as a function of x, for the state $\rho_{ab}(x)$.

The CSTRE criterion is also useful in identifying entanglement in $d_1 \times d_2$ dimensional states as can be seen through the example of a qubit-qutrit (2×3) X

state [67, 68]. The state $\rho^X_{ab}(x)$ is given by [67, 68]

$$\rho^X_{ab}(x) = \frac{1}{8}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 8x \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 8x & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

where $0 \leq x \leq 1/4$ and its subsystem $\rho_b = \mathrm{Tr}_a(\rho^X_{ab}(x))$ corresponding to the qutrit is found to be $\frac{1}{8}\,\mathrm{diag}(3,\,2,\,3)$. The subsystem $\rho_a = \mathrm{Tr}_b(\rho^X_{ab}(x))$ corresponding to the qubit is found to be $\frac{1}{2}\,\mathrm{diag}(1,\,1)$.

The distinct eigenvalues of the the state $\rho^X_{ab}(x)$ are

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \frac{1}{8}, \quad \lambda_5 = \frac{1-4x}{4}, \quad \lambda_6 = \frac{1+4x}{4}$$

The partially transposed density matrix of the state $\rho^X_{ab}(x)$, is given as follows

$$\rho^T = \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{8} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{8} & x & 0 & 0 \\ 0 & 0 & x & \frac{1}{8} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{8} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}$$

Denoting the eigenvalues of $(\rho^T)^2$ as α_i^2, one gets

$$\begin{aligned} \alpha_1 &= \alpha_2 = \frac{1}{4}, \qquad \alpha_3 = \alpha_4 = \frac{1}{8} \\ \alpha_5 &= \frac{(8x-1)}{8}, \qquad \alpha_6 = \frac{(1+8x)}{8}. \end{aligned} \tag{3.33}$$

and the negativity of partial transpose $N(\rho) = (\sum_i \alpha_i - 1)/2$ can be readily evaluated. The variation of $N(\rho)$ with x is as shown in the Fig. 3.20. The PPT separability range of the state $\rho^X_{ab}(x)$ is obtained as $(0,\ 0.125)$.

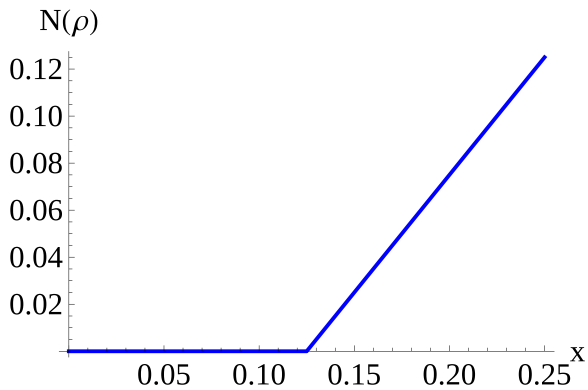

FIGURE 3.20: The plot of negativity of partial transpose of the state $\rho_{ab}^{X}(x)$ as a function of x.

The AR criterion to evaluate the separability range in $\rho_{ab}^{X}(x)$, with respect to qutrit, is given by

$$\begin{aligned} S_q^T(a|b) &= \frac{1}{q-1}\left[1 - \frac{\mathrm{Tr}\left[(\rho_{ab})^q\right]}{\mathrm{Tr}\left[(\rho_b)^q\right]}\right] \\ &= \frac{1}{q-1}\left[1 - \frac{\left(\frac{1-4x}{4}\right)^q + \left(\frac{1+4x}{4}\right)^q + 4\left(\frac{1}{8}\right)^q}{2\left(\frac{3}{8}\right)^q + \left(\frac{2}{8}\right)^q}\right] \end{aligned}$$

One can readily obtain $(0,\ 0.125)$ as the AR separability range, in the limit $q \to \infty$. This AR separability range clearly matches with the PPT separability range.

Now the CSTRE criterion is employed to obtain the separability range in $\rho_{ab}^{X}(x)$. As the single qutrit density matrix of $\rho_{ab}^{X}(x)$ is already in the diagonal form, one can directly evaluate the matrix

$$\begin{aligned} \Gamma &= \left(I_2 \otimes\ \mathrm{diag}\left(\left(\frac{3}{8}\right)^{\frac{1-q}{2q}}, \left(\frac{1}{4}\right)^{\frac{1-q}{2q}}, \left(\frac{3}{8}\right)^{\frac{1-q}{2q}}\right)\right)\rho_{ab}^{X}(x) \\ &\times \left(I_2 \otimes\ \mathrm{diag}\left(\left(\frac{3}{8}\right)^{\frac{1-q}{2q}}, \left(\frac{1}{4}\right)^{\frac{1-q}{2q}}, \left(\frac{3}{8}\right)^{\frac{1-q}{2q}}\right)\right) \end{aligned}$$

The non-zero eigenvalues of Γ are found to be,

$$\gamma_1 = \gamma_2 = 2^{-\frac{2+q}{q}}, \; \gamma_3 = \gamma_4 = 3^{\frac{1-q}{q}}\; 8^{\frac{-1}{q}}$$

$$\gamma_5 = (1-4x)\; 2^{\frac{q-3}{q}}\; 3^{\frac{1-q}{q}}, \; \gamma_6 = (1+4x)\; 2^{\frac{q-3}{q}}\; 3^{\frac{1-q}{q}}.$$

The expression for CSTRE $\tilde{D}_q^T(\rho_{ab}^X(x)||\rho_b)$ can be evaluated as

$$\tilde{D}_q^T(\rho_{ab}||\rho_b) = \frac{\sum_i \gamma_i^q - 1}{1-q}$$

The variation of $\tilde{D}_q^T(\rho_{ab}^X(x)||\rho_b)$ as a function of x, for different values of q is shown by the Fig. 3.21.

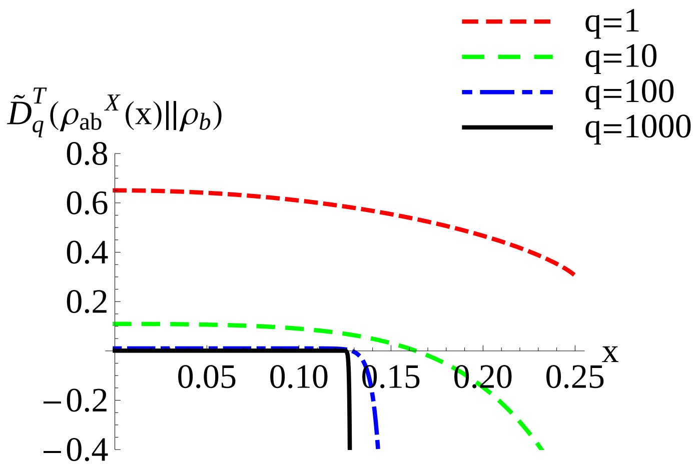

FIGURE 3.21: The CSTRE $\tilde{D}_q^T(\rho_{ab}^X(x)||\rho_b)$ as a function of x for $\rho_{ab}^X(x)$.

From Fig. 3.21 one can obtain $(0,\ 0.125)$ as the CSTRE separability range [63]. This matches exactly with that obtained through PPT criterion and AR-criterion. As the single qubit density matrix ρ_a of $\rho_{ab}^X(x)$ is already in the diagonal form, the sandwiched matrix

$$\Gamma = \left(\text{diag}\left(\left(\frac{1}{2}\right)^{\frac{1-q}{2q}}, \left(\frac{1}{2}\right)^{\frac{1-q}{2q}}\right) \otimes I_3\right)\; \rho_{ab}^X(x) \left(\text{diag}\left(\left(\frac{1}{2}\right)^{\frac{1-q}{2q}}, \left(\frac{1}{2}\right)^{\frac{1-q}{2q}}\right) \otimes I_3\right)$$

The eigenvalues of Γ are given by

$$\gamma_1 = 2^{-\frac{1+2q}{q}} \;; \quad \text{(4-fold degenerate)}$$

$$\gamma_2 = (1-4x)\, 2^{-\frac{1+q}{q}}, \; \gamma_3 = (1+4x)\, 2^{-\frac{1+q}{q}}.$$

One can now readily evaluate the expression for CSTRE $\tilde{D}_q^T(\rho_{ab}^X(x)||\rho_a)$ as

$$\tilde{D}_q^T(\rho_{ab}^X(x)||\rho_a) = \frac{\sum_i \gamma_i^q - 1}{1-q}$$

The variation of $\tilde{D}_q^T(\rho_{ab}^X(x)||\rho_a)$ as a function of x, for different values of q is shown by Fig. 3.22. From Fig. 3.22 one can observe that $\tilde{D}_q^T(\rho_{ab}^X(x)||\rho_a) \geq 0$ for all values

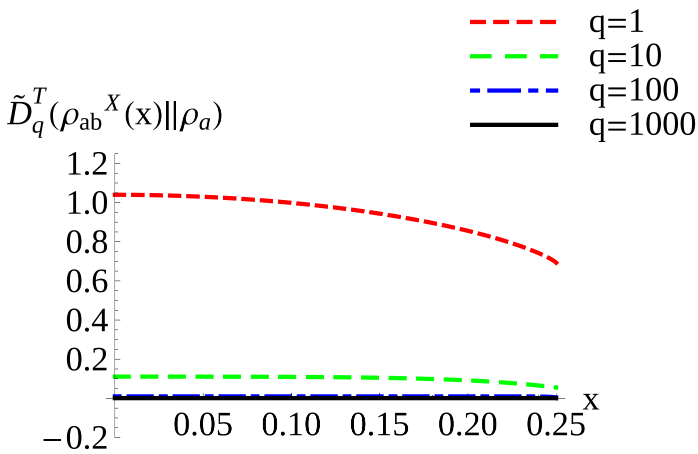

FIGURE 3.22: The CSTRE $\tilde{D}_q^T(\rho_{ab}^X(x)||\rho_a)$ as a function of x for $\rho_{ab}^X(x)$.

of $x \in (0,\, 1/4)$ thus failing to capture the entanglement in the state [63]. This example illustrates the need for suitable choice of marginals in making effective use of CSTRE criterion.

3.6 Summary

In this chapter, using the result that negative values of the conditional version of sandwiched Tsallis relative entropy necessarily imply quantum entanglement in a bipartite state and considering the limit $q \to \infty$ in Tsallis entropy, the separability range of the symmetric one-parameter family of noisy N-qubit W, GHZ, $\mathrm{W\bar{W}}$ states is obtained, in their $1 : N-1$ partition. For the symmetric one-parameter family of noisy W-states it is shown that the CSTRE criterion provides a stricter $1 : N-1$ separability range when compared to that obtained through AR q-conditional entropy approach. The non-commutativity of the single qubit marginal with the original density matrix of the noisy N-qubit W states is seen to be the reason behind the supremacy of CSTRE criterion over AR criterion. The $1 : N-1$ separability range, obtained using CSTRE criterion, for the one-parameter family of noisy GHZ states is shown to match with that through AR criterion. This is due to the maximally mixed, thereby commuting nature of the single qubit density matrix with the original density matrix in the symmetric one parameter family of noisy N-qubit GHZ states. It is thus illustrated that CSTRE criterion is a non-commuting generalization of the AR criterion and its equivalence with the results of AR criterion in the commuting cases, wherein the marginals are maximally mixed. In view of the fact that the $1 : N-1$ separability ranges through CSTRE and PPT criterion match with each other, the work here has provided the $1 : N-1$ PPT separability range for the symmetric one-parameter families of states considered here. The analysis, using AR- and CSTRE criteria, of the symmetric one parameter family of noisy state involving the state $|\mathrm{W\bar{W}}\rangle$, an equal superposition of W, obverse W states, has revealed an interesting feature that the 3-qubit state of this family shows a different entanglement feature than its higher qubit counterparts. Further illustrations on the applicability of CSTRE criterion to $d \times d$ as well as $d_1 \times d_2$ states is given by considering a two qutrit isotropic state and a qubit-qutrit X state.

* * * * * * * * * * * * *

Chapter 4

Biseparability of noisy pseudopure and Werner-like one parameter families of states using conditional quantum relative Tsallis entropy

In this Chapter, the CSTRE criterion is used to witness entanglement in noisy one parameter families of N-qubit pseudopure states [69] and the N-qubit generalizations of Werner-like one parameter states [19] involving W, GHZ states. It is shown that the non-commutative CSTRE criterion is both necessary and sufficient to detect entanglement in the $(1 : N-1)$ partitions of the one parameter families of noisy multiqubit states explored here.

Chapter 4 is organized as under: Sec. 4.1 defines the Pseudopure family of states and in Secs. 4.1.1–4.1.4, the CSTRE criterion is employed to identify $1:2$, $1:3$, $1:4$, $1:5$ separability range respectively in the psuedopure family of states involving 3-, 4-, 5-, 6-qubit W states. In Sec. 4.1.5, using the results of Secs. 4.1.1–4.1.4, the $1 : N-1$ CSTRE separability range for any $N \geq 3$ is obtained for pseudopure family of states involving N-qubit W states. The $1 : N-1$ CSTRE separability range obtained is shown to be necessary and sufficient for the pseudopure family of W states, in Sec. 4.1.6. Similar analysis to obtain the $1:N-1$ separability range of Pseudopure family containing GHZ states, and establish its necessary and sufficient status is carried out in Sec. 4.2. Werner-like family of noisy states involving W- or GHZ states is defined in Sec. 4.3 and their $1:N-1$ separability range is obtained using CSTRE criterion. The necessity and sufficiency of the separability ranges for Werner-like family of states involving W-, GHZ states is established in Secs. 4.3.5, 4.3.6 respectively. For each family of states considered, the $1:N-1$ separability range obtained using CSTRE criterion is compared with the $1:N-1$ separability ranges obtained using AR- and PPT criteria.

4.1 One parameter family of N qubit Pseudopure states involving W states

The pseudopure (PP) families of states are formed by mixing any pure state with white noise [69]. They have played a crucial role in demonstrating quantum information processing possibilities in liquid state NMR spectroscopy [70, 71]. In Ref. [69], different measures of quantum correlations of bipartite $d\times d$ pseudopure (PP) states of the form

$$\rho_\phi^{\rm PP}(x)=\frac{1-x}{d^2-1}\left[(I_d\otimes I_d)-|\phi\rangle\langle\phi|\right]+x|\phi\rangle\langle\phi| \tag{4.1}$$

(where $|\phi\rangle$ is any arbitrary $d\times d$ pure entangled state and $0\leq x\leq 1$ denotes the noisy parameter) are examined. An investigation of entanglement in the $1:N-1$ partition of the N qubit PP states, constructed using W and GHZ states, is carried out here based on the CSTRE approach. An expression for the $1:N-1$ separability range in the N-qubit pseudopure states involving W, GHZ states is found out using AR-, CSTRE criteria and it is shown that CSTRE criterion always fares better than AR criterion. The $1:N-1$ separability range identified through PPT criterion is shown to match with that obtained using CSTRE criterion.

The one parameter family of N-qubit pseudopure states

$$\rho^{\rm PP}_{{\rm W}_N}(x) = \frac{1-x}{2^N-1}\left(I_2^{\otimes N} - |{\rm W}_N\rangle\langle {\rm W}_N|\right) + x|{\rm W}_N\rangle\langle {\rm W}_N|$$

obtained by considering the pure state $|\phi\rangle$ in Eq. (4.1) to be the N-qubit W state:

$$|{\rm W}_N\rangle = \frac{1}{\sqrt{N}}[|1_1 0_2\cdots 0_N\rangle + |0_1 1_2\cdots 0_N\rangle + \cdots + \cdots + |0_1 0_2 0_3\cdots 1_N\rangle]$$

and the $d\times d$ matrix $I_d\otimes I_d$ replaced by its multiqubit counterpart $I_2^{\otimes N}$.

To make use of CSTRE approach for the determination of $1:N-1$ separability range of the pseudopure (PP) states $\rho^{\rm PP}_{{\rm W}_N}(x)$, the $1:N-1$ separability range when $N=3,\,4,\,5,\,6$ is explicitly evaluated in Secs. 4.1.1 to 4.1.4 and the result is generalized for arbitrary N in Sec. 4.1.5.

4.1.1 $1:2$ separability in one parameter family of 3-qubit pseudopure W-states

The one parameter family of 3-qubit pseudopure W-states are given by

$$\rho^{\rm PP}_{{\rm W}_3}(x) = \frac{1-x}{7}\left(I_8 - |{\rm W}_3\rangle\langle {\rm W}_3|\right) + x|{\rm W}_3\rangle\langle {\rm W}_3| \tag{4.2}$$

Here, I_8 denotes the 8×8 identity matrix and $|{\rm W}_3\rangle$ is the 3-qubit W-state.

The density matrix of the state $\rho^{\rm PP}_{{\rm W}_3}(x)$ is given by

$$\rho^{\rm PP}_{{\rm W}_3}(x) = \begin{pmatrix} \frac{1-x}{7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2+5x}{21} & \frac{8x-1}{21} & 0 & \frac{8x-1}{21} & 0 & 0 & 0 \\ 0 & \frac{8x-1}{21} & \frac{2+5x}{21} & 0 & \frac{8x-1}{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-x}{7} & 0 & 0 & 0 & 0 \\ 0 & \frac{8x-1}{21} & \frac{8x-1}{21} & 0 & \frac{2+5x}{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-x}{7} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1-x}{7} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1-x}{7} \end{pmatrix} \tag{4.3}$$

The distinct non-zero eigenvalues of $\rho^{\rm PP}_{\rm W_3}(x)$ are given by

$$\lambda_1 = \lambda_2 = \cdots \lambda_7 = \frac{1-x}{7}, \quad \lambda_8 = x. \tag{4.4}$$

In the $1:2$ partition, the single qubit marginal of $\rho^{\rm PP}_{\rm W_3}(x)$ forms the part A, and it is given by

$$\rho_A = \begin{pmatrix} \frac{2(5+2x)}{21} & 0 \\ 0 & \frac{(11-4x)}{21} \end{pmatrix}$$

The density matrix corresponding to the remaining two qubit marginal of $\rho^{\rm PP}_{\rm W_3}(x)$ forms the part B and it is given by

$$\rho_B = \begin{pmatrix} \frac{5+2x}{21} & 0 & 0 & 0 \\ 0 & \frac{5+2x}{21} & \frac{8x-1}{21} & 0 \\ 0 & \frac{8x-1}{21} & \frac{5+2x}{21} & 0 \\ 0 & 0 & 0 & \frac{2(1-x)}{7} \end{pmatrix} \tag{4.5}$$

The eigenvalues of ρ_B are given by

$$\eta_1 = \eta_2 = \frac{2(1-x)}{7}, \qquad \eta_3 = \frac{2(2+5x)}{21}, \quad \eta_4 = \frac{5+2x}{21}.$$

The entropy $S(B)$ of subsystem ρ_B and the entropy $S(A,\,B)$ of the global state $\rho^{\rm PP}_{\rm W_3}(x)$ are obtained respectively as

$$S(A,\,B) = \sum_i -\lambda_i \log_2 \lambda_i, \qquad S(B) = \sum_i -\eta_k \log_2 \eta_k \tag{4.6}$$

and the conditional entropy $S(A|B) = S(A,\,B) - S(B)$ in the $1:2$ partition of the state can readily be evaluated. The monotonically decreasing nature of $S(A|B)$ and the identification of its zero (the point where $S(A,\,B)$ changes value from positive to negative) at $x = 0.7390$, gives the $1:2$ von Neumann separability range of $\rho^{\rm PP}_{\rm W_3}(x)$ as $(0,\, 0.7390)$.

The evaluation of the AR q-conditional entropy in the $1:2$ partition of $\rho^{\rm PP}_{\rm W_3}(x)$ leads to

$$\begin{aligned} S_q^T(A|B) &= \frac{1}{q-1}\left[1-\frac{\mathrm{Tr}\,[(\rho_{\rm AB})^{\rm q}]}{\mathrm{Tr}\,[(\rho_{\rm B})^{\rm q}]}\right] \\ &= \frac{1}{q-1}\left(1-\frac{7\left(\frac{1-x}{7}\right)^q+(x)^q}{2\left(\frac{2-2x}{7}\right)^q+\left(\frac{4+10x}{21}\right)^q+\left(\frac{5+2x}{21}\right)^q}\right) \end{aligned}$$

On identifying the zero of $S_q^T(A|B)$ in the limit $q\to\infty$, the $1:2$ separability range of $\rho^{\rm PP}_{\rm W_3}(x)$ is found to be $(0,\ 0.3636)$.

As ρ_B given in Eq.(4.5) is not a diagonal matrix, there is a need for the construction of an unitary matrix U_B to diagonalize it in order to evaluate the quantity $\mathrm{Tr}\,\Gamma^{\rm q}$, Γ being the sandwiched matrix

$$\Gamma = (I_2\otimes\rho_B)^{\frac{1-q}{2q}}\,\rho^{\rm PP}_{\rm W_3}(x)\,(I_2\otimes\rho_B)^{\frac{1-q}{2q}}.$$

It can be seen that

$$U_B=\begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 1 & 0 & 0 & 0\\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\end{pmatrix}$$

The non-zero eigenvalues γ_i of Γ are now evaluated as the eigenvalues of the unitarily equivalent matrix $\Gamma_U=(I_2\otimes U_B)\Gamma(I_2\otimes U_B)^\dagger$ and they are given by

$$\begin{aligned} \gamma_1 &= 2^{\frac{1-q}{q}}\left(\frac{1-x}{7}\right)^{\frac{1}{q}} \quad \text{(4-fold degenerate)}, \qquad (4.7)\\ \gamma_2 &= \left(\frac{1-x}{7}\right)\left(\frac{5+2x}{21}\right)^{\frac{1-q}{q}}, \qquad \gamma_3=\left(\frac{1-x}{7}\right)\left(\frac{4+10x}{21}\right)^{\frac{1-q}{q}}, \\ \gamma_{4/5} &= (21)^{\frac{-1}{q}}\left(\frac{1}{2}\right)\left(\alpha\,a+\beta\,b\pm\sqrt{(\alpha\,a+\beta\,b)^2+504x(x-1)\,\alpha\,\beta}\right). \end{aligned}$$

with $\quad\alpha=(5+2x)^{\frac{1-q}{q}},\quad \beta=(4+10x)^{\frac{1-q}{q}},\quad a=(2+5x),\quad b=(1+13x).$

Now the expression for CSTRE $\tilde{D}_q^T(\rho_{\mathrm{W}_3}^{\mathrm{PP}}(x)||\rho_B)$ in the $1:2$ partition of $\rho_{\mathrm{W}_3}^{\mathrm{PP}}(x)$ can be evaluated through the expression

$$\tilde{D}_q^T(\rho_{\mathrm{W}_3}^{\mathrm{PP}}(x)||\rho_B) = \frac{\sum_i \gamma_i^q - 1}{1-q}.$$

The variation of conditional entropies $S_q^T(A|B)$, $\tilde{D}_q^T(\rho_{\mathrm{W}_3}^{\mathrm{PP}}(x)||\rho_B)$ both in the limit $q \to \infty$ and the von-Neumann conditional entropy with the parameter x is shown in Fig. 4.1. It can be seen that $(0,\ 0.3083)$ is the $1:2$ separability range of

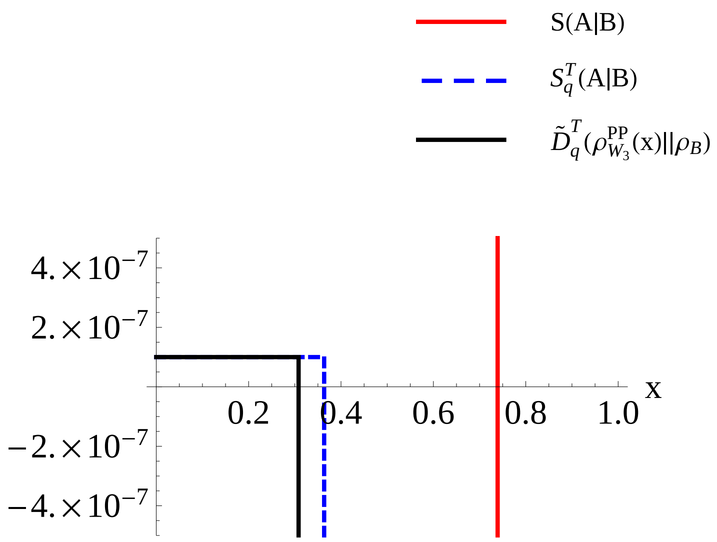

FIGURE 4.1: Plot illustrating the monotonic decreasing nature of $S(A|B)$, $\lim_{q\to\infty} S_q^T(A|B)$, $\lim_{q\to\infty} \tilde{D}_q^T(\rho_{\mathrm{W}_3}^{\mathrm{PP}}(x)||\rho_B)$ with x, in the $1:2$ partition of $\rho_{\mathrm{W}_3}^{\mathrm{PP}}(x)$.

$\rho_{\mathrm{W}_3}^{\mathrm{PP}}(x)$ using CSTRE criterion [72] and it fares better than the separability ranges $(0,\ 0.3636)$, $(0,\ 0.7390)$ obtained using AR-, von-Neumann conditional entropy criteria respectively.

The $1:2$ partially transposed density matrix of $\rho^{\rm PP}_{\rm W_3}(x)$ is given by

$$\rho^T = \begin{pmatrix} \frac{1-x}{7} & 0 & 0 & 0 & 0 & \frac{8x-1}{21} & \frac{8x-1}{21} & 0 \\ 0 & \frac{2+5x}{21} & \frac{8x-1}{21} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{8x-1}{21} & \frac{2+5x}{21} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-x}{7} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2+5x}{21} & 0 & 0 & 0 \\ \frac{8x-1}{21} & 0 & 0 & 0 & 0 & \frac{1-x}{7} & 0 & 0 \\ \frac{8x-1}{21} & 0 & 0 & 0 & 0 & 0 & \frac{1-x}{7} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1-x}{7} \end{pmatrix} \quad (4.8)$$

The square root of eigenvalues α_i^2 of $(\rho^T)^2$ are given by

$$\alpha_1 = \frac{(1-x)}{7} \quad \text{(4-fold degenerate)}, \quad (4.9)$$

$$\alpha_2 = \frac{(2+5x)}{21}, \quad \alpha_3 = \frac{(1+13x)}{21},$$

$$\alpha_{4/5} = \frac{1}{21}\sqrt{\left(11 + x(137x-50) \pm 6\sqrt{2}(1-9x+8x^2)\right)}$$

With the negativity of partial transpose being given by

$$N(\rho) = \frac{\|\rho^T\| - 1}{2}, \qquad ||\rho^T|| = \text{trace norm} = \sum_i \alpha_i$$

one can obtain $(0,\ 0.3083)$ as the $1:2$ PPT separability range of $\rho^{\rm PP}_{\rm W_3}(x)$, matching with that obtained using CSTRE criteria.

4.1.2 $1:3$ separability in one parameter family of 4-qubit pseudopure W-states

The one parameter family of 4-qubit pseudopure W-states are given by

$$\rho^{\rm PP}_{\rm W_4}(x) = \frac{1-x}{15}\left(I_{16} - |{\rm W_4}\rangle\langle{\rm W_4}|\right) + x|{\rm W_4}\rangle\langle{\rm W_4}|. \quad (4.10)$$

Here, I_{16} denotes the 16×16 identity matrix and $|\mathrm{W}_4\rangle$ is the 4-qubit W-state. The distinct nonzero eigenvalues of the state $\rho^{\rm PP}_{\mathrm{W}_4}(x)$ are

$$\lambda_1=\lambda_2=\cdots\lambda_{15}=\frac{1-x}{15},\quad \lambda_{16}=x. \tag{4.11}$$

The single qubit marginal of $\rho^{\rm PP}_{\mathrm{W}_4}(x)$, in its $1:3$ partition is diagonal with diagonal entries (eigenvalues) $\frac{29+16x}{60}$ and $\frac{31-16x}{60}$. The remaining three qubit marginal of the state $\rho^{\rm PP}_{\mathrm{W}_4}(x)$ is given by

$$\rho_B=\begin{pmatrix}\frac{7+8x}{60} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & \frac{7+8x}{60} & \frac{-1+16x}{60} & 0 & \frac{-1+16x}{60} & 0 & 0 & 0\\ 0 & \frac{-1+16x}{60} & \frac{7+8x}{60} & 0 & \frac{-1+16x}{60} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{2(1-x)}{15} & 0 & 0 & 0 & 0\\ 0 & \frac{-1+16x}{60} & \frac{-1+16x}{60} & 0 & \frac{7+8x}{60} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{2(1-x)}{15} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{2(1-x)}{15} & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2(1-x)}{15}\end{pmatrix}$$

and its non-zero eigenvalues are

$$\eta_1=\eta_2=\cdots=\eta_6=\frac{2(1-x)}{15},\quad \eta_7=\frac{1+8x}{12},\quad \eta_8=\frac{7+8x}{60}.$$

The $1:3$ von Neumann conditional entropy $S(A|B)=S(A,\,B)-S(B)$ of the state $\rho^{\rm PP}_{\mathrm{W}_4}(x)$ can now be evaluated and on identifying its zero, $(0,\ 0.6963)$ is obtained as the the $1:3$ von Neumann separability range of $\rho^{\rm PP}_{\mathrm{W}_4}(x)$.

The AR q-conditional entropy of $\rho^{\rm PP}_{\mathrm{W}_4}(x)$ system in its $1:3$ partition is given by

$$S^T_q(A|B)=\frac{1}{q-1}\left(1-\frac{15\left(\frac{1-x}{15}\right)^q+(x)^q}{6\left(\frac{2(1-x)}{15}\right)^q+\left(\frac{1+8x}{12}\right)^q+\left(\frac{7+8x}{60}\right)^q}\right) \tag{4.12}$$

Finding the zero of $S^T_q(A|B)$, in the limit $q\to\infty$, one can obtain $(0,\ 0.25)$ as the $1:3$ AR separability range of the state $\rho^{\rm PP}_{\mathrm{W}_4}(x)$.

In order to evaluate the CSTRE to obtain the $1:3$ separability range of the state $\rho^{\rm PP}_{\mathrm{W}_4}(x)$, the three qubit marginal ρ_B is to be diagonalized through a unitary matrix U_B. The unitary matrix U_B is given by

$$U_B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{6}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

This unitary matrix leads to the evaluation of $\Gamma_U = (I_2 \otimes U_B)\Gamma(I_2 \otimes U_B)^\dagger$ and its eigenvalues γ_i which are the same as that of the eigenvalues of sandwiched matrix

$$\Gamma = (I_2 \otimes \rho_B)^{\frac{1-q}{2q}} \rho^{\rm PP}_{\rm W_4}(x)(I_2 \otimes \rho_B)^{\frac{1-q}{2q}}.$$

The non-zero eigenvalues of Γ_U (hence of Γ) are found to be

$$\begin{aligned} \gamma_1 &= (2)^{\frac{1-q}{q}} \left(\frac{1-x}{15}\right)^{\frac{1}{q}} \qquad \text{(12-fold degenerate)}, \qquad (4.13) \\ \gamma_2 &= \left(\frac{1-x}{15}\right)\left(\frac{7+8x}{60}\right)^{\frac{1-q}{q}}, \quad \gamma_3 = \left(\frac{1-x}{15}\right)\left(\frac{1+8x}{12}\right)^{\frac{1-q}{q}}, \\ \gamma_{4/5} &= (60)^{\frac{-1}{q}}\left(\frac{1}{2}\right)\left[\alpha\, a + \beta\, b \pm \sqrt{(\alpha\, a + \beta\, b)^2 + 1920x(x-1)\ \alpha\, \beta}\right]. \end{aligned}$$

where $\quad \alpha = (7+8x)^{\frac{1-q}{q}},\ \beta = (5+40x)^{\frac{1-q}{q}}, \quad a = (3+12x), \quad b = (1+44x).$

One can now readily evaluate the expression for CSTRE $\tilde{D}^T_q(\rho^{\rm PP}_{\rm W_4}(x)||\rho_B)$ in the $1:3$ partition using the relation

$$\tilde{D}^T_q(\rho^{\rm PP}_{\rm W_4}(x)||\rho_B) = \frac{\sum_i \gamma_i^q - 1}{1-q}$$

The variation of all the three conditional entropies $S(A|B)$, $\lim_{q\to\infty} S^T_q(A|B)$, $\lim_{q\to\infty} \tilde{D}^T_q(\rho^{\rm PP}_{\rm W_4}(x)||\rho_B)$ with the parameter x is shown in Fig. 4.2. One can obtain $(0,\ 0.1807)$ as the $1:3$ CSTRE separability range of $\rho^{\rm PP}_{\rm W_4}(x)$ [72] which is stronger

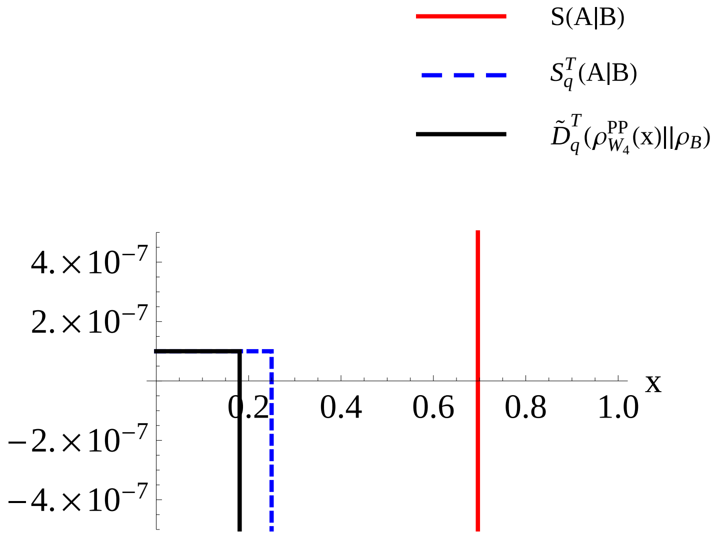

FIGURE 4.2: Plot of the conditional entropies $\lim_{q\to\infty} \tilde{D}_q^T(\rho_{\mathrm{W}_4}^{\mathrm{PP}}(x)||\rho_B)$, $\lim_{q\to\infty} S_q^T(A|B)$, $S(A|B)$ of $\rho_{\mathrm{W}_4}^{\mathrm{PP}}(x)$, in its $1:3$ partition, as a function of x.

than the corresponding range (0, 0.25) obtained using AR-separability criterion.

On explicitly evaluating the partially transposed density matrix ρ^T in the $1:3$ partition of $\rho_{\mathrm{W}_4}^{\mathrm{PP}}(x)$ and the eigenvalues α_i^2 of $\left(\rho^T\right)^2$, one has,

$$\begin{aligned}
\alpha_1 &= \frac{(1-x)}{15} \quad \text{(12-fold degenerate)}, \qquad (4.14)\\
\alpha_2 &= \frac{(1+4x)}{20}, \qquad \alpha_3 = \frac{(1+44x)}{60},\\
\alpha_{4/5} &= \frac{1}{460}\sqrt{\left(19+16x(49x-8)\pm 8\sqrt{3}(1-17x+16x^2)\right)}.
\end{aligned}$$

The negativity of partial transpose $N(\rho) = \frac{1}{2}(-1+\sum_i \alpha_i)$ is thereby evaluated. The $1:3$ PPT separability range of $\rho_{\mathrm{W}_4}^{\mathrm{PP}}(x)$ is seen to be $(0,\ 0.1807)$ thus matching with the corresponding CSTRE range.

4.1.3 $1:4$ separability in one parameter family of 5-qubit pseudopure W-states

The one parameter family of 5-qubit pseudopure W-states are given by

$$\rho^{\rm PP}_{\rm W_5}(x) = \frac{1-x}{31}\,\left(I_{32} - |{\rm W}_5\rangle\langle {\rm W}_5|\right) + x|{\rm W}_5\rangle\langle {\rm W}_5| \tag{4.15}$$

Here, I_{32} denotes the 32×32 identity matrix and $|{\rm W}_5\rangle$ is the 5-qubit W-state. The nonzero eigenvalues of the state $\rho^{\rm PP}_{\rm W_5}(x)$ are

$$\lambda_1 = \lambda_2 = \cdots \lambda_{31} = \frac{1-x}{31}, \qquad \lambda_{32} = x. \tag{4.16}$$

While the single qubit marginal ρ_A of $\rho^{\rm PP}_{\rm W_5}(x)$ is diagonal with eigenvalues $\frac{76+48x}{155}$, $\frac{79-48x}{155}$, the distinct non-zero eigenvalues of the four qubit marginal ρ_B are given by

$$\eta_1 = \eta_2 = \cdots = \eta_{14} = \frac{2-2x}{31}, \quad \eta_{15} = \frac{9+22x}{155}, \quad \eta_{16} = \frac{2(3+59x)}{155}.$$

The $1:4$ von Neumann conditional entropy

$$S(A|B) = S(A,B) - S(B) = \sum_i -\lambda_i \log_2 \lambda_i + \sum_k \eta_k \log_2 \eta_k$$

of the state $\rho^{\rm PP}_{\rm W_5}(x)$ can be readily evaluated using the eigenvalues λ_i, η_k. The zero of the monotonically decreasing function $S(A|B)=0$ occurs at $x=0.6723$ yielding $(0,\ 0.6723)$ as the $1:4$ von Neumann separability range of $\rho^{\rm PP}_{\rm W_5}(x)$.

The AR q-conditional entropy of $\rho^{\rm PP}_{\rm W_5}(x)$ in its $1:4$ partition is given by

$$S^T_q(A|B) = \frac{1}{q-1}\left(1 - \frac{31\left(\frac{1-x}{31}\right)^q + (x)^q}{14\left(\frac{2(1-x)}{31}\right)^q + \left(\frac{9+22x}{155}\right)^q + \left(\frac{2(3+59x)}{155}\right)^q}\right) \tag{4.17}$$

In the limit $q\to\infty$, the zero of $S^T_q(A|B)$ occurs at $x=0.1621$ thus yielding $(0,\ 0.1621)$ as the $1:4$ separability range of the state $\rho^{\rm PP}_{\rm W_5}(x)$.

The unitary U_B which diagonalizes the four qubit marginal ρ_B is given by

$$U_B = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\sqrt{3}}{2} & \frac{-1}{2\sqrt{3}} & 0 & \frac{-1}{2\sqrt{3}} & 0 & 0 & 0 & \frac{-1}{2\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{\frac{2}{3}} & 0 & \frac{-1}{\sqrt{6}} & 0 & 0 & 0 & \frac{-1}{\sqrt{6}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

and facilitates the evaluation of the eigenvalues γ_i of $\Gamma_U = (I_2 \otimes U_B)\Gamma(I_2 \otimes U_B)^\dagger$ where $\Gamma = (I_2 \otimes \rho_B)^{\frac{1-q}{2q}} \rho^{\rm PP}_{\rm W_5}(x)(I_2 \otimes \rho_B)^{\frac{1-q}{2q}}$ is the sandwiched matrix. The non-zero eigenvalues of Γ are seen to be

$$\begin{aligned}
\gamma_1 &= (2)^{\frac{1-q}{q}} \left(\frac{1-x}{31}\right)^{\frac{1}{q}} \quad \text{28-fold degenerate,} \qquad (4.18) \\
\gamma_2 &= \left(\frac{1-x}{31}\right)\left(\frac{9+22x}{155}\right)^{\frac{1-q}{q}}, \ \gamma_3 = \left(\frac{1-x}{31}\right)\left(\frac{6+118x}{155}\right)^{\frac{1-q}{q}}, \\
\gamma_{4/5} &= (155)^{\frac{-1}{q}} \left(\frac{1}{2}\right)\left[\alpha\, a + \beta\, b \pm \sqrt{(\alpha\, a + \beta\, b)^2 + 6200x(x-1)\, \alpha\, \beta}\right].
\end{aligned}$$

where $\quad \alpha = (9+22x)^{\frac{1-q}{q}},\ \beta = (6+118x)^{\frac{1-q}{q}},\ a = (4+27x),\ b = (1+123x).$

One can now readily evaluate the expression for CSTRE $\tilde{D}_q^T(\rho_{\mathrm{W}_5}^{\mathrm{PP}}(x)||\rho_B)$ in the $1:4$ partition as

$$\tilde{D}_q^T(\rho_{\mathrm{W}_5}^{\mathrm{PP}}(x)||\rho_B) = \frac{\sum_i \gamma_i^q - 1}{1-q}$$

Fig. 4.3 illustrates the stricter $1:4$ separability range $0 \le x \le 0.1014$ for $\rho_{\mathrm{W}_5}^{\mathrm{PP}}(x)$

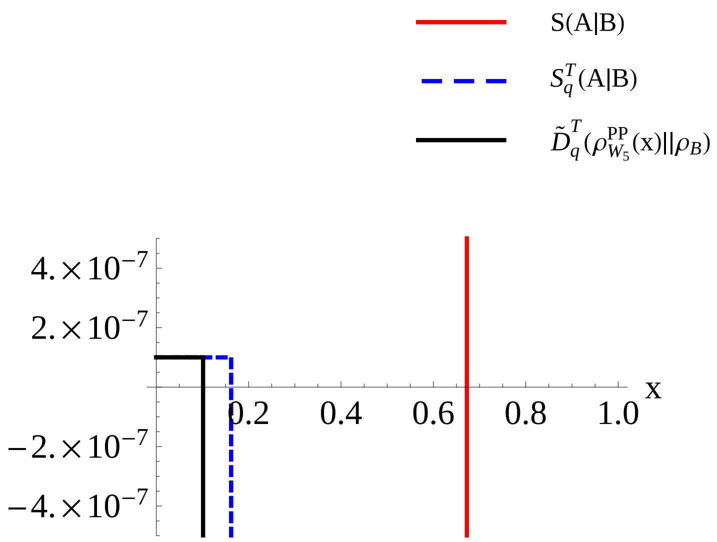

FIGURE 4.3: Identification of the zeroes of $\lim_{q\to\infty} \tilde{D}_q^T(\rho_{\mathrm{W}_5}^{\mathrm{PP}}(x)||\rho_B)$, $\lim_{q\to\infty} S_q^T(A|B)$ and $S(A|B)$ to obtain the $1:4$ separability range of $\rho_{\mathrm{W}_5}^{\mathrm{PP}}(x)$ using different entropic criteria

through the identification of the zero of the function $\lim_{q\to\infty} \tilde{D}_q^T(\rho_{\mathrm{W}_5}^{\mathrm{PP}}(x)||\rho_B)$ [72] in comparison with the corresponding separability ranges obtained using AR-criterion and Von-Neumann conditional entropy criterion.

Evaluating the partially transposed density matrix ρ^T of $\rho_{\mathrm{W}_5}^{\mathrm{PP}}(x)$ in its $1:4$ partition, the square root α_i of the eigenvalues α_i^2 of $\left(\rho^T\right)^2$ are determined to be

$$\alpha_1 = \frac{(1-x)}{31} \quad \text{(28-fold degenerate)}, \qquad \alpha_2 = \frac{(4+27x)}{155},$$

$$\alpha_3 = \frac{(3+59x)}{155}, \qquad \alpha_4 = \frac{7-69x}{155}, \qquad \alpha_5 = \frac{1+123x}{155}.$$

Identifying the zero of the monotonically increasing function $N(\rho)=\frac{1}{2}(-1+\sum_i \alpha_i)$ at $x=0.1014$, one obtains $(0,\ 0.1014)$ as the $1:4$ PPT separability range of $\rho^{\rm PP}_{\rm W_5}(x)$ and it matches with the CSTRE separability range.

4.1.4 $1:5$ separability in one parameter family of 6-qubit pseudopure W-states

The one parameter family of 6-qubit pseudopure W-states are given by

$$\rho^{\rm PP}_{\rm W_6}(x)=\frac{1-x}{63}\,\left(I_{64}-|{\rm W}_6\rangle\langle {\rm W}_6|\right)+x|{\rm W}_6\rangle\langle {\rm W}_6| \tag{4.19}$$

Here, I_{64} denotes the 64×64 identity matrix and $|{\rm W}_6\rangle$ is the 6-qubit W-state The distinct nonzero eigenvalues of the state $\rho^{\rm PP}_{\rm W_6}(x)$ are

$$\lambda_1=\lambda_2=\cdots\lambda_{63}=\frac{1-x}{63},\quad \lambda_{64}=x. \tag{4.20}$$

The single qubit marginal ρ_A of $\rho^{\rm PP}_{\rm W_6}(x)$ is diagonal with eigenvalues $\frac{187+128x}{378}$, $\frac{191-128x}{378}$ and the non-zero eigenvalues of the remaining five qubit marginal ρ_B are

$$\eta_1=\eta_2=\cdots=\eta_{30}=\frac{2-2x}{63},\quad \eta_{31}=\frac{1+44x}{54},\quad \eta_{32}=\frac{11+52x}{378}.$$

The conditional entropy $S(A|B)=S(A,\,B)-S(B)=-\sum_i \lambda_i\,\log_2\lambda_i+\sum_k \eta_k\,\log_2\eta_k$ in the $1:5$ partition of the state $\rho^{\rm PP}_{\rm W_6}(x)$ can readily be evaluated and on identifying its zero at $x=0.6621$ one obtains $(0,\ 0.6621)$ as the $1:5$ von Neumann separability range of $\rho^{\rm PP}_{\rm W_6}(x)$.

The AR q-conditional entropy of $\rho^{\rm PP}_{\rm W_6}(x)$ in its $1:5$ partition is given by

$$S_q^T(A|B)=\frac{1}{q-1}\left(1-\frac{63\left(\frac{1-x}{63}\right)^q+(x)^q}{30\left(\frac{2(1-x)}{63}\right)^q+\left(\frac{1+44x}{54}\right)^q+\left(\frac{11+52x}{378}\right)^q}\right) \tag{4.21}$$

In the limit $q\to\infty$ identifying the zero of the monotonically decreasing function $S_q^T(A|B)$ occurs at $x=0.1$ and the $1:5$ separability range of the state $\rho^{\rm PP}_{\rm W_6}(x)$ is obtained as $(0,\ 0.1)$.

On explicitly evaluating the unitary matrix U_B that diagonalizes the 5-qubit marginal ρ_B of $\rho^{\rm PP}_{{\rm W}_6}(x)$ one can evaluate $\Gamma_U=(I_2\otimes U_B)\Gamma(I_2\otimes U_B)^\dagger$ and its eigenvalues γ_i. $\Gamma=(I_2\otimes\rho_B)^{\frac{1-q}{2q}}\rho^{\rm PP}_{{\rm W}_6}(x)(I_2\otimes\rho_B)^{\frac{1-q}{2q}}$ being the sandwiched matrix of $\rho^{\rm PP}_{{\rm W}_6}(x)$ in its $1:5$ partition, one gets

$$\begin{aligned}
\gamma_1 &= (2)^{\frac{1-q}{q}}\left(\frac{1-x}{63}\right)^{\frac{1}{q}} \qquad \text{(60-fold degenerate)}, \qquad (4.22)\\
\gamma_2 &= \left(\frac{1-x}{63}\right)\left(\frac{11+52x}{378}\right)^{\frac{1-q}{q}}, \quad \gamma_3=\left(\frac{1-x}{63}\right)\left(\frac{7+308x}{378}\right)^{\frac{1-q}{q}},\\
\gamma_{4/5} &= (378)^{\frac{-1}{q}}\left(\frac{1}{2}\right)\left[\alpha\ a+\beta\ b\pm\sqrt{(\alpha\ a+\beta\ b)^2+18144x(x-1)\ \alpha\ \beta}\right].
\end{aligned}$$

with $\qquad \alpha=(11+52x)^{\frac{1-q}{q}},\ \beta=(7+308x)^{\frac{1-q}{q}},\ \ a=(5+58x),\ \ b=(1+314x).$

as the eigenvalues of Γ_U and hence of Γ. The expression for CSTRE $\tilde{D}^T_q(\rho^{\rm PP}_{{\rm W}_6}(x)||\rho_B)$ in the $1:5$ partition can now be evaluated as

$$\tilde{D}^T_q(\rho^{\rm PP}_{{\rm W}_6}(x)||\rho_B)=\frac{\sum_i\gamma_i^q-1}{1-q}$$

The variation of the two q-conditional entropies $\tilde{D}^T_q(\rho^{\rm PP}_{{\rm W}_6}(x)||\rho_B)$, $S^T_q(A|B)$ in the limit $q\to\infty$ and the von-Neumann conditional entropy $S(A|B)$ as a function of x is shown in Fig. 4.4. The zero of the function $\lim_{q\to\infty}\ \tilde{D}^T_q(\rho^{\rm PP}_{{\rm W}_6}(x)||\rho_B)$ [72] is seen to occur at $x=0.0552$ and $(0,\ 0.0552)$ is obtained as the $1:5$ CSTRE separability range of the state $\rho^{\rm PP}_{{\rm W}_6}(x)$.

The square root of the eigenvalues of $\left(\rho^T\right)^2$, ρ^T being the partially transposed density matrix of $\rho^{\rm PP}_{{\rm W}_6}(x)$, in its $1:5$ partition, are found to be

$$\begin{aligned}
\alpha_1 &= \frac{(1-x)}{63} \qquad \text{(60-fold degenerate)},\\
\alpha_2 &= \frac{(5+58x)}{378}, \qquad \alpha_3=\frac{(1+314x)}{378},\\
\alpha_{4/5} &= \frac{1}{378}\sqrt{41+4x(5129x-178)\pm12\sqrt{5}(1-65x+64x^2)}.
\end{aligned}$$

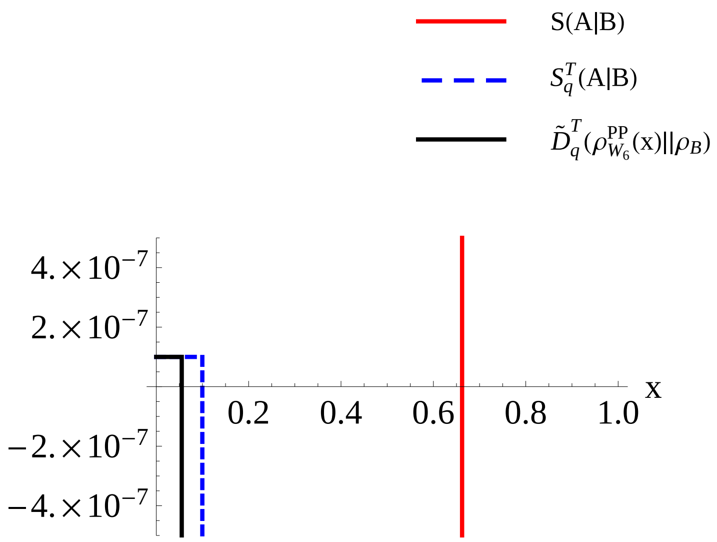

FIGURE 4.4: The conditional entropies $\lim_{q\to\infty}\tilde{D}_q^T(\rho_{\mathrm{W}_6}^{\mathrm{PP}}(x)||\rho_B)$, $\lim_{q\to\infty} S_q^T(A|B)$, $S(A|B)$ as a function of x for $\rho_{\mathrm{W}_6}^{\mathrm{PP}}(x)$, in its $1:5$ partition.

On identifying the zero of the monotonically increasing function $N(\rho) = (-1 + \sum_i \alpha_i)/2$ at $x = 0.0552$, one can obtain $(0,\ 0.0552)$ as the $1:5$ PPT separability range of $\rho_{\mathrm{W}_6}^{\mathrm{PP}}(x)$ and it is identical to the CSTRE separability range.

4.1.5 $1:N-1$ separability in pseudopure states involving N-qubit W states

As the focus is on finding the $1:N-1$ separability range of the W family of PP states $\rho_{\mathrm{W}_N}^{\mathrm{PP}}(x)$ using CSTRE criterion, an evaluation of the eigenvalues γ_i of the sandwiched matrix

$$\Gamma \;=\; (I_2\otimes\rho_B)^{\frac{1-q}{2q}}\,\rho_{\mathrm{W}_N}^{\mathrm{PP}}(x)\;\,(I_2\otimes\rho_B)^{\frac{1-q}{2q}}$$

with $\rho_B = \mathrm{Tr}_1[\rho_{\mathrm{W}_N}^{\mathrm{PP}}(x)]$ being the $N-1$ qubit marginal of $\rho_{\mathrm{W}_N}^{\mathrm{PP}}(x)$, needs to be carried out.

The following explicit structure of the eigenvalues γ_i (for $N \geq 3$) is obtained by generalizing the form of γ_i obtained for the states $\rho_{\mathrm{W}_N}^{\mathrm{PP}}(x)$ when $N = 3,\,4,\,5,\,6$

(See Eqs: (4.7), (4.13), (4.18), (4.22))

$$\begin{aligned}
\gamma_1 &= (2)^{\frac{1-q}{q}}\left(\frac{1-x}{2^N-1}\right)^{\frac{1}{q}}, \quad \left(2^N-4\right) \text{ fold-degenerate;} \\
\gamma_2 &= \left(\frac{1-x}{2^N-1}\right)\left[\frac{(2N-1)+\left(\sum_{j=3}^N 2^{j-1}-2(N-2)\right)x}{N\left(2^N-1\right)}\right]^{\frac{1-q}{q}}, \\
\gamma_3 &= \left(\frac{1-x}{2^N-1}\right)\left[\frac{(N+1)+\left(\sum_{j=3}^N 2^{j-1}+(N-2)\left(2^N-2\right)\right)x}{N\left(2^N-1\right)}\right]^{\frac{1-q}{q}}, \\
\gamma_{4/5} &= \left[N\left(2^N-1\right)\right]^{\frac{-1}{q}}\left(\frac{1}{2}\right)\left[\alpha\ a+\beta\ b \pm\sqrt{(\alpha\ a+\beta\ b)^2+8N^2(2^N-1)x(x-1)\ \alpha\ \beta}\right].
\end{aligned} \tag{4.23}$$

where

$$\begin{aligned}
\alpha &= \left[2N-1+\left(\sum_{j=3}^N 2^{j-1}-2(N-2)\right)x\right]^{\frac{1-q}{q}}, \\
\beta &= \left[N+1+\left(\sum_{j=3}^N 2^{j-1}+(N-2)\left(2^N-2\right)\right)x\right]^{\frac{1-q}{q}}, \\
a &= (N-1)+\left(\sum_{j=3}^N 2^{j-1}-N+4\right)x, \\
b &= 1+\left(\sum_{j=3}^N 2^{j-1}+(N-2)\left(2^N-2\right)+N\right)x.
\end{aligned} \tag{4.24}$$

Substituting the values of γ_i in the expression for CSTRE given by

$$\widetilde{D}_q^T\left(\rho_{AB}||\rho_B\right)=\frac{\left(\sum_i \gamma_i^q\right)-1}{1-q}, \tag{4.25}$$

a numerical estimation of the $1:N-1$ CSTRE separability range for $N=3,\ 4,\ 5,\ 6$ has been carried out. This results in the separability range for the noisy parameter x to be $(0,\ 0.3083)$, $(0,\ 0.1807)$, $(0,\ 0.1014)$, $(0,\ 0.0552)$ in the $1:2$, $1:3$, $1:4$, $1:5$ partitions of the noisy state $\rho^{\rm PP}_{{\rm W}_N}(x)$ with $N=3$, $N=4$, $N=5$, $N=6$ respectively. The results obtained based on the CSTRE, along with the corresponding cut-off value of the parameter x obtained using the AR- and the

PPT criteria are listed in Table 4.1. This offers a direct comparison of different approaches, each leading to the threshold values of the parameter x (beyond which the noisy state is found to be entangled). From Table 4.1 it is clearly seen that,

TABLE 4.1: Comparison of the $1 : N-1$ separability range $(0, x_0)$ of the state $\rho^{\rm PP}_{{\rm W}_N}(x)$, for $N = 3,\ 4,\ 5,\ 6$ through the threshold values x_0 obtained through different separability criteria

Number of qubits (N)	von Neumann conditional entropy	AR q-conditional entropy	CSTRE	PPT
3	0.7390	0.3636	0.3083	0.3083
4	0.6963	0.25	0.1807	0.1807
5	0.6723	0.1621	0.1014	0.1014
6	0.6621	0.1	0.0552	0.0552

for the noisy state $\rho^{\rm PP}_{{\rm W}_N}(x)$, CSTRE provides stricter separability range than the AR-criterion. Moreover, the CSTRE separability range matches identically with the PPT separability range.

In general, the CSTRE criterion (in the limit $q \to \infty$) leads to,

$$0 \leq x \leq \frac{N+\sqrt{N-1}}{N+2^N\sqrt{N-1}} \tag{4.26}$$

for the $(1 : N-1)$ separability range of the noisy N qubit PP state $\rho^{\rm PP}_{{\rm W}_N}(x)$ for $N \geq 3$ [72]. Alternately, in the parameter region

$$\frac{N+\sqrt{N-1}}{N+2^N\sqrt{N-1}} < x \leq 1,$$

the CSTRE method witnesses entanglement in the $(1 : N-1)$ bipartition of the noisy state $\rho^{\rm PP}_{{\rm W}_N}(x$.

4.1.6 Necessary and sufficient condition for $1 : N-1$ separability of $\rho^{\rm PP}_{{\rm W}_N}(x)$

The pseudopure family of states (see Eq. (4.1)) with the pure entangled state $|\phi\rangle$ expressed in terms of Schmidt co-efficients u_i, i.e.,

$$|\,\phi\,\rangle = \sum_{i=1}^{d} u_i\,|\,i_A i_B\rangle, \tag{4.27}$$

with $u_1 \geq u_2 \geq \cdots \geq u_d \geq 0$ are shown to be separable iff [69, 73]

$$0 \leq x \leq \frac{1+u_1u_2}{1+d^2\,(u_1u_2)} \tag{4.28}$$

It is to be recalled here that the Schmidt coefficients are the positive square roots of the eigenvalues of any of the subsystems of a pure state $|\phi\rangle$. For the PP state $\rho^{\rm PP}_{{\rm W}_N}(x)$ with $(1 : N-1)$ bipartition under investigation, the Schmidt coefficients are the square roots of the eigenvalues of the single qubit marginal of the N qubit W state. They are given by [72],

$$u_1 = \sqrt{\frac{N-1}{N}}, \quad u_2 = \frac{1}{\sqrt{N}}. \tag{4.29}$$

Substituting the values of u_1, u_2 in (4.29) and replacing d^2 by 2^N in Eq. (4.28), one can recover the result in Eq. (4.26) for the separability range. This establishes that the CSTRE approach serves as both necessary and sufficient to detect entanglement in the $(1 : N-1)$ partition of the PP state $\rho^{\rm PP}_{{\rm W}_N}(x)$.

4.2 One parameter family of N-qubit pseudopure GHZ-states

The noisy one parameter family of N qubit PP states $\rho^{\rm PP}_{{\rm GHZ}_N}(x)$ is given by

$$\rho^{\rm PP}_{{\rm GHZ}_N}(x) = \frac{1-x}{2^N-1}\left(I_2^{\otimes N} - |{\rm GHZ}_N\rangle\langle{\rm GHZ}_N|\right) + x|{\rm GHZ}_N\rangle\langle{\rm GHZ}_N|. \tag{4.30}$$

where,

$$|\mathrm{GHZ}_N\rangle = \frac{1}{\sqrt{2}}\left(|0_1 0_2 \cdots 0_N\rangle + |1_1 1_2 \cdots 1_N\rangle\right)$$

To find the $1:N-1$ separability range of $\rho^{\mathrm{PP}}_{\mathrm{GHZ}_N}(x)$ using CSTRE approach, the corresponding separability ranges for $\rho^{\mathrm{PP}}_{\mathrm{GHZ}_N}(x)$ when $N = 3,\ 4,\ 5,\ 6$ are explicitly evaluated in Secs. 4.2.1–4.2.4. The results are generalized in Sec. 4.2.5, to obtain the separability range in $\rho^{\mathrm{PP}}_{\mathrm{GHZ}_N}(x)$ for arbitrary N.

4.2.1 $1:2$ separability in one parameter family of 3-qubit pseudopure GHZ-states

The one parameter family of 3-qubit pseudopure GHZ-states are defined as

$$\rho^{\mathrm{PP}}_{\mathrm{GHZ}_3}(x) = \frac{1-x}{7}\left(I_8 - |\mathrm{GHZ}_3\rangle\langle \mathrm{GHZ}_3|\right) + x|\mathrm{GHZ}_3\rangle\langle \mathrm{GHZ}_3| \tag{4.31}$$

Here, I_8 denotes the 8×8 identity matrix and $|\mathrm{GHZ}_3\rangle$ is the 3-qubit GHZ-state. The density matrix of the state $\rho^{\mathrm{PP}}_{\mathrm{GHZ}_3}(x)$ is given by

$$\rho^{\mathrm{PP}}_{\mathrm{GHZ}_3}(x) = \begin{pmatrix} \frac{1+6x}{14} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{8x-1}{14} \\ 0 & \frac{1-x}{7} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-x}{7} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-x}{7} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-x}{7} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-x}{7} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1-x}{7} & 0 \\ \frac{8x-1}{14} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+6x}{14} \end{pmatrix} \tag{4.32}$$

The non-zero eigenvalues of $\rho^{\mathrm{PP}}_{\mathrm{GHZ}_3}(x)$ are

$$\lambda_1 = \lambda_2 = \cdots \lambda_7 = \frac{1-x}{7},\ \ \lambda_8 = x. \tag{4.33}$$

The single qubit marginal of $\rho^{\mathrm{PP}}_{\mathrm{GHZ}_3}(x)$, in the $1:2$ partition is maximally mixed and is given by

$$\rho_A = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The density matrix ρ_B corresponding to the remaining two qubit marginal of $\rho^{\rm PP}_{\rm GHZ_3}(x)$ is

$$\rho_B = \frac{1}{14}\,{\rm diag}\,(3+4x,\ 4-4x,\ 4-4x,\ 3+4x) \tag{4.34}$$

The AR q-conditional entropy for $\rho^{\rm PP}_{\rm GHZ_3}(x)$ is seen to be

$$S_q^T(A|B) = \frac{1}{q-1}\left(1 - \frac{7\left(\frac{1-x}{7}\right)^q + (x)^q}{2\left(\frac{2-2x}{7}\right)^q + 2\left(\frac{3+4x}{14}\right)^q}\right) \tag{4.35}$$

Through obtaining the zero of the monotonically decreasing function $S_q^T(A|B)$ in the limit $q\to\infty$, $(0,\ 0.3)$ is obtained as the $1:2$ AR separability range of the state $\rho^{\rm PP}_{\rm GHZ_3}(x)$.

As the two qubit marginal ρ_B is already in the diagonal form, one can evaluate the sandwiched matrix Γ as

$$\Gamma = \left(I_2\otimes\ \rho_B^{\frac{1-q}{2q}}\right)\rho^{\rm PP}_{\rm GHZ_3}(x)\,\left(I_2\otimes\ \rho_B^{\frac{1-q}{2q}}\right)$$

and the non-zero eigenvalues of Γ are evaluated to be

$$\begin{aligned}
\gamma_1 &= \left(\frac{1-x}{7}\right)\left(\frac{2-2x}{7}\right)^{\frac{1-q}{q}}, \quad \text{(4-fold degenerate)}, \\
\gamma_2 &= \left(\frac{1-x}{7}\right)\left(\frac{3+4x}{14}\right)^{\frac{1-q}{q}} \quad \text{(3-fold degenerate)}, \\
\gamma_3 &= x\left(\frac{3+4x}{14}\right)^{\frac{1-q}{q}}.
\end{aligned} \tag{4.36}$$

One can now readily evaluate the expression for CSTRE $\tilde{D}_q^T(\rho^{\rm PP}_{\rm GHZ_3}(x)||\rho_B)$ in the $1:2$ partition using

$$\tilde{D}_q^T(\rho^{\rm PP}_{\rm GHZ_3}(x)||\rho_B) = \frac{\sum_i \gamma_i^q - 1}{1-q}$$

It can be readily seen that the zero of $\lim_{\rm q\to\infty}\tilde{\rm D}_{\rm q}^{\rm T}(\rho^{\rm PP}_{\rm GHZ_3}({\rm x})||\rho_{\rm B})$ occurs at $x = 0.3$ and hence $(0, 0.3)$ is the $1:2$ separability range of $\rho^{\rm PP}_{\rm GHZ_3}(x)$ using CSTRE approach.

The $1:2$ partially transposed density matrix of $\rho^{\rm PP}_{{\rm GHZ}_3}(x)$ is given by

$$\rho^T = \begin{pmatrix} \frac{1+6x}{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1-x}{7} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-x}{7} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-x}{7} & \frac{8x-1}{14} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{8x-1}{14} & \frac{1-x}{7} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-x}{7} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1-x}{7} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+6x}{14} \end{pmatrix}. \tag{4.37}$$

The eigenvalues of $\left(\rho^T\right)^2$ being α_i^2, one gets

$$\begin{aligned} \alpha_1 &= \frac{(1-x)}{7}, \quad \text{(4-fold degenerate)}, \\ \alpha_2 &= \frac{(1+6x)}{14}, \quad \text{(3-fold degenerate)}, \quad \alpha_3 = \frac{(3-10x)}{14}. \end{aligned} \tag{4.38}$$

The negativity of partial transpose $N(\rho) = (-1 + \sum_i \alpha_i)/2$ is seen to be non-zero for values of $x \geq 0.3$ thus yielding $(0,\ 0.3)$ as the PPT separability range of $\rho^{\rm PP}_{{\rm GHZ}_3}(x)$, in its $1:2$ partition.

4.2.2 $1:3$ separability in one parameter family of four qubit pseudopure GHZ-states

The one parameter family of 4-qubit pseudopure GHZ-states are given by

$$\rho^{\rm PP}_{{\rm GHZ}_4}(x) = \frac{1-x}{15}\left(I_{16} - |{\rm GHZ}_4\rangle\langle {\rm GHZ}_4|\right) + x|{\rm GHZ}_4\rangle\langle{\rm GHZ}_4| \tag{4.39}$$

Here, I_{16} denotes the 16×16 identity matrix and $|{\rm GHZ}_4\rangle$ is the 4-qubit GHZ-state. The non zero eigenvalues of the state $\rho^{\rm PP}_{{\rm GHZ}_4}(x)$ are

$$\lambda_1 = \lambda_2 = \cdots \lambda_{15} = \frac{1-x}{15}, \quad \lambda_{16} = x. \tag{4.40}$$

The single qubit marginal of $\rho^{\rm PP}_{\rm GHZ_4}(x)$, in the $1:3$ partition is a maximally mixed state $I_2/2$. The remaining three qubit marginal of the state $\rho^{\rm PP}_{\rm GHZ_4}(x)$ is a diagonal matrix with diagonal elements (eigenvalues of ρ_B) being given by

$$\eta_1=\eta_2=\cdots=\eta_6=\frac{2(1-x)}{15}, \qquad \eta_7=\eta_8=\frac{1+4x}{10}.$$

The AR q-conditional entropy for $\rho^{\rm PP}_{\rm GHZ_4}(x)$, in its $1:3$ partition is given by

$$S_q^T(A|B)=\frac{1}{q-1}\left(1-\frac{15\left(\frac{1-x}{15}\right)^q+(x)^q}{6\left(\frac{2-2x}{15}\right)^q+2\left(\frac{1+4x}{10}\right)^q}\right) \tag{4.41}$$

The zero of $S_q^T(A|B)$ occurs at $x=0.1666$ and hence $(0,\ 0.1666)$ is the $1:3$ AR separability range of the state $\rho^{\rm PP}_{\rm GHZ_4}(x)$ using AR criterion.

As ρ_B is in the diagonal form, the sandwiched matrix

$$\Gamma=\left(I_2\otimes\rho_B^{\frac{1-q}{2q}}\right)\,\rho^{\rm PP}_{\rm GHZ_4}(x)\,\left(I_2\otimes\rho_B^{\frac{1-q}{2q}}\right)$$

and its eigenvalues γ_i can be readily evaluated. The non-zero eigenvalues of Γ are found to be,

$$\begin{aligned}\gamma_1 &= \left(\frac{1-x}{15}\right)\left(\frac{2-2x}{15}\right)^{\frac{1-q}{q}}\ \text{(12-fold degenerate)}, \\ \gamma_2 &= \left(\frac{1-x}{15}\right)\left(\frac{3+12x}{30}\right)^{\frac{1-q}{q}}\ \text{(3-fold degenerate)}, \\ \gamma_3 &= x\left(\frac{3+12x}{30}\right)^{\frac{1-q}{q}}.\end{aligned} \tag{4.42}$$

The expression for CSTRE $\tilde{D}_q^T(\rho^{\rm PP}_{\rm GHZ_4}(x)||\rho_B)$ in the $1:3$ partition is evaluated using

$$\tilde{D}_q^T(\rho^{\rm PP}_{\rm GHZ_4}(x)||\rho_B)=\frac{\sum_i\gamma_i^q-1}{1-q}$$

The Fig. 4.5 illustrate the variation of the von-Neumann conditional entropy $S(A|B)$ and $\lim_{q\to\infty}\tilde{D}_q^T(\rho^{\rm PP}_{\rm GHZ_4}(x)||\rho_B)$ as functions of x. The $1:3$ separability range of the state $\rho^{\rm PP}_{\rm GHZ_4}(x)$ is obtained as $0\leq x\leq 0.1666$ through identifying the zero of the function $\lim_{q\to\infty} \tilde{D}_q^T(\rho_{\mathrm{GHZ}_4}^{\mathrm{PP}}(x)||\rho_B)$ [72].

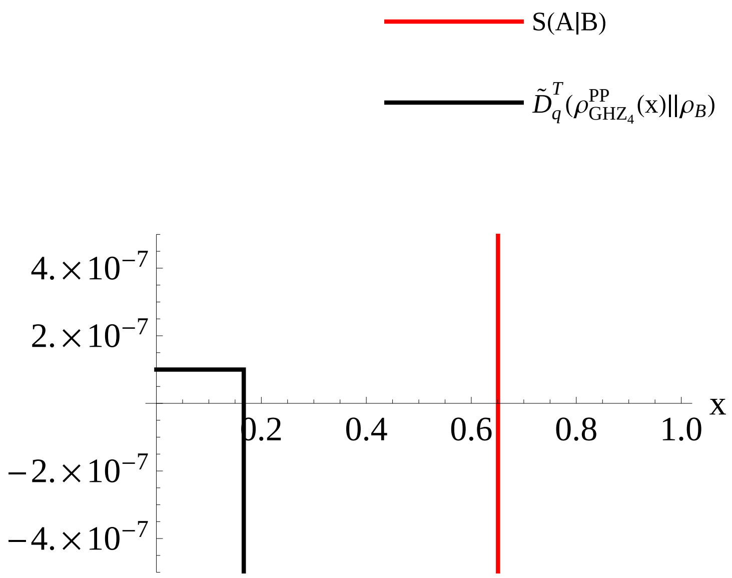

FIGURE 4.5: Variation of the Von-Neumann conditional entropy $S(A|B)$ and the CSTRE $\tilde{D}_q^T(\rho_{\mathrm{GHZ}_4}^{\mathrm{PP}}(x)||\rho_B)$ in the $1:3$ partition of $\rho_{\mathrm{GHZ}_4}^{\mathrm{PP}}(x)$ when $q \to \infty$, as a function of x.

With α_i^2 being the eigenvalues of $\left(\rho^T\right)^2$, ρ^T being the partially transposed density matrix in the $1:3$ partition of $\rho_{\mathrm{GHZ}_4}^{\mathrm{PP}}(x)$, one has

$$\alpha_1 = \frac{(1-x)}{15}, \quad \text{(12-fold degenerate)}, \tag{4.43}$$

$$\alpha_2 = \frac{(1+14x)}{30}, \quad \text{(3-fold degenerate)}, \qquad \alpha_3 = \frac{(1-6x)}{10}.$$

and $N(\rho) = (-1+\sum_i \alpha_i)/2$ is negativity of partial transpose. $N(\rho)$ is seen to be zero till $x = 0.1666$ and greater than zero thereafter. Thus $(0,\ 0.1666)$ is obtained as the PPT separability range of $\rho_{\mathrm{GHZ}_4}^{\mathrm{PP}}(x)$, in its $1:3$ partition.

4.2.3 1:4 separability in one parameter family of five qubit pseudopure GHZ-states

The one parameter family of 5-qubit pseudopure GHZ-states are given by

$$\rho^{\rm PP}_{{\rm GHZ}_5}(x) = \frac{1-x}{31}\,\left(I_{32} - |{\rm GHZ}_5\rangle\langle {\rm GHZ}_5|\right) + x|{\rm GHZ}_5\rangle\langle {\rm GHZ}_5| \qquad (4.44)$$

where I_{32} denotes the 32×32 identity matrix and $|{\rm GHZ}_5\rangle$ is the 5-qubit GHZ-state. The distinct non-zero eigenvalues of the state $\rho^{\rm PP}_{{\rm GHZ}_5}(x)$ are

$$\lambda_1 = \lambda_2 = \cdots \lambda_{31} = \frac{1-x}{31}, \quad \lambda_{32} = x. \qquad (4.45)$$

The distinct non-zero eigenvalues of tne four qubit marginal $\rho_B = {\rm Tr}_1\,\rho^{\rm PP}_{{\rm GHZ}_5}(x)$ are

$$\eta_1 = \eta_2 = \cdots \eta_{14} = \frac{2-2x}{31}, \quad \eta_{15} = \eta_{16} = \frac{3+28x}{62}.$$

The AR q-conditional entropy for the state $\rho^{\rm PP}_{{\rm GHZ}_5}(x)$ in its $1:4$ partition is given by

$$S_q^T(A|B) = \frac{1}{q-1}\left(1 - \frac{31\left(\frac{1-x}{31}\right)^q + (x)^q}{14\left(\frac{2-2x}{31}\right)^q + 2\left(\frac{3+28x}{62}\right)^q}\right) \qquad (4.46)$$

The zero of $S_q^T(A|B) = 0$ reveals that the state $\rho^{\rm PP}_{{\rm GHZ}_5}(x)$ is separable in the range $(0,\ 0.0882)$ in its $1:4$ partition in the limit $q \rightarrow \infty$.

The four qubit marginal ρ_B of $\rho^{\rm PP}_{{\rm GHZ}_5}(x)$ is a diagonal matrix and hence the eigenvalues of the sandwiched matrix $\Gamma = \left(I_2 \otimes\ (\rho_B^{\frac{1-q}{2q}})\right)\,\rho^{\rm PP}_{{\rm GHZ}_5}(x)\,\left(I_2 \otimes\ (\rho_B^{\frac{1-q}{2q}})\right)$ are readily evaluated to be

$$\begin{aligned}
\gamma_1 &= \left(\frac{1-x}{31}\right)\left(\frac{2-2x}{31}\right)^{\frac{1-q}{q}} \ \text{(28-fold degenerate)}, &(4.47)\\
\gamma_2 &= \left(\frac{1-x}{31}\right)\left(\frac{3+28x}{62}\right)^{\frac{1-q}{q}} \ \text{(3-fold degenerate)},\\
\gamma_3 &= x\left(\frac{3+28x}{62}\right)^{\frac{1-q}{q}}.
\end{aligned}$$

The expression for CSTRE $\tilde{D}_q^T(\rho^{\rm PP}_{\rm GHZ_5}(x)||\rho_B)$ in the $1:4$ partition can now be obtained using

$$\tilde{D}_q^T(\rho^{\rm PP}_{\rm GHZ_5}(x)||\rho_B) = \frac{\sum_i \gamma_i^q - 1}{1-q}.$$

It can be seen that $(0,\ 0.0882)$ is the $1:4$ CSTRE separability range for $\rho^{\rm PP}_{\rm GHZ_5}(x)$.

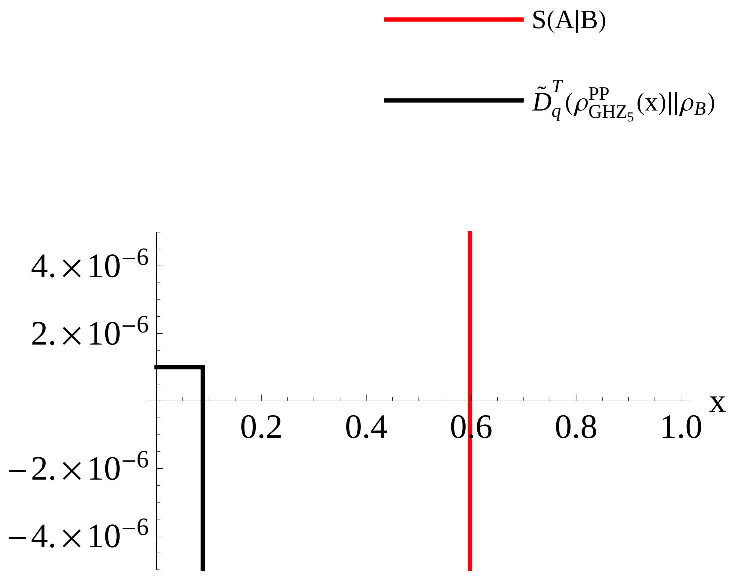

FIGURE 4.6: Variation of the Von-Neumann conditional entropy $S(A|B)$ and the CSTRE $\tilde{D}_q^T(\rho^{\rm PP}_{\rm GHZ_5}(x)||\rho_B)$ in the $1:4$ partition of $\rho^{\rm PP}_{\rm GHZ_5}(x)$ when $q\to\infty$, as a function of x.

If α_i^2 are the eigenvalues of $\left(\rho^T\right)^2$, it can be seen that

$$\alpha_1 = \frac{(1-x)}{31} \quad \text{(28-fold degenerate)}, \tag{4.48}$$

$$\alpha_2 = \frac{(1+30x)}{62} \quad \text{(3-fold degenerate)}, \qquad \alpha_3 = \frac{(3-34x)}{62}.$$

The negativity of partial transpose $N(\rho) = (-1+\sum_i \alpha_i)/2$ is seen to be greater than zero in the range $(0{,}0882, 1)$ thus yielding $(0,\ 0.0882)$ as the $1:4$ PPT separability range of $\rho^{\rm PP}_{\rm GHZ_5}(x)$.

4.2.4 $1:5$ separability in one parameter family of six qubit pseudopure GHZ-states

The one parameter family of 6-qubit pseudopure GHZ-states are given by

$$\rho^{\rm PP}_{{\rm GHZ}_6}(x) = \frac{1-x}{63}\left(I_{64} - |{\rm GHZ}_6\rangle\langle{\rm GHZ}_6|\right) + x|{\rm GHZ}_6\rangle\langle{\rm GHZ}_6| \tag{4.49}$$

Here, I_{64} denotes the 64×64 identity matrix and $|{\rm GHZ}_6\rangle$ is the 6-qubit GHZ-state. The distinct non-zero eigenvalues of the state $\rho^{\rm PP}_{{\rm GHZ}_6}(x)$ are

$$\lambda_1 = \lambda_2 = \cdots \lambda_{63} = \frac{1-x}{63}, \qquad \lambda_{64} = x. \tag{4.50}$$

The five qubit marginal $\rho_B = {\rm Tr}_1\, \rho^{\rm PP}_{{\rm GHZ}_6}(x)$ is diagonal and its eigenvalues are

$$\eta_1 = \eta_2 = \cdots \eta_{30} = \frac{2-2x}{63}, \qquad \eta_{31} = \eta_{32} = \frac{1+20x}{42}.$$

The AR q-conditional entropy [40] for the state $\rho^{\rm PP}_{{\rm GHZ}_6}(x)$ in its $1:5$ partition is given by

$$S^T_q(A|B) = \frac{1}{q-1}\left(1 - \frac{63\left(\frac{1-x}{63}\right)^q + (x)^q}{30\left(\frac{2-2x}{63}\right)^q + 2\left(\frac{1+20x}{42}\right)^q}\right) \tag{4.51}$$

Identifying the zero of $S^T_q(A|B) = 0$ in the limit $q\to\infty$, the $1:5$ separability range of the state $\rho^{\rm PP}_{{\rm GHZ}_6}(x)$ is obtained as $(0,\ 0.0454)$.

Due to the diagonal nature of five qubit marginal ρ_B, the sandwiched matrix $\Gamma = \left(I_2\otimes\ \rho_B^{\frac{1-q}{2q}}\right)\rho^{\rm PP}_{{\rm GHZ}_6}(x)\left(I_2\otimes\ \rho_B^{\frac{1-q}{2q}}\right)$ and its eigenvalues γ_i can readily be evaluated.

$$\begin{aligned}
\gamma_1 &= \left(\frac{1-x}{63}\right)\left(\frac{2-2x}{63}\right)^{\frac{1-q}{q}} \text{ (60-fold degenerate)}, \\
\gamma_2 &= \left(\frac{1-x}{63}\right)\left(\frac{3+60x}{126}\right)^{\frac{1-q}{q}} \text{ (3-fold degenerate)}, \\
\gamma_3 &= x\left(\frac{3+60x}{126}\right)^{\frac{1-q}{q}}.
\end{aligned} \tag{4.52}$$

The conditional Tsallis relative entropy of the state $\rho^{\rm PP}_{{\rm GHZ}_6}(x)$ in its $1:5$ partition is evaluated using the relation $\tilde{D}^T_q(\rho^{\rm PP}_{{\rm GHZ}_6}(x)||\rho_B) = \left(\sum_i \gamma_i^q - 1\right)/(1-q)$ and Fig. 4.7 illustrates the variation of $\tilde{D}^T_q(\rho^{\rm PP}_{{\rm GHZ}_6}(x)||\rho_B)$ as a function of x in the limit $q \rightarrow \infty$. The $1:5$ separability range of the state $\rho^{\rm PP}_{{\rm GHZ}_6}(x)$ is obtained as $0 \leq x \leq 0.0454$

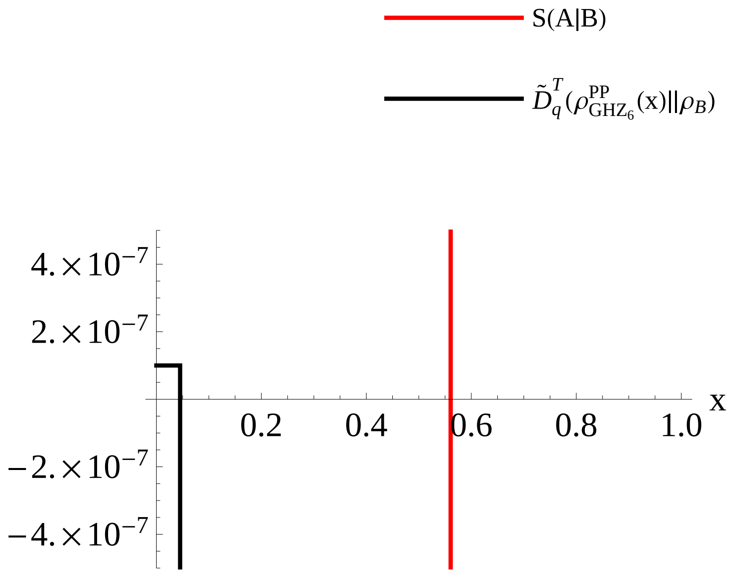

Figure 4.7: Plot of the conditional entropies $S(A|B)$ and $\lim_{q\rightarrow\infty} \tilde{D}^T_q(\rho^{\rm PP}_{{\rm GHZ}_6}(x)||\rho_B)$ of $\rho^{\rm PP}_{{\rm GHZ}_6}(x)$, in its $1:5$ partition, as a function of x.

through identifying the zero of the function $\lim_{q\rightarrow\infty} \tilde{D}^T_q(\rho^{\rm PP}_{{\rm GHZ}_6}(x)||\rho_B)$ [72].

Evaluating the partially transposed density matrix ρ^T in the $1:5$ partition of the state $\rho^{\rm PP}_{{\rm GHZ}_6}(x)$, the trace norm $||\rho^T|| = \sum_i \alpha_i$ and negativity of partial transpose $N(\rho) = (||\rho^T||-1)/2$ are obtained using the eigenvalues α_i^2 of $\left(\rho^T\right)^2$: It can be seen that

$$\alpha_1 = \frac{(1-x)}{63}, \quad \text{(60-fold degenerate)}, \tag{4.53}$$

$$\alpha_2 = \frac{(1+62x)}{126}, \quad \text{(3-fold degenerate)}, \qquad \alpha_3 = \frac{(1-22x)}{42}.$$

It can be seen that $(0,\ 0.0454)$ as the $1:5$ PPT separability range of $\rho^{\rm PP}_{{\rm GHZ}_6}(x)$ matching with the corresponding CSTRE as well as AR-separability range.

4.2.5 $1:N-1$ separability in $\rho^{\mathrm{PP}}_{\mathrm{GHZ}_N}(x)$

On observing the structure of γ_i, the eigenvalues of the sandwiched matrix

$$\Gamma = \left(I_2 \otimes\ \rho_B^{\frac{1-q}{2q}}\right) \rho^{\mathrm{PP}}_{\mathrm{GHZ}_N}(x) \left(I_2 \otimes\ \rho_B^{\frac{1-q}{2q}}\right)$$

when $N = 3,\ 4,\ 5,\ 6,$ an explicit structure of the non-zero eigenvalues γ_i of the sandwiched matrix Γ for any $N \geq 3$ can be arrived at [72] and they are given by

$$\gamma_1 = \left[\frac{1-x}{2^N-1}\right]\left[\frac{2\,(1-x)}{2^N-1}\right]^{\frac{1-q}{q}},, \quad (2^N-4)\text{-fold degenerate} \tag{4.54}$$

$$\gamma_2 = \left[\frac{1-x}{2^N-1}\right]\left[\frac{3+\left(\sum_{j=3}^N 2^{j-1}\right)x}{\sum_{j=1}^N 2^j}\right]^{\frac{1-q}{q}}, \quad \text{3-fold degenerate}$$

$$\gamma_3 = x\left[\frac{3+\left(\sum_{j=3}^N 2^{j-1}\right)x}{\sum_{j=1}^N 2^j}\right]^{\frac{1-q}{q}}.$$

In general for any $N \geq 3$, one can obtain the following bound (See Table 4.2)

$$0 \leq x \leq \frac{3}{2^N+2} \tag{4.55}$$

in the limit $q \to \infty$, within which the PP state $\rho^{\mathrm{PP}}_{\mathrm{GHZ}_N}(x)$ is separable.

TABLE 4.2: The comparison of the $1:N-1$ separability threshold values x_0 of the state $\rho^{\mathrm{PP}}_{\mathrm{GHZ}_N}(x)$, for $N = 3,\ 4,\ 5,\ 6$ obtained through different separability criteria.

Number of qubits (N)	von Neumann conditional entropy	AR q-conditional entropy	CSTRE	PPT
3	0.7225	0.3	0.3	0.3
4	0.6509	0.1666	0.1666	0.1666
5	0.5976	0.0882	0.0882	0.0882
6	0.5606	0.0454	0.0454	0.0454

The CSTRE separability range obtained in Eq. (4.55) is seen to match with that obtained using AR-criterion and also PPT criterion (See Table 4.2). The matching of the AR-separability range with the CSTRE separability range is due to the maximally mixed nature and hence commutativity of the single qubit marginal with the global state $\rho^{\rm PP}_{{\rm GHZ}_5}(x)$. But though the CSTRE and AR criteria result in the same separability threshold for the noisy parameter x, they approach the cut-off value with different convergence rates, which is depicted in Fig. 4.8, for the specific case of $N=6$. The separability range in Eq. (4.55) is also seen to

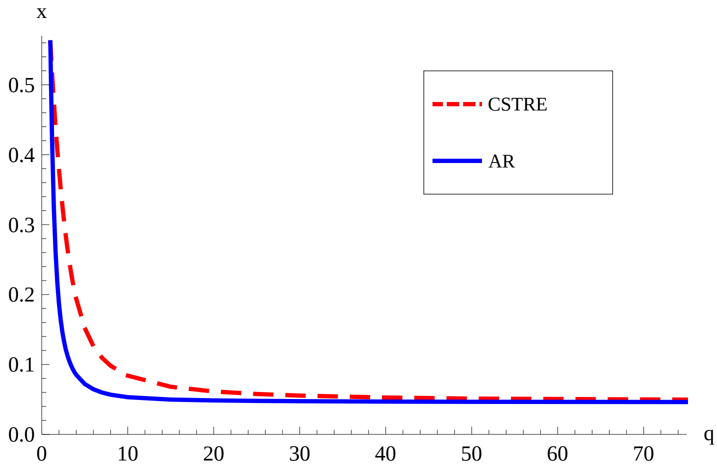

FIGURE 4.8: Implicit plots of $\tilde{D}^T_q(\rho^{\rm PP}_{{\rm GHZ}_6}(x)||\rho_B)=0$ and the Abe-Rajagopal q-conditional entropy $S^T_q(A|B)=0$ as a function of q in the $1:5$ partition of the state $\rho^{\rm PP}_{{\rm GHZ}_6}(x)$. This demonstrates the relatively slower convergence of the noisy parameter x to the cut-off value 0.04545 in the case of the CSTRE approach, when compared with that of the AR method.

match identically with the necessary and sufficient condition (See Eq. (4.28) for separability. This is readily seen on substituting the Schmidt coefficients $u_1 = u_2 = 1/\sqrt{2}$ associated with the $(1:N-1)$ partition of the GHZ state. Thus, the CSTRE method is found to serve as a necessary and sufficient condition to detect entanglement in the $1:N-1$ partition of the N qubit PP state $\rho^{\rm PP}_{{\rm GHZ}_N}(x)$.

4.3 Werner-like one parameter family of N qubit W states

It can be recalled here (See Eq. (1.5)) that an example of a two-qubit mixed state is referred to as Werner state and it is given by

$$\rho_w = \frac{(1-x)I_4}{4} + x|\phi_1\rangle\langle\phi_1|, \quad |\phi_1\rangle = \frac{1}{\sqrt{2}}[|00\rangle + |11\rangle] \tag{4.56}$$

The state ρ_w is entangled when $\frac{1}{3} < x \leq 1$ and is separable when $0 \leq x \leq \frac{1}{3}$.

The N-qubit generalizations of the state ρ_w can be termed as Werner-like one parameter family of states and they are given by

$$\rho_{\Phi_N}(x) \;=\; (1-x)\frac{I_2^{\otimes N}}{2^N} + x\,|\Phi\rangle\langle\Phi|\,, \quad 0 \leq x \leq 1 \tag{4.57}$$

When the pure entangled state $|\Phi\rangle$ corresponds to the N-qubit W state (See Eq. (2.24)), the noisy state

$$\rho_{\mathrm{W}_N}(x) \;=\; (1-x)\frac{I_2^{\otimes N}}{2^N} + \;x\;|\mathrm{W}_N\rangle\langle\mathrm{W}_N|\,. \tag{4.58}$$

is obtained. In order to carry out the task of identifying the 1 : $N-1$ separability range of the state $\rho_{\mathrm{W}_N}(x)$ via the CSTRE method, one needs to evaluate the 2^N eigenvalues γ_i of the sandwiched matrix

$$\Gamma = (I_2 \otimes \rho_B)^{\frac{1-q}{2q}}\rho_{\mathrm{W}_N}(x)(I_2 \otimes \rho_B)^{\frac{1-q}{2q}}$$

where $\rho_B = \mathrm{Tr}_1[\rho_{\mathrm{W}_N}(x)]$ is the $N-1$ qubit marginal of $\rho_{\mathrm{W}_N}(x)$ obtained by tracing over the first qubit. In Secs. 4.3.1 to 4.3.4, the form of γ_i is obtained for $N = 3, 4, 5, 6$ and the generalized form of these eigenvalues for any N is obtained in Sec. 4.3.5.

4.3.1 Separability in the $1:2$ partition of three qubit Werner-like family involving W-states

The Werner-like one parameter family of 3-qubit W-states given by

$$\rho_{\mathrm{W}_3}(x) = (1-x)\frac{I_8}{8} + \; x \; \left|\mathrm{W}_3\right\rangle \left\langle \mathrm{W}_3\right| . \tag{4.59}$$

has non-zero eigenvalues λ_k where

$$\lambda_1 = \lambda_2 = \cdots \lambda_7 = \frac{1-x}{8}, \;\; \lambda_8 = \frac{1+7x}{8}. \tag{4.60}$$

The single qubit marginal of $\rho_{\mathrm{W}_3}(x)$, though diagonal, does not correspond to maximally mixed state. The density matrix corresponding to the remaining two qubit marginal of $\rho_{\mathrm{W}_3}(x)$ is explicitly given by

$$\rho_B = \begin{pmatrix} \frac{3+x}{12} & 0 & 0 & 0 \\ 0 & \frac{3+x}{12} & \frac{x}{3} & 0 \\ 0 & \frac{x}{3} & \frac{3+x}{12} & 0 \\ 0 & 0 & 0 & \frac{1-x}{4} \end{pmatrix} \tag{4.61}$$

The eigenvalues of ρ_B are

$$\eta_1 = \eta_2 = \frac{1-x}{4}, \;\; \eta_3 = \frac{3+x}{12}, \;\; \eta_4 = \frac{3+5x}{12}.$$

The AR q-conditional entropy for $\rho_{\mathrm{W}_3}(x)$ turns out to be

$$S_q^T(A|B) = \frac{1}{q-1}\left(1 - \frac{7\left(\frac{1-x}{8}\right)^q + \left(\frac{1+7x}{8}\right)^q}{2\left(\frac{1-x}{4}\right)^q + \left(\frac{3+x}{12}\right)^q + \left(\frac{3+5x}{12}\right)^q}\right) \tag{4.62}$$

On identifying the zero of the monotonically decreasing function $\lim_{\mathrm{q}\to\infty} \mathrm{S}_{\mathrm{q}}^{\mathrm{T}}(\mathrm{A}|\mathrm{B})$ one can observe that the state $\rho_{\mathrm{W}_3}(x)$ is separable in the range $(0,\ 0.2727)$ according to AR-criterion.

The unitary operator U_B which diagonalizes ρ_B is as follows

$$U_B = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

This unitary matrix facilitates the evaluation of $\Gamma_U = (I_2 \otimes U_B)\rho_{\mathrm{W}_3}(x)(I_2 \otimes U_B)^\dagger$ which has the same eigenvalues as that of the sandwiched matrix

$$\Gamma = \left(I_2 \otimes \; \rho_B^{\frac{1-q}{2q}}\right) \rho_{\mathrm{W}_3}(x) \left(I_2 \otimes \; \rho_B^{\frac{1-q}{2q}}\right).$$

The non-zero eigenvalues γ_i of Γ_U and hence of Γ are obtained as

$$\begin{aligned} \gamma_1 &= \left(\frac{1-x}{8}\right)\left(\frac{1-x}{4}\right)^{\frac{1-q}{q}} \text{ (4-fold degenerate)}, \qquad (4.63) \\ \gamma_2 &= \left(\frac{1-x}{8}\right)\left(\frac{3+x}{12}\right)^{\frac{1-q}{q}}, \ \lambda_3 = \left(\frac{1-x}{8}\right)\left(\frac{3+5x}{12}\right)^{\frac{1-q}{q}}, \\ \gamma_{4/5} &= \left(\frac{1}{4}\right)(12)^{\frac{-1}{q}}\left(\alpha\ a + \beta\ b \pm \sqrt{(\alpha\ a+\beta\ b)^2 + 512x^2\ \alpha\ \beta}\right). \end{aligned}$$

where $\qquad \alpha = (3+x)^{\frac{1-q}{q}},\ \beta = (3+5x)^{\frac{1-q}{q}},\ a = (3+5x),\ b = (3+13x).$

One can now readily evaluate the expression for CSTRE $\tilde{D}_q^T(\rho_{\mathrm{W}_3}(x)||\rho_B)$ in the $1:2$ partition using $\tilde{D}_q^T(\rho_{\mathrm{W}_3}(x)||\rho_B) = \left(\sum_i \gamma_i^q - 1\right)/(1-q)$. The $1:2$ separability range of the state $\rho_{\mathrm{W}_3}(x)$ is obtained as $0 \leq x \leq 0.2095$ through identifying the zero of the function $\lim_{q\to\infty} \tilde{D}_q^T(\rho_{\mathrm{W}_3}(x)||\rho_B)$ at $x = 0.2095$ [72].

4.3.2 Separability in the $1:3$ partition of four qubit Werner-like family involving W-states

The distinct non-zero eigenvalues of the 4-qubit state $\rho_{\mathrm{W}_4}(x)$ are

$$\lambda_1 = \lambda_2 = \cdots \lambda_{15} = \frac{1-x}{16},\ \lambda_{16} = \frac{1+15x}{16}. \qquad (4.64)$$

The three qubit marginal of the state $\rho_{\mathrm{W}_4}(x)$, obtained by tracing over the first qubit of $\rho_{\mathrm{W}_4}(x)$ is explicitly given by

$$\rho_B = \begin{pmatrix} \frac{1+x}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1+x}{8} & \frac{x}{4} & 0 & \frac{x}{4} & 0 & 0 & 0 \\ 0 & \frac{x}{4} & \frac{1+x}{8} & 0 & \frac{x}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-x}{8} & 0 & 0 & 0 & 0 \\ 0 & \frac{x}{4} & \frac{x}{4} & 0 & \frac{1+x}{8} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-x}{8} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1-x}{8} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1-x}{8} \end{pmatrix}$$

The distinct non-zero eigenvalues of ρ_B are

$$\eta_1 = \eta_2 = \cdots \eta_6 = \frac{1-x}{8}, \; \eta_7 = \frac{1+x}{8}, \qquad \eta_8 = \frac{1+5x}{8}.$$

The AR q-conditional entropy [40] for the state $\rho_{\mathrm{W}_4}(x)$ in its $1:3$ partition is given by

$$S_q^T(A|B) = \frac{1}{q-1}\left(1 - \frac{15\left(\frac{1-x}{16}\right)^q + \left(\frac{1+15x}{16}\right)^q}{6\left(\frac{1-x}{8}\right)^q + \left(\frac{1+x}{8}\right)^q + \left(\frac{1+5x}{8}\right)^q}\right) \tag{4.65}$$

Identifying the zero of the monotonically decreasing function $\lim_{\mathrm{q}\rightarrow\infty} \mathrm{S}_{\mathrm{q}}^{\mathrm{T}}(\mathrm{A}|\mathrm{B})$ at $x = 0.2$, one can obtain $(0,\ 0.2)$ as the $1:3$ AR separability range of the state $\rho_{\mathrm{W}_4}(x)$.

The unitary U_B which diagonalizes the three qubit marginal ρ_B is given by

$$U_B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & 0 & \frac{2}{\sqrt{6}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \end{pmatrix}$$

The non-zero eigenvalues γ_i of $\Gamma_U = (I_2 \otimes U_B)\Gamma(I_2 \otimes U_B)^\dagger$ which are same as the eigenvalues of the sandwiched matrix $\Gamma = \left(I_2 \otimes\, \rho_B^{\frac{1-q}{2q}}\right) \rho_{\mathrm{W}_4}(x) \left(I_2 \otimes\, \rho_B^{\frac{1-q}{2q}}\right)$ are seen to be,

$$\gamma_1 = \left(\frac{1-x}{16}\right)\left(\frac{1-x}{8}\right)^{\frac{1-q}{q}} \quad \text{(12-fold degenerate)}, \tag{4.66}$$

$$\gamma_2 = \left(\frac{1-x}{16}\right)\left(\frac{4+4x}{32}\right)^{\frac{1-q}{q}}, \ \gamma_3 = \left(\frac{1-x}{16}\right)\left(\frac{4+20x}{32}\right)^{\frac{1-q}{q}},$$

$$\gamma_{4/5} = \left(\frac{1}{4}\right)(32)^{\frac{-1}{q}}\left(\alpha\ a + \beta\ b \pm \sqrt{(\alpha\ a+\beta\ b)^2 + 3072x^2\ \alpha\ \beta}\right).$$

where $\qquad \alpha = (4+4x)^{\frac{1-q}{q}},\ \beta = (4+20x)^{\frac{1-q}{q}},\ a = (4+12x),\ b = (4+44x).$

One can now readily evaluate the expression for CSTRE $\tilde{D}_q^T(\rho_{\mathrm{W}_4}(x)||\rho_B)$ in its 1 : 3 partition using $\tilde{D}_q^T(\rho_{\mathrm{W}_4}(x)||\rho_B) = \frac{\sum_i \gamma_i^q - 1}{1-q}$. The 1 : 3 separability range of the state $\rho_{\mathrm{W}_4}(x)$ is obtained as $0 \leq x \leq 0.1261$ through identifying the zero of the function $\lim_{q\to\infty} \tilde{D}_q^T(\rho_{\mathrm{W}_4}(x)||\rho_B)$ [72].

4.3.3 Separability in the 1 : 4 partition of five qubit Werner-like family involving W-states

The Werner-like one parameter family involving 5-qubit W-states is given by

$$\rho_{\mathrm{W}_5}(x) = (1-x)\frac{I_{32}}{32} + \ x\ \left|\mathrm{W}_5\right\rangle\left\langle \mathrm{W}_5\right|. \tag{4.67}$$

and its nonzero eigenvalues are

$$\lambda_1 = \frac{1-x}{32} \quad \text{(31-fold degenerate)}, \quad \lambda_2 = \frac{1+31x}{32}. \tag{4.68}$$

The AR q-conditional entropy for the state $\rho_{\mathrm{W}_5}(x)$, in its 1 : 4 partition, is given by

$$S_q^T(A|B) = \frac{1}{q-1}\left(1 - \frac{31\left(\frac{1-x}{32}\right)^q + \left(\frac{1+31x}{32}\right)^q}{14\left(\frac{1-x}{16}\right)^q + \left(\frac{5+11x}{80}\right)^q + \left(\frac{5+59x}{80}\right)^q}\right) \tag{4.69}$$

In the limit $q \to \infty$, the zero of $S_q^T(A|B)$ reveals that the state $\rho_{\mathrm{W}_5}(x)$ is separable in the range $(0,\ 0.1351)$ in its $1:4$ partition.

The unitary U_B which diagonalizes the four qubit marginal ρ_B is given by

$$U_B = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{3}{\sqrt{12}} & \frac{-1}{\sqrt{12}} & 0 & \frac{-1}{\sqrt{12}} & 0 & 0 & 0 & \frac{-1}{\sqrt{12}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{2}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{6}} & 0 & 0 & 0 & \frac{-1}{\sqrt{6}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

The non-zero eigenvalues of the matrix Γ_U which is unitarily equivalent to the sandwiched matrix $\Gamma = \left(I_2 \otimes \rho_B^{\frac{1-q}{2q}}\right) \rho_{\mathrm{W}_5}(x) \left(I_2 \otimes \rho_B^{\frac{1-q}{2q}}\right)$ are found to be

$$\begin{aligned}
\gamma_1 &= \left(\frac{1-x}{32}\right)\left(\frac{1-x}{16}\right)^{\frac{1-q}{q}} \text{ (28-fold degenerate)}, \qquad (4.70) \\
\gamma_2 &= \left(\frac{1-x}{32}\right)\left(\frac{5+11x}{80}\right)^{\frac{1-q}{q}}, \quad \gamma_3 = \left(\frac{1-x}{32}\right)\left(\frac{5+59x}{80}\right)^{\frac{1-q}{q}}, \\
\gamma_{4/5} &= \left(\frac{1}{4}\right)(80)^{\frac{-1}{q}} \left(\alpha\ a + \beta\ b \pm \sqrt{(\alpha\ a+\beta\ b)^2 + 16384x^2\ \alpha\ \beta}\right).
\end{aligned}$$

where $\quad \alpha = (5+11x)^{\frac{1-q}{q}},\ \beta = (5+59x)^{\frac{1-q}{q}},\ a = (5+27x),\ b = (5+123x).$

One can now readily evaluate the expression for CSTRE using $\tilde{D}_q^T(\rho_{\mathrm{W}_5}(x)||\rho_B) = \frac{\sum_i \gamma_i^q - 1}{1-q}$ and obtain $\tilde{D}_q^T(\rho_{\mathrm{W}_5}(x)||\rho_B)$ as a function of x. From the CSTRE criterion one can obtain $(0,\ 0.0724)$ as the separability range for $\rho_{\mathrm{W}_5}(x)$.

4.3.4 Separability in the $1:5$ partition of six qubit Werner-like family involving W-states

The Werner-like one parameter family involving 6-qubit W-state is given by

$$\rho_{\mathrm{W}_6}(x) = (1-x)\frac{I_{64}}{64} + \ x \ \left|\mathrm{W}_6\right\rangle \left\langle \mathrm{W}_6\right|. \tag{4.71}$$

where I_{64} denotes the 64×64 identity matrix and $|\mathrm{W}_6\rangle$ is the 6-qubit W-state.

The non zero eigenvalues of the state $\rho_{\mathrm{W}_6}(x)$ and its 5-qubit marginal ρ_B are respectively given by

$$\lambda_1 = \frac{1-x}{64} \quad \text{(63-fold degenerate)}, \quad \lambda_2 = \frac{1+63x}{64} \tag{4.72}$$

$$\eta_1 = \eta_2 = \cdots \eta_{32} = \frac{1-x}{32}, \quad \eta_{33} = \frac{3+13x}{96}, \quad \eta_{34} = \frac{3+77x}{96}.$$

The AR q-conditional entropy for the state $\rho_{\mathrm{W}_6}(x)$ in its $1:5$ partition is given by

$$S_q^T(A|B) = \frac{1}{q-1}\left(1 - \frac{63\left(\frac{1-x}{64}\right)^q + \left(\frac{1+63x}{64}\right)^q}{30\left(\frac{1-x}{32}\right)^q + \left(\frac{3+13x}{96}\right)^q + \left(\frac{3+77x}{96}\right)^q}\right) \tag{4.73}$$

On obtaining the zero of $S_q^T(A|B)$ in the limit $q \to \infty$, one can obtain $(0,\ 0.0857)$ as the $1:5$ separability range of the state $\rho_{\mathrm{W}_6}(x)$ using AR criterion.

The unitary matrix U_B which diagonalizes the five qubit marginal ρ_B helps in the evaluation of Γ_U, the unitary equivalent of the sandwiched matrix Γ and the non-zero eigenvalues of Γ_U (hence of Γ) are seen to be

$$\gamma_1 = \left(\frac{1-x}{64}\right)\left(\frac{1-x}{32}\right)^{\frac{1-q}{q}} \text{ (60-fold degenerate)}, \tag{4.74}$$

$$\gamma_2 = \left(\frac{1-x}{64}\right)\left(\frac{6+26x}{192}\right)^{\frac{1-q}{q}}, \quad \gamma_3 = \left(\frac{1-x}{64}\right)\left(\frac{6+154x}{192}\right)^{\frac{1-q}{q}},$$

$$\gamma_{4/5} = \left(\frac{1}{4}\right)(192)^{\frac{-1}{q}}\left(\alpha\ a+\beta\ b \pm \sqrt{(\alpha\ a+\beta\ b)^2+81920x^2\ \alpha\ \beta}\right).$$

where $\alpha=(6+26x)^{\frac{1-q}{q}},\ \beta=(6+154x)^{\frac{1-q}{q}},\ a=(6+58x),\ b=(6+314x).$

Using the relation $\tilde{D}_q^T(\rho_{\mathrm{W}_6}(x)||\rho_B) = \frac{\sum_i \gamma_i^q - 1}{1-q}$, the expression for CSTRE $\tilde{D}_q^T(\rho_{\mathrm{W}_6}(x)||\rho_B)$ in its $1:5$ partition can be evaluated. The variation of the different conditional entropies including $\tilde{D}_q^T(\rho_{\mathrm{W}_6}(x)||\rho_B)$ as a function of x is shown in Fig. 4.9. The $1:5$ separability range of the state $\rho_{\mathrm{W}_6}(x)$ is obtained as $0 \le x \le 0.0857$ through identifying the zero of the function $\lim_{q\to\infty} \tilde{D}_q^T(\rho_{\mathrm{W}_6}(x)||\rho_B)$ [72].

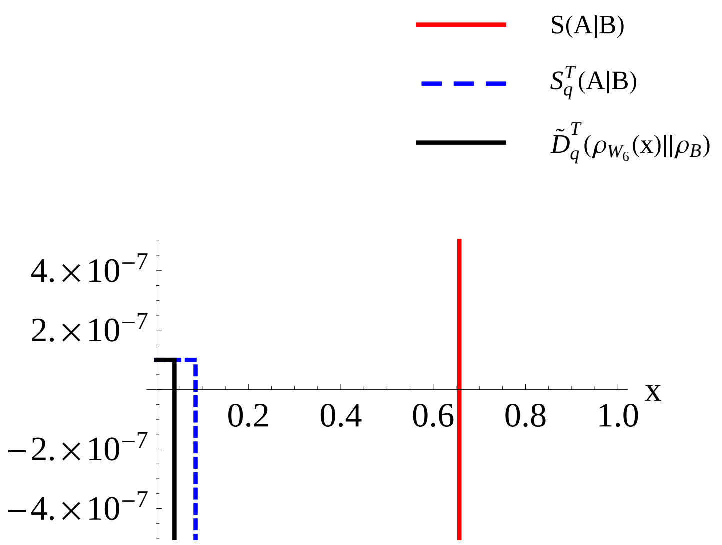

FIGURE 4.9: Plot of the conditional entropies $S(A|B)$ and $\lim_{q\to\infty} \tilde{D}_q^T(\rho_{\mathrm{W}_6}(x)||\rho_B)$ of $\rho_{\mathrm{W}_6}(x)$, in its $1:5$ partition, as a function of x.

The structure of γ_i obtained for $N=3,\ 4,\ 5,\ 6$ helps in the generalization of their form for any $N \ge 3$ and this task is carried out in the following.

4.3.5 $1:N-1$ separability in Werner like one parameter family involving N-qubit W states

The state corresponding to Werner like one-parameter family involving N-qubit W states is given by

$$\rho_{\mathrm{W}_N}(x) \;=\; (1-x)\frac{I_2^{\otimes N}}{2^N} + \; x \;\left|\mathrm{W}_N\right\rangle\left\langle \mathrm{W}_N\right|.$$

In order to carry on with the task of identifying the $1:N-1$ separability range of the state $\rho_{\mathrm{W}_N}(x)$ via the CSTRE method, the 2^N eigenvalues γ_i of the sandwiched matrix

$$\Gamma = (I_2\otimes\rho_B)^{\frac{1-q}{2q}}\rho_{\mathrm{W}_N}(x)(I_2\otimes\rho_B)^{\frac{1-q}{2q}}$$

are to be evaluated and an observation of the structure of γ_i obtained for $N = 3,\ 4,\ 5,\ 6$ helps in their generalization for any $N\geq 3$. It can be seen that the eigenvalues of the sandwiched matrix Γ corresponding to $1:N-1$ partition of the state $\rho_{\mathrm{W}_N}(x)$ are explicitly given by

$$\begin{aligned}
\gamma_1 &= \left(\frac{1-x}{2^N}\right)\left[\frac{1-x}{2^{N-1}}\right]^{\frac{1-q}{q}}; \qquad (2^N-4) \text{ fold-degenerate}\\
\gamma_2 &= \left(\frac{1-x}{2^N}\right)\left[\frac{N+\left(\sum_{j=3}^{N}2^{j-2}-(N-2)\right)x}{N\;2^{N-1}}\right]^{\frac{1-q}{q}};\\
\gamma_3 &= \left(\frac{1-x}{2^N}\right)\left[\frac{N+\left(\sum_{j=3}^{N}2^{j-2}+(N-2)\left(2^{N-1}-1\right)\right)x}{N\;2^{N-1}}\right]^{\frac{1-q}{q}}; \qquad (4.75)\\
\gamma_{4/5} &= \frac{1}{4}\left(2^{N-1}N\right)^{\frac{-1}{q}}\left[\alpha\;a+\beta\;b\pm\sqrt{(\alpha\;a-\beta\;b)^2+2^{2N+2}(N-1)x^2\;\alpha\;\beta}\right]
\end{aligned}$$

where

$$\begin{aligned}
\alpha &= \left[N + \left(\sum_{j=3}^{N} 2^{j-2} - (N-2)\right) x\right]^{\frac{1-q}{q}}, \\
\beta &= \left(N + \left[\sum_{j=3}^{N} 2^{j-2} + (N-2)(2^{N-1}-1)\right) x\right]^{\frac{1-q}{q}}, \\
a &= N + \left(\sum_{j=3}^{N} 2^{j-2} - (N-2) + 2^{N-1}\right) x, \\
b &= N + \left(\sum_{j=3}^{N} 2^{j-2} + 2^{N-1}(2N-3) - (N-2)\right) x.
\end{aligned} \tag{4.76}$$

Substituting for γ_i in Eq. (4.25), a numerical estimation of the separability ranges in the 1 : 2, 1 : 3, 1 : 4, 1 : 5 bipartitions of the noisy states $\rho_{\mathrm{W}_3}(x)$, $\rho_{\mathrm{W}_4}(x)$, $\rho_{\mathrm{W}_5}(x)$, $\rho_{\mathrm{W}_6}(x)$ is carried out. The separability threshold value of the parameter x obtained using CSTRE approach, along with the corresponding results from PPT criteria and also those inferred via the positivity of the corresponding von Neumann and the AR-conditional entropies are tabulated in (see Table 4.3). It

TABLE 4.3: The $1:N-1$ separability threshold value of the noisy parameter x in the states $\rho_{\mathrm{W}_N}(x)$ for $N = 3,\ 4,\ 5,\ 6$, obtained through different separability criteria.

Number of qubits	von Neumann conditional entropy	AR q-conditional entropy	CSTRE	PPT
3	0.7018	0.2727	0.2095	0.2095
4	0.6760	0.2	0.1261	0.1261
5	0.6618	0.1351	0.0724	0.0724
6	0.6567	0.0857	0.0402	0.0402

is readily seen that the result based on the positivity of the CSTRE is stronger than the one obtained from the positivity of the von Neumann, AR conditional entropies. Further, it is observed that the CSTRE result agrees with that identified from the PPT criterion. In general, the CSTRE approach is found to lead to the separability range

$$0 \leq x \leq \frac{N}{N + 2^N\,\sqrt{N-1}} \tag{4.77}$$

for the $1 : N-1$ partitions of the state $\rho_{\mathrm{W}_N}(x)$ for $N \geq 3$.

One can recall here that the noisy N-qubit state $\rho_{\Phi_N}(x)$ in Eq. (4.57) is known to be separable iff [73]

$$0 \leq x \leq \frac{1}{2^N\, u_1\, u_2 + 1} \tag{4.78}$$

where u_1 and u_2 are the two largest Schmidt coefficients of the pure entangled state $\vert\Phi_N\rangle$ under bipartition. In the specific case of $(1 : N-1)$ partition of the state $\rho_{\mathrm{W}_N}(x)$, on substituting the corresponding Schmidt coefficients (see Eq. (4.29))

$$u_1 = \sqrt{\frac{N-1}{N}}, \quad u_2 = \frac{1}{\sqrt{N}}$$

in Eq. (4.78), one can recognize that the separability range reveals a clear agreement with Eq. (4.77) obtained via the CSTRE approach. This establishes that the CSTRE method serves as necessary and sufficient for inferring separability in the Werner-like one parameter family of W states also.

4.3.6 $1 : N-1$ separability in Werner-like noisy states involving N-qubit GHZ states

To investigate the $(1 : N-1)$ separability range in the one parameter family of noisy Werner-like N qubit GHZ states,

$$\rho_{\mathrm{GHZ}_N}(x) \;\; = \;\; (1-x)\frac{I_2^{\otimes N}}{2^N} + \; x\; \vert\mathrm{GHZ}_N\rangle\,\langle\mathrm{GHZ}_N\vert \tag{4.79}$$

the eigenvalues Γ_i of the sandwiched matrix

$$\Gamma = (I_2\otimes\rho_B)^{\frac{1-q}{2q}}\rho_{\mathrm{GHZ}_N}(x)(I_2\otimes\rho_B)^{\frac{1-q}{2q}} \tag{4.80}$$

are evaluated. Here $\rho_B = \mathrm{Tr}_1[\rho_{\mathrm{GHZ}_N}(x)]$ is the subsystem density matrix of $\rho_{\mathrm{GHZ}_N}(x)$ obtained by tracing over its first qubit. The eigenvalues γ_i of the sandwiched matrix Γ for the cases $N = 3,\ 4,\ 5,\ 6$ are explicitly evaluated and they

TABLE 4.4: The non-zero eigenvalues γ_i of the sandwiched matrix $(I_2\otimes\rho_B)^{\frac{1-q}{2q}}\rho_{\mathrm{GHZ}_N}(x)(I_2\otimes\rho_B)^{\frac{1-q}{2q}}$ for $N=3$ *to* 6

Number of qubits (N)	γ_1 (2^N-4) fold degenerate	γ_2 3 fold degenerate	γ_3
$N=3$	$\left(\frac{1-x}{8}\right)\left(\frac{1-x}{4}\right)^{\frac{1-q}{q}}$	$\left(\frac{1-x}{8}\right)\left(\frac{1+x}{4}\right)^{\frac{1-q}{q}}$	$\left(\frac{1+7x}{8}\right)\left(\frac{1+x}{4}\right)^{\frac{1-q}{q}}$
$N=4$	$\left(\frac{1-x}{16}\right)\left(\frac{1-x}{8}\right)^{\frac{1-q}{q}}$	$\left(\frac{1-x}{16}\right)\left(\frac{1+3x}{8}\right)^{\frac{1-q}{q}}$	$\left(\frac{1+15x}{16}\right)\left(\frac{1+3x}{8}\right)^{\frac{1-q}{q}}$
$N=5$	$\left(\frac{1-x}{32}\right)\left(\frac{1-x}{16}\right)^{\frac{1-q}{q}}$	$\left(\frac{1-x}{32}\right)\left(\frac{1+7x}{16}\right)^{\frac{1-q}{q}}$	$\left(\frac{1+31x}{32}\right)\left(\frac{1+7x}{16}\right)^{\frac{1-q}{q}}$
$N=6$	$\left(\frac{1-x}{64}\right)\left(\frac{1-x}{32}\right)^{\frac{1-q}{q}}$	$\left(\frac{1-x}{64}\right)\left(\frac{1+15x}{32}\right)^{\frac{1-q}{q}}$	$\left(\frac{1+63x}{64}\right)\left(\frac{1+15x}{32}\right)^{\frac{1-q}{q}}$

are given in Table 4.4. On observing the structure of the eigenvalues γ_i in Table 4.4, it is possible to arrive at the form of the eigenvalues for any $N\geq 3$.

$$
\begin{aligned}
\gamma_1 &= \left[\frac{1-x}{2^N}\right]\left[\frac{1-x}{2^{N-1}}\right]^{\frac{1-q}{q}}; \quad (2^N-4)\text{-fold degenerate;} \\
\gamma_2 &= \left[\frac{1-x}{2^N}\right]\left[\frac{1+\left(2^{N-2}-1\right)x}{2^{N-1}}\right]^{\frac{1-q}{q}}; \quad \text{3-fold degenerate} \\
\gamma_3 &= \left[\frac{1+\left(2^{N}-1\right)x}{2^{N}}\right]\left[\frac{1+\left(2^{N-2}-1\right)x}{2^{N-1}}\right]^{\frac{1-q}{q}}.
\end{aligned}
\tag{4.81}
$$

Substituting Eq. (4.81) in Eq. (4.25) we find that positivity of CSTRE as $q\to\infty$ requires the following bound

$$
0\leq x\leq \frac{1}{2^{N-1}+1}. \tag{4.82}
$$

on the noisy parameter x. This result agrees with the $1:N-1$ separability range obtained based on the commutative AR method too in the case of $\rho_{\mathrm{GHZ}_N}(x)$. However, the convergence towards the threshold value of the parameter $x\to\frac{1}{2^{N-1}+1}$ in the limit $q\to\infty$ based on the CSTRE method is slower compared to that of the AR approach. This is illustrated in Fig. 4.10 in the specific case of $N=6$.

Moreover, substituting the Schmidt coefficients $u_1 = u_2 = 1/\sqrt{2}$ associated with

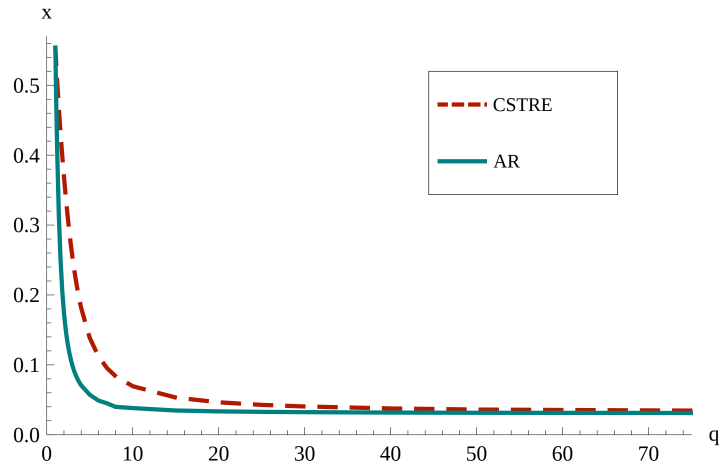

FIGURE 4.10: Implicit plots of $\tilde{D}_q^T(\rho_{\mathrm{GHZ}_6}(x)||\rho_B) = 0$ as a function of q and the AR q-conditional entropy $S_q^T(A|B) = 0$ for $\rho_{\mathrm{GHZ}_6}(x)$ in its $1:5$ partition. The convergence of the parameter x to its bound 0.0303 under the CSTRE criterion is slower compared to that of the AR method.

the $(1 : N-1)$ partition of the GHZ state in Eq. (4.78), reveals that the range Eq. (4.82) for the parameter x obtained from CSTRE approach is both necessary and sufficient for the separability in the $(1 : N-1)$ bipartition of the state $\rho_{\mathrm{GHZ}_N}(x)$. Table 4.5 compares the threshold values of x, obtained using different separability criteria, above which the state $\rho_{\mathrm{GHZ}_N}(x)$ is entangled.

TABLE 4.5: Comparison of the $1 : N-1$ separability threshold value of the noisy parameter x in the states $\rho_{\mathrm{GHZ}_N}(x)$ using different separability criteria

Number of qubits	von Neumann conditional entropy	AR q-conditional entropy	CSTRE	PPT
3	0.6829	0.2	0.2	0.2
4	0.6276	0.1111	0.1111	0.1111
5	0.5846	0.0588	0.0588	0.0588
6	0.5536	0.0303	0.0303	0.0303

4.4 Summary

In this chapter, the $1 : N-1$ separability range in the noisy N qubit states of the pseudopure family and Werner-like family involving W, GHZ states[1] is investigated using the CSTRE approach. It is shown that in all the families of states considered here, the $1 : N-1$ separability range obtained using CSTRE criterion matches with the corresponding PPT separability range. In both pseudopure, Werner-like family of states involving W states the AR-criterion is shown to yield a weaker separability range than the one obtained using CSTRE, and PPT criteria. In both these families involving GHZ states, the $1 : N-1$ separability ranges obtained using AR-. CSTRE- and PPT criteria match with one another. The matching of the results due to AR- and CSTRE criteria is attributed to the maximally mixed nature of the single qubit marginal of the noisy families involving GHZ states. It is shown that the positivity of the CSTRE in the limit $q \to \infty$ is both necessary and sufficient for the $1 : N-1$ separability in the one parameter family of noisy pseudopure, Werner-like states.

* * * * * * * * * * * * *

[1] It is to be noted here that both PP and Werner-like noisy one parameter family of states considered here are not permutation symmetric states, as they do not get restricted only to the $N+1$ dimensional symmetric subspace of the 2^N Hilbert space of N-qubits. This is because the completely random state $I_2^{\otimes N}/2^N$, which is mixed with the pure symmetric N qubit states in $\rho^{\rm PP}_{{\rm W}_N}(x)$, $\rho^{\rm PP}_{{\rm GHZ}_N}(x)$, $\rho_{{\rm W}_N}(x)$, $\rho_{{\rm GHZ}_N}(x)$, is not confined to the $N+1$ dimensional subspace of permutation symmetric N qubit states

Chapter 5

One parameter family of N-qudit Werner-Popescu states: Bipartite separability using Conditional quantum relative Tsallis entropy

In Chapters 2, 3, 4, it has been shown that CSTRE criterion gives strictest separability range in the $1 : N-1$ partition of the one parameter family of N-qubit mixed W and GHZ states. In Chapter 3, an illustration of the utility of CSTRE criterion to detect entanglement in qubit-qutrit and qutrit-qutrit states has also been explored. It is of interest to check whether CSTRE approach can be employed to obtain the bipartitie separability range of mixed multipartite states with higher dimensions. In this Chapter, the CSTRE method is employed to obtain the $1 : N-1$ separability range of the N-qudit Werner- Popescu type of states. It is observed that in the limit $q \to \infty$, a $1 : N-1$ separability range that matches with the one obtained using an algebraic necessary and sufficient condition [54, 55] for separability. Further, a comparison of the convergence of the parameter x with increasing values of q in the implicit plots of AR q-conditional entropy and CSTRE is carried out [74].

This Chapter is organized in three sections. Sec. 5.1 defines the one-parameter family of N-qudit Werner-Popescu states and details the attempts in the literature to identify the bipartite separability in these states. The CSTRE criterion is employed in Sec. 5.2 to obtain the $1:N-1$ separability range of these states. A comparison of CSTRE and AR-criterion for the state under consideration is also carried out in Sec. 5.2. Section 5.3 contains the summary of the Chapter.

5.1 N-qudit Werner-Popescu states

The density matrix of the one parameter family of the Werner-Popescu-type state with N-qudits [35] is defined as follows

$$\begin{aligned} \rho_N^d(x) &= \rho\,(A_1,\,A_2,\,\ldots A_N) \\ &= \frac{1-x}{d^N} I_d(A_1)\otimes I_d(A_2)\otimes\ldots I_d(A_N) + \; x\left|\Phi_d^N\right\rangle\,\left\langle\Phi_d^N\right| \end{aligned} \tag{5.1}$$

Here $0 \le x \le 1$ and $I_d(A_i)$, $i = 1, 2, \ldots, N$ are $d\times d$ unit matrices belonging to the subsystem space of each qudit A_i, $i = 1, 2, \ldots, N$. The pure state $\left|\Phi_d^N\right\rangle$ is given by

$$\left|\Phi_d^N\right\rangle = \frac{1}{\sqrt{d}}\sum_{k=0}^{d-1}\,|k\rangle_{A_1}\otimes|k\rangle_{A_2}\otimes\ldots\otimes|k\rangle_{A_N}\,. \tag{5.2}$$

and it is an analogue of GHZ state to d-level systems. Notice that when $d = 2$, i.e., for qubits, $k = 0, 1$ and Eq. (5.2) reduces to the N-qubit GHZ state (See Eq. (2.57))

$$|\mathrm{GHZ}_N\rangle = \frac{1}{\sqrt{2}}\left(|0_1 0_2\cdots 0_N\rangle + |1_1 1_2\cdots 1_N\rangle\right)$$

The eigenvalues of $\rho_N^d(x)$ are given by

$$\begin{aligned} \lambda_1 &= \frac{1-x}{d^N} \qquad\qquad [(d^N-1)\text{fold degenerate}], \\ \lambda_2 &= \frac{1+(d^N-1)x}{d^N}. \end{aligned} \tag{5.3}$$

The focus here is on finding the $1:N-1$ separability range of $\rho_N^d(x)$ using CSTRE criterion.

5.2 $1:N-1$ separability of $\rho_N^d(x)$ using CSTRE

Denoting the first qubit as subsystem A and the remaining $N-1$ qubits as subsystem B, the density matrix of the $N-1$ qubit marginal is given by

$$\rho_B \;=\; = \mathrm{Tr}_{A_1}\, \rho\,(A_1,\, A_2, \ldots, A_N) = \mathrm{Tr}_{A_1}\, \rho_N^d(x)$$

It can be seen that the eigenvalues η_i of the $N-1$ qubit marginal ρ_B of $\rho_N^d(x)$, obtained by reducing over the first qubit, are given by

$$\eta_1 = \frac{1-x}{d^{N-1}} \qquad [(d^{N-1}-d) - \text{fold degenerate}],$$

$$\eta_2 = \frac{1+(d^{N-2}-1)x}{d^{N-1}} \qquad [d - \text{fold degenerate}] \tag{5.4}$$

Also, the subsystem ρ_A, the single qudit marginal of $\rho_N^d(x)$, corresponds to the maximally mixed state I_d/d, I_d being $d\times d$ unit matrix.

In order to find the separability range of the state ρ_N^d in its $1:N-1$ partition using CSTRE criterion, one needs to evaluate the eigenvalues γ_i of the sandwiched matrix

$$\Gamma = (I_A\otimes\rho_B)^{\frac{1-q}{2q}}\; \rho_N^d(x)\;\; (I_A\otimes\rho_B)^{\frac{1-q}{2q}} \tag{5.5}$$

so that

$$\widetilde{D}_q^T\left(\rho_N^d(x)||\rho_B\right) = \frac{\sum_i \gamma_i^q - 1}{1-q} \tag{5.6}$$

can be evaluated. Thus, in the evaluation of $\widetilde{D}_q^T\left(\rho_N^d(x)||\rho_B\right)$, the non-negative eigenvalues γ_i play a crucial role. Before generalizing the form of the eigenvalues γ_i for N-qudits, an analysis of their form for different $N (N = 2,\,3,\,4,\,5)$ and d ($d = 3,\,4,\,5,\,6$) is carried out to arrive at a generalization for any N, d [74]. Table 5.1 provides the explicitly evaluated non-zero eigenvalues of the sandwiched matrix Γ for different values of N and d [74]. It can be readily seen from Table 5.1 that, there are only three distinct non-zero eigenvalues for the sandwiched matrix Γ. A careful observation of the eigenvalues $\gamma_i, i = 1,\,2,\,3$ leads towards the generalization of the eigenvalues of sandwiched matrix $(I_A\otimes\rho_B)^{\frac{1-q}{2q}}\; \rho_N^d(x)\;\; (I_A\otimes\rho_B)^{\frac{1-q}{2q}}$ for $N\geq 3$. The generalized eigenvalues γ_i of the sandwiched matrix Γ are given in the following [74]:

$$\begin{aligned}
\gamma_1 &= \left(\frac{1-x}{d^N}\right)\left(\frac{1-x}{d^{N-1}}\right)^{\frac{1-q}{q}}, \qquad \left(d^N-d^2\right) - \text{fold degenerate} \\
\gamma_2 &= \left(\frac{1-x}{d^N}\right)\left(\frac{1+\left(d^{N-2}-1\right)x}{d^{N-1}}\right)^{\frac{1-q}{q}}, \qquad \left(d^2-1\right) - \text{ fold degenerate} \\
\gamma_3 &= \left(\frac{1+\left(d^{N}-1\right)x}{d^N}\right)\left(\frac{1+\left(d^{N-2}-1\right)x}{d^{N-1}}\right)^{\frac{1-q}{q}}, \quad \text{non-degenerate.} \qquad (5.7)
\end{aligned}$$

The $1:N-1$ separability range of the state $\rho_N^d(x)$, for each combination of $N=2,3,4,5$ and $d=3,4,5,6$ obtained using CSTRE approach allows us to generalize this range to any N and d. Table 5.2 gives the values of x below which the state $\rho_N^d(x)$ is separable. Using Table 5.2, the following $1:N-1$ separability range is conjectured for the one parameter family of N-qudit Werner-Popescu-states [74].

$$0 \leq x \leq \frac{1}{1+d^{N-1}} \qquad (5.8)$$

One can note that the $1:N-1$ separability range given in Eq.(5.8) is the same as that obtained in Ref. [35], using the AR-criterion. In fact, the existence of maximally mixed single qubit density matrix is the reason behind the equivalence of separability ranges in CSTRE and AR-criteria. Such a situation occurs in the case of symmetric one parameter family of noisy GHZ states (Sec. 3.3), psuedopure family containing GHZ states (Sec. 4.2) and Werner-like family of states containing GHZ states (Sec. 4.3.6). In all these states, the single qubit density matrix turns out to be $I_2/2$ thus implying that the non-commutative CSTRE approach yields the results equivalent to commutative AR-approach [63]. It is important to notice here that, using algebraic methods [54, 55] it has been shown that Eq.(5.8) is actually the necessary and sufficient condition for separability.

Fig.5.1 gives an illustration of the monotonic decrease of $\tilde{D}_q^T(\rho_4^{(3)}(x)||\rho_B)$ with increasing x in the $q\to\infty$. From Fig. 5.2 it can be seen that $\tilde{D}_q^T(\rho_4^{(3)}(x)||\rho_B)$ is negative for $x>0.5633$ when $q=1$ (separability range through von Neumann conditional entropy), whereas it is negative for $x>0.0357$ in the limit $q\to\infty$ (separability range through CSTRE criterion).

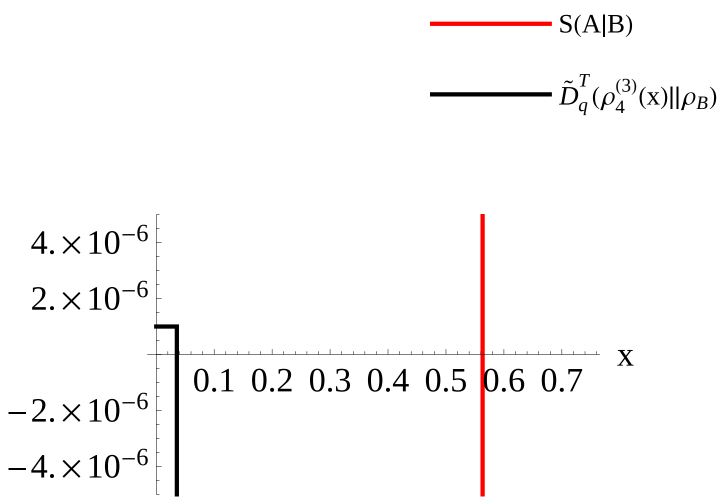

FIGURE 5.1: The variation of conditional form of sandwiched Tsallis relative entropy $\tilde{D}_q^T(\rho_4^{(3)}(x)||\rho_B)$ in the $1:3$ partition of 4-qutrit Werner-Popescu states $\rho_4^{(3)}(x)$ (N=4, $d=3$), with respect to x, in the limit $q \to \infty$.

Even though the separability range of $\rho_N^d(x)$, obtained using both CSTRE and AR-conditional entropy are same, there is a rapid convergence of the parameter x with increasing values of q in the case of AR q-conditional entropy. This feature is illustrated in Figs. 5.2, 5.3. Table 5.3 provides the values of the parameter x at which CSTRE, AR q-conditional entropy becomes zero, when $q=2$, for different d and N. From Table 5.3 one can easily note that the parameter x is rapidly decreasing in AR method even for $q=2$ thus confirming its relatively rapid convergence in comparison with that of CSTRE in the limit $q \to \infty$.

It is also evident from Table 5.2 that the separability range decreases with the number of subsystems i.e., with the increase of N for any given d. This feature is illustrated in Figs. 5.4, 5.5. A comparison of Figs. 5.4, 5.5 illustrates that for any given N, the separability range decreases with increasing d. Thus a state of the Werner-Popescu family is entangled throughout the parameter range x if its constituents are qudits with larger d. More qudits in the state also helps the state to be entangled in the whole parameter range.

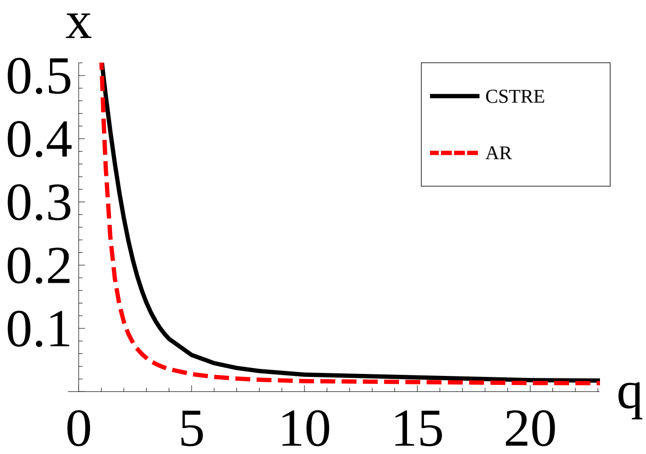

FIGURE 5.2: The comparison between implicit plots of $\tilde{D}_q^T(\rho_5^{(3)}(x)||\rho_B) = 0$ and $S_q^T(A|B) = 0$, as a function of q in the $1 : 4$ partition of the 5-qutrit ($N = 5$, $d = 3$) state $\rho_5^{(3)}(x)$. A rapid decrease in the value of x, in comparison with $\tilde{D}_q^T(\rho_5^{(3)}(x)||\rho_B)$ can be observed in the case of $S_q^T(A|B)$.

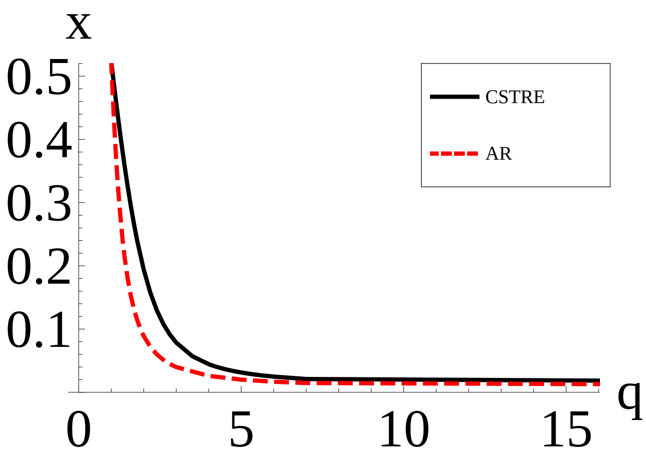

FIGURE 5.3: The comparison between implicit plots of $\tilde{D}_q^T(\rho_4^{(5)}(x)||\rho_B) = 0$ and $S_q^T(A|B) = 0$, as a function of q for 4-partite ($N = 4$), 5-level ($d = 5$) Werner-Popescu states $\rho_4^{(5)}(x)$.

5.3 Summary

In this Chapter, the CSTRE criterion is employed to find out the $1 : N-1$ separability range of N-party Werner-Popescu states containing d-level quantum systems. It is observed that the $1 : N-1$ separability range obtained through both CSTRE and AR q conditional entropy criteria match with each other for

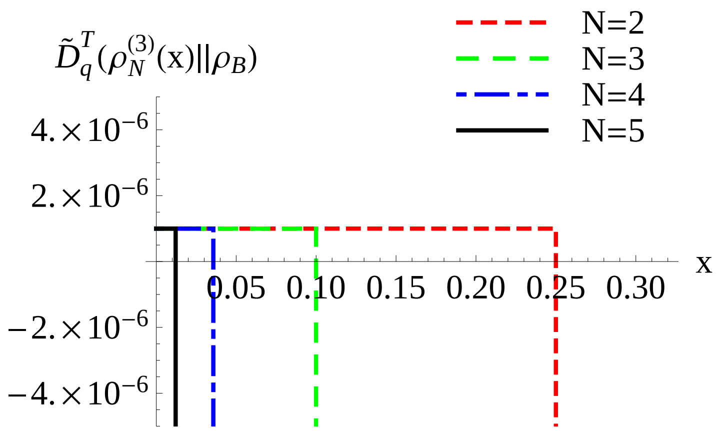

FIGURE 5.4: The graph of CSTRE $\tilde{D}_q^T(\rho_N^{(3)}(x)||\rho_B) = 0$ versus x for different values of N when $d = 3$. The decrease of the separability range with N, for any given d is clearly seen.

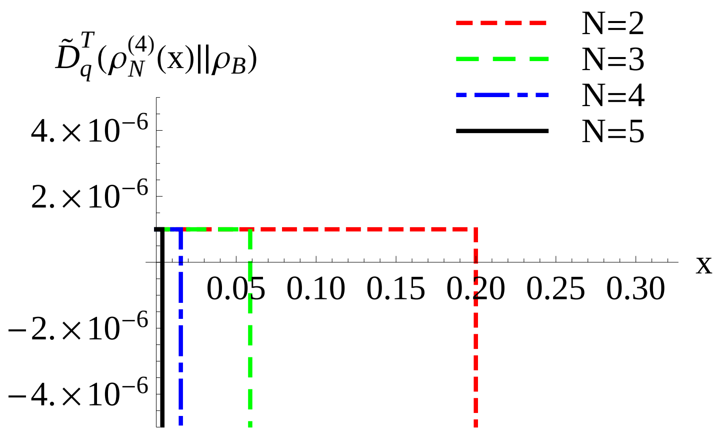

FIGURE 5.5: The graph of CSTRE $\tilde{D}_q^T(\rho_N^{(4)}(x)||\rho_B) = 0$ versus x for different values of N when $d = 4$.

these states. This separability range is seen to match with that obtained using an algebraic necessary and sufficient condition for separability. The relatively smoother convergence of the parameter x with respect to q in the limit $q \to \infty$ is observed in the case of implicit plots of CSTRE in comparison with the convergence in the case of AR q-conditional entropy.

* * * * * * * * * * * * *

TABLE 5.1: The non-zero eigenvalues γ_i of the sandwiched matrix $(I_A \otimes \rho_B)^{\frac{1-q}{2q}}\ \rho_N^d(x)\ (I_A \otimes \rho_B)^{\frac{1-q}{2q}}$.

Number of levels (d)	Number of parties (N)	γ_1 $\left(d^N - d^2\right)$ fold degenerate	γ_2 $\left(d^2-1\right)$ fold degenerate	γ_3
3	2	-	$\left(\frac{1-x}{9}\right)\left(\frac{1}{3}\right)^{\frac{1-q}{q}}$	$\left(\frac{1+8x}{9}\right)\left(\frac{1}{3}\right)^{\frac{1-q}{q}}$
	3	$\left(\frac{1-x}{27}\right)\left(\frac{1-x}{9}\right)^{\frac{1-q}{q}}$	$\left(\frac{1-x}{27}\right)\left(\frac{1+2x}{9}\right)^{\frac{1-q}{q}}$	$\left(\frac{1+26x}{27}\right)\left(\frac{1+2x}{9}\right)^{\frac{1-q}{q}}$
	4	$\left(\frac{1-x}{81}\right)\left(\frac{1-x}{27}\right)^{\frac{1-q}{q}}$	$\left(\frac{1-x}{81}\right)\left(\frac{1+8x}{27}\right)^{\frac{1-q}{q}}$	$\left(\frac{1+80x}{81}\right)\left(\frac{1+8x}{27}\right)^{\frac{1-q}{q}}$
	5	$\left(\frac{1-x}{243}\right)\left(\frac{1-x}{81}\right)^{\frac{1-q}{q}}$	$\left(\frac{1-x}{243}\right)\left(\frac{1+26x}{81}\right)^{\frac{1-q}{q}}$	$\left(\frac{1+242x}{243}\right)\left(\frac{1+26x}{81}\right)^{\frac{1-q}{q}}$
4	2	-	$\left(\frac{1-x}{16}\right)\left(\frac{1}{4}\right)^{\frac{1-q}{q}}$	$\left(\frac{1+15x}{16}\right)\left(\frac{1}{4}\right)^{\frac{1-q}{q}}$
	3	$\left(\frac{1-x}{64}\right)\left(\frac{1-x}{16}\right)^{\frac{1-q}{q}}$	$\left(\frac{1-x}{64}\right)\left(\frac{1+3x}{16}\right)^{\frac{1-q}{q}}$	$\left(\frac{1+63x}{64}\right)\left(\frac{1+3x}{16}\right)^{\frac{1-q}{q}}$
	4	$\left(\frac{1-x}{256}\right)\left(\frac{1-x}{64}\right)^{\frac{1-q}{q}}$	$\left(\frac{1-x}{256}\right)\left(\frac{1+15x}{64}\right)^{\frac{1-q}{q}}$	$\left(\frac{1+255x}{256}\right)\left(\frac{1+15x}{64}\right)^{\frac{1-q}{q}}$
	5	$\left(\frac{1-x}{1024}\right)\left(\frac{1-x}{256}\right)^{\frac{1-q}{q}}$	$\left(\frac{1-x}{1024}\right)\left(\frac{1+63x}{256}\right)^{\frac{1-q}{q}}$	$\left(\frac{1+1023x}{1024}\right)\left(\frac{1+63x}{256}\right)^{\frac{1-q}{q}}$
5	2	-	$\left(\frac{1-x}{25}\right)\left(\frac{1}{5}\right)^{\frac{1-q}{q}}$	$\left(\frac{1+24x}{25}\right)\left(\frac{1}{5}\right)^{\frac{1-q}{q}}$
	3	$\left(\frac{1-x}{125}\right)\left(\frac{1-x}{25}\right)^{\frac{1-q}{q}}$	$\left(\frac{1-x}{125}\right)\left(\frac{1+4x}{25}\right)^{\frac{1-q}{q}}$	$\left(\frac{1+124x}{125}\right)\left(\frac{1+4x}{25}\right)^{\frac{1-q}{q}}$
	4	$\left(\frac{1-x}{625}\right)\left(\frac{1-x}{125}\right)^{\frac{1-q}{q}}$	$\left(\frac{1-x}{625}\right)\left(\frac{1+24x}{125}\right)^{\frac{1-q}{q}}$	$\left(\frac{1+624x}{625}\right)\left(\frac{1+24x}{125}\right)^{\frac{1-q}{q}}$
	5	$\left(\frac{1-x}{3125}\right)\left(\frac{1-x}{625}\right)^{\frac{1-q}{q}}$	$\left(\frac{1-x}{3125}\right)\left(\frac{1+124x}{625}\right)^{\frac{1-q}{q}}$	$\left(\frac{1+3124x}{3125}\right)\left(\frac{1+124x}{625}\right)^{\frac{1-q}{q}}$
6	2	-	$\left(\frac{1-x}{36}\right)\left(\frac{1}{6}\right)^{\frac{1-q}{q}}$	$\left(\frac{1+35x}{36}\right)\left(\frac{1}{6}\right)^{\frac{1-q}{q}}$
	3	$\left(\frac{1-x}{216}\right)\left(\frac{1-x}{36}\right)^{\frac{1-q}{q}}$	$\left(\frac{1-x}{216}\right)\left(\frac{1+5x}{36}\right)^{\frac{1-q}{q}}$	$\left(\frac{1+215x}{216}\right)\left(\frac{1+5x}{36}\right)^{\frac{1-q}{q}}$
	4	$\left(\frac{1-x}{1296}\right)\left(\frac{1-x}{216}\right)^{\frac{1-q}{q}}$	$\left(\frac{1-x}{1296}\right)\left(\frac{1+35x}{216}\right)^{\frac{1-q}{q}}$	$\left(\frac{1+1295x}{1296}\right)\left(\frac{1+35x}{216}\right)^{\frac{1-q}{q}}$
	5	$\left(\frac{1-x}{7776}\right)\left(\frac{1-x}{1296}\right)^{\frac{1-q}{q}}$	$\left(\frac{1-x}{7776}\right)\left(\frac{1+215x}{1296}\right)^{\frac{1-q}{q}}$	$\left(\frac{1+7775x}{7776}\right)\left(\frac{1+215x}{1296}\right)^{\frac{1-q}{q}}$

Table 5.2: The comparison of the $1 : N - 1$ separability range of the state $\rho_N^d(x)$, for various compositions of d and N obtained through CSTRE criterion.

Number of levels (d)	Number of parties (N)	CSTRE separability range
3	2	(0, 0.25)
	3	(0, 0.1)
	4	(0, 0.0357)
	5	(0, 0.0121)
4	2	(0, 0.2)
	3	(0, 0.0588)
	4	(0, 0.0153)
	5	(0, 0.0039)
5	2	(0, 0.1666)
	3	(0, 0.0384)
	4	(0, 0.0079)
	5	(0, 0.0016)
6	2	(0, 0.1428)
	3	(0, 0.0270)
	4	(0, 0.0046)
	5	(0, 0.0007)

Table 5.3: The comparison of the value of x for $q = 2$, obtained through different criteria

Criterion	3-level			4-level			5-level		
	3-party	4-party	5-party	3-party	4-party	5-party	3-party	4-party	5-party
CSTRE	0.3837	0.3114	0.2744	0.3108	0.2396	0.2116	0.2610	0.1943	0.1730
AR	0.3162	0.1889	0.1104	0.2425	0.1240	0.0623	0.1961	0.0890	0.0399

Chapter 6

Non-Spectral nature of the Conditional version of Sandwiched Tsallis Relative Entropy

This chapter outlines a special characteristic of the CSTRE criterion which the other entropic criteria do not have. It is shown that the CSTRE criterion can characterize entanglement in those bipartite states where the knowledge of the eigenvalues of the state and its marginals fail to identify the entanglement in the state. This non-spectral feature of the CSTRE criterion is illustrated here using two isospectral states, states having same local and global spectra (Sec. 6.1). An attempt to characterize the hidden entanglement in several entangled states having positive partial transpose, the so-called bound entangled states, is also carried out in this Chapter (Sec. 6.2). A discussion on the results of the chapter is given in Sec. 6.3.

6.1 Illustration of Nonspectral nature of the CSTRE criterion

Generalized entropies serve as a measure of disorder in a given quantum state and negative values of traditional versions of generalized conditional entropies point towards more global order than local order in a composite system. Separable states are more locally ordered than globally as the eigen spectra of the whole composite separable state is majorized by that of its reduced systems [43]. Negative values of conditional entropies reflect the contrasting feature that local spectra *need not* majorize global spectra in entangled states. However, separability criteria based purely on the spectra of composite and reduced states are shown to be insufficient [43]. This feature was illustrated through an example of an isospectral [1] pair of two qubit states ρ_{AB} and ϱ_{AB}, which share same local and global spectra [43]. The states are explicitly given by

$$\rho_{AB} = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.1}$$

$$\varrho_{AB} = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}. \tag{6.2}$$

The non-zero eigenvalues of the state ρ_{AB} are,

$$\lambda_1 = \frac{2}{3}, \quad \lambda_2 = \frac{1}{3} \tag{6.3}$$

[1]Isospectral states are states of a composite system with same set of non-zero eigenvalues for the composite density matrix and the subsystem density matrices.

and the single qubit marginal obtained by tracing over either the first or the second qubit of ρ_{AB} is seen to be diagonal. That is,

$$\rho_A = \mathrm{Tr}_B\, \rho_{AB}, \quad \rho_B = \mathrm{Tr}_A\, \rho_{AB}, \quad \rho_A = \rho_B$$

and

$$\rho_B = \frac{1}{3}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \rho_A \tag{6.4}$$

It can be readily seen that $S(A,\, B) = S(A) = S(B)$ and hence the von-Neumann conditional entropies $S(A|B) = S(A,\, B) - S(B)$, $S(B|A) = S(A,\, B) - S(A)$ are zero. Similarly, the isospectral nature of the state ρ_{AB} implies that the Abe-Rajagopal q-conditional entropy defined by

$$S_q^T(A|B) = \frac{1}{q-1}\left[1 - \frac{\mathrm{Tr}\,(\rho_{AB})^q}{\mathrm{Tr}\,(\rho_B)^q}\right]$$

turns out to be zero as $\mathrm{Tr}\,(\rho_{AB})^q = \mathrm{Tr}\,(\rho_A)^q$.

To make use of the CSTRE criterion to detect the entanglement in the state ρ_{AB}, the power of each diagonal element of ρ_B is raised by $\frac{1-q}{2q}$, and the sandwiched matrix

$$\Gamma = (I_2 \otimes \rho_B)^{\frac{1-q}{2q}}\, \rho_{AB}\, (I_2 \otimes \rho_B)^{\frac{1-q}{2q}}\,.$$

is evaluated. The eigenvalues of the sandwiched matrix Γ are seen to be

$$\gamma_1 \;=\; 2^{\frac{1-q}{q}} 3^{-\frac{1}{q}}, \quad \gamma_2 = \left(1 + 2^{\frac{1-q}{q}}\right) 3^{-\frac{1}{q}}$$

and the Conditional Sandwiched Tsallis Relative Entropy of the state ρ_{AB} is given by

$$\begin{aligned} \tilde{D}_q^T(\rho_{AB}||\rho_B) &= \frac{\sum_i \gamma_i^q - 1}{1-q} \\ &= \frac{\left(1 + 2^{\frac{1-q}{q}}\right)^q + 2^{1-q} - 3}{3(1-q)}. \end{aligned} \tag{6.5}$$

A plot of $\tilde{D}_q^T(\rho_{AB}||\rho_B)$ as a function of q is given in Fig. 6.1. It can be readily seen through Fig. 6.1 that $\tilde{D}_q^T(\rho_{AB}||\rho_B)$ is negative for all values of q, except for

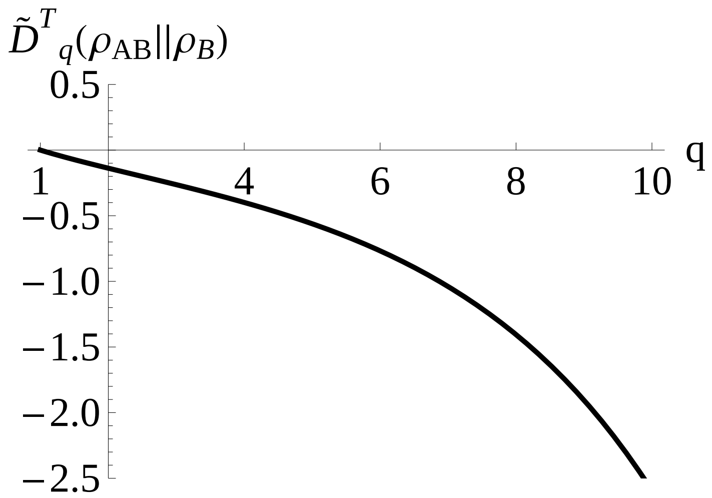

FIGURE 6.1: A plot of CSTRE $\tilde{D}^T_q(\rho_{AB}||\rho_B)$ of the two qubit entangled state in Eq. (6.1) as a function of q. It is readily seen that CSTRE is negative for all values of $q > 1$ indicating that the state is entangled.

$q = 1$. Thus, CSTRE criterion is successful in identifying the entanglement in an isospectral state whereas the spectra-dependent entropic criteria using von-Neumann conditional entropy or AR q-conditional entropy are unable to detect the entanglement in the isospectral entangled state ρ_{AB}.

It is worth observing here that ρ_{AB} does not commute with the state $I_A \otimes \rho_B$ or $\rho_A \otimes I_B$ despite the subsystem density matrix $\rho_A (= \rho_B)$ being diagonal. Thus the non-commutative CSTRE criterion is able to capture the entanglement in states having same local, global spectra and thereby exhibiting its non-spectral nature.

It can be readily seen that the partially transposed density matrix of the state ρ_{AB} given by

$$\rho^T_{AB} = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

has only one negative eigenvalue $\frac{1}{6}\left(1 - \sqrt{5}\right)$ and hence the negativity of partial transpose $N(\rho) = \left|\frac{1}{6}\left(1 - \sqrt{5}\right)\right| = 0.206$ thus being able to quantify the entanglement in ρ_{AB}.

The two-qubit state ϱ_{AB} is readily seen to be separable as it is equal to its own partial transpose and being a two-qubit state, the positivity of partial transpose is a necessary and sufficient condition for separability. The subsystem density matrix $\varrho_A(= \varrho_B)$ is diagonal with eigenvalues $\frac{1}{2}$, $\frac{2}{3}$ same as that of the non-zero eigenvalues of ϱ_{AB}. While the von-Neumann conditional entropy and AR q-conditional entropy are easily seen to be zero for this separable state, the vanishing of the conditional sandwiched relative entropy $\tilde{D}_q^T(\varrho_{AB}||\varrho_B)$ is established below. In fact, the eigenvalues γ_i of the sandwiched matrix $\Gamma = (I_2 \otimes \varrho_B)^{\frac{1-q}{2q}}\ \varrho_{AB}\ (I_2 \otimes \varrho_B)^{\frac{1-q}{2q}}$ are seen to be

$$\gamma_1 = \left(\frac{2}{3}\right)^{\frac{1}{q}}, \quad \gamma_2 = \left(\frac{1}{3}\right)^{\frac{1}{q}} \tag{6.6}$$

leading to $\gamma_1^q + \gamma_2^q = 1$ and hence $\tilde{D}_q^T(\varrho_{AB}||\varrho_B) = 0$. for all $q \geq 1$. Notice that the separable state ϱ_{AB} commutes with its reduced density matrices unlike the entangled state ρ_{AB} thus emphasizing the utility of CSTRE criterion when non-commuting global, local systems are involved.

Through the example of an isospectral entangled state, it is established that the sandwiched conditional Tsallis entropy is capable of detecting entanglement in isospectral states and hence the approach proves to be superior to any spectral disorder criteria.

6.2 Bound Entangled States: An attempt to detect entanglement using CSTRE criterion

According to Peres-Horodecki separability criterion, negative eigenvalues of the partially transposed density matrix indicates entanglement in bipartite systems of dimension 2×2 and 2×3. But in higher dimensions there exist some entangled states with positive partial transpose and they are called **bound entangled states** [23, 24].

The CSTRE criterion is shown to have non-spectral features [62] unlike its commuting counterpart, the AR-criterion, thus being able to identity entanglement in isospectral states [43]. In fact, with the observation [75] that quantum witnesses constructed using spectral criteria cannot identify entanglement in the so-called bound entangled states [23, 24], an effort has been made to identify entanglement in several bound entangled states using the non-spectral CSTRE criterion.

1. **Smolin State[76]**: A four-qubit state ρ_S is defined by [76]

$$\rho_{\rm S} = \sum_{i=1}^{4} \left|\phi^i\right\rangle\left\langle\phi^i\right|_{AD} \otimes \left|\phi^i\right\rangle\left\langle\phi^i\right|_{BC} \tag{6.7}$$

where $|\phi^i\rangle \in \{|\phi^\pm\rangle, |\varphi^\pm\rangle\}$ are the two qubit Bell states

$$\left|\phi^\pm\right\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle \pm |11\rangle\right), \quad \left|\varphi^\pm\right\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle \pm |10\rangle\right).$$

The state ρ_S is termed *Smolin state* and is shown to be *bound entangled* in its 2 : 2 partition [76]. For the state $\rho_{\rm S}$ there exists only one eigenvalue $\frac{1}{4}$ with four fold degeneracy. To check whether the new entropic separability criterion based on conditional version of sandwiched Tsallis relative entropy (CSTRE) detects entanglement in ρ_S, consider the two-qubit marginal of $\rho_{\rm S}$

$$\rho_B = {\rm Tr}_{1,2}\,\rho_{\rm S} = {\rm Tr}_{3,4}\,\rho_{\rm S} = \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} \tag{6.8}$$

As ρ_B is a maximally mixed state of the two-qubit system, it can be expected that the non-commutative CSTRE criterion gives the same results as that of commutative AR-criterion and hence may not detect the bound entanglement in the Smolin state $\rho_{\rm S}$. In order to verify this, the eigenvalues of the sandwiched matrix $\Gamma = (I_4 \otimes \rho_B)^{\frac{1-q}{2q}} \rho_{\rm S} (I_4 \otimes \rho_B)^{\frac{1-q}{2q}}$ are evaluated and they are given by

$$\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 4^{\frac{-1}{q}}$$

As $\sum_{i=1}^4 \gamma_i^q = 4\left(4^{\frac{-q}{q}}\right) = 1$, the expression for CSTRE given by $\tilde{D}_q^T(\rho_{\rm S}||\rho_B) = \frac{\sum_i \gamma_i^q - 1}{1-q}$ turns out to be zero for all values of $q > 1$. Thus one can conclude that CSTRE criterion fails to detect entanglement in the bound entangled state $\rho_{\rm S}$.

2. **Horodecki's $3\otimes 3$ bound entangled state**: The two-qutrit state $\rho_{\rm H}(x)$ given by [77]

$$\rho_{\rm H}(x)=\frac{2}{7}\left|\phi\right\rangle\left\langle\phi\right|+\frac{x}{7}\sigma_{+}+\frac{5-x}{7}\sigma_{-},\quad a\in[0,\ 5] \tag{6.9}$$

where

$$\begin{aligned}\left|\phi\right\rangle &= \frac{1}{\sqrt{3}}\sum_{k=0}^{2}\left|kk\right\rangle, \quad \sigma_{+}=\frac{1}{3}\left(\left|01\right\rangle\left\langle01\right|+\left|12\right\rangle\left\langle12\right|+\left|20\right\rangle\left\langle20\right|\right), \\ \sigma_{-} &= \frac{1}{3}\left(\left|10\right\rangle\left\langle10\right|+\left|21\right\rangle\left\langle21\right|+\left|02\right\rangle\left\langle02\right|\right)\end{aligned} \tag{6.10}$$

is found to be bound entangled state [77]. Here, $|0\rangle$, $|1\rangle$, $|2\rangle$ form the basis vectors of the qutrit space. The distinct non-zero eigenvalues of the state $\rho_{\rm H}(x)$ are given by

$$\lambda_1=\frac{2}{7},\ \lambda_2=\lambda_3=\lambda_4=\frac{5-x}{21}\quad\text{and}\quad\lambda_5=\lambda_6=\lambda_7=\frac{x}{21}$$

Both the single qutrit reduced states of $\rho_{\rm H}(x)$ are seen to be $I_3/3$ and hence are maximally mixed. It thus appears that the CSTRE criterion may not be able to detect the bound entanglement in $\rho_{\rm H}$ owing to the maximally mixed and hence commuting nature of the single qutrit marginals with $\rho_{\rm H}(x)$.

On evaluating the eigenvalues of the sandwiched matrix $\Gamma=(I_3\otimes\rho_B)^{\frac{1-q}{2q}}\ \rho_{\rm H}(x)\ (I_3\otimes\rho_B)^{\frac{1-q}{2q}}$, one gets,

$$\begin{aligned}\gamma_1 &= \frac{2}{7}\ 3^{\frac{-1+q}{q}} \\ \gamma_2 &= \frac{(x-5)}{7}\ 3^{\frac{-1}{q}}\quad\text{(3-fold degenerate)} \\ \gamma_3 &= \frac{x}{7}\ 3^{\frac{-1}{q}}\quad\text{(3-fold degenerate)}\end{aligned} \tag{6.11}$$

One can now readily evaluate the expression for CSTRE given by

$$\tilde{D}_q^T(\rho_{\rm H}(x)||\rho_B) = \frac{\sum_i \gamma_i^q - 1}{1-q}$$

The Fig. 6.2 illustrates the variation of $\tilde{D}_q^T(\rho_{\rm H}(x)||\rho_B)$ with respect to x for different values of q. From Fig. 6.2 one can observe that $\tilde{D}_q^T(\rho_{\rm H}(x)||\rho_B)$ remains non-negative for each value of q. Owing to the fact that $\rho_A = \rho_B = I_3/3$, one gets $\tilde{D}_q^T(\rho_{\rm H}||\rho_A) = \tilde{D}_q^T(\rho_{\rm H}||\rho_B) \geq 0$ for all $q \geq 1$. Thus in the case of $\rho_{\rm H}(x)$, the CSTRE is unable to detect the bound entanglement.

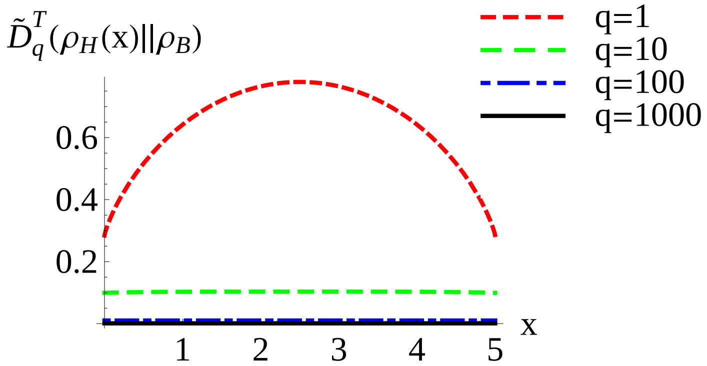

FIGURE 6.2: The CSTRE $\tilde{D}_q^T(\rho_{\rm H}(x)||\rho_B)$ as a function of x for different value of q in the state $\rho_{\rm H}(x)$.

3. **Horodecki's $4 \otimes 4$ bound entangled state**: The $4 \otimes 4$ bound entangled state [78] $\rho_{\rm H4}$ is given by

$$\rho_{\rm H4} = \sum_{i=0}^{3} q_i \left|\psi_i\right\rangle \left\langle\psi_i\right| \otimes \rho^{(i)}$$

where

$$\rho^{(0)} = \frac{1}{2}\left(\left|00\right\rangle\left\langle 00\right| + \left|\psi_2\right\rangle\left\langle\psi_2\right|\right), \;\; \rho^{(1)} = \frac{1}{2}\left(\left|11\right\rangle\left\langle 11\right| + \left|\psi_3\right\rangle\left\langle\psi_3\right|\right), \;\; \rho^{(2/3)} = \left|\chi_\pm\right\rangle\left\langle\chi_\pm\right|$$

with

$$\left|\psi_{0/1}\right\rangle = \frac{(\left|00\right\rangle \pm \left|11\right\rangle)}{\sqrt{2}}, \quad \left|\psi_{2/3}\right\rangle = \frac{(\left|01\right\rangle \pm \left|10\right\rangle)}{\sqrt{2}}, \quad \left|\chi_\pm\right\rangle = \frac{(\left|00\right\rangle \pm \left|\psi_0.\right\rangle)}{\sqrt{2 \pm \sqrt{2}}},$$

The mixing parameters q_i, $i = 0, 1, 2, 3$ are given by

$$\left\{\frac{p}{2}, \frac{p}{2}, \frac{(1-p)}{2}, \frac{(1-p)}{2}\right\} \quad \text{with} \;\; p = \frac{\sqrt{2}}{\left(1+\sqrt{2}\right)}.$$

On substituting the value of p, the distinct non-zero eigenvalues of the state ρ_{H4} are

$$\lambda_1 = \lambda_2 = 0.2071 \quad \text{and} \quad \lambda_3 = \lambda_4 = \lambda_5 = 0.1464$$

The marginal ρ_B of ρ_{H4} is seen to be

$$\rho_B = \mathrm{Tr}_1\, \rho_{\mathrm{H4}} = \begin{pmatrix} 0.3535 & 0 & 0 & 0 \\ 0 & 0.1464 & 0 & 0 \\ 0 & 0 & 0.1464 & 0 \\ 0 & 0 & 0 & 0.3535 \end{pmatrix} \tag{6.12}$$

The eigenvalues of the sandwiched matrix $\Gamma = (I_4 \otimes \rho_B)^{\frac{1-q}{2q}}\ \rho_{\mathrm{H4}}\ (I_4 \otimes \rho_B)^{\frac{1-q}{2q}}$ are given by

$$\begin{aligned} \gamma_1 &= \left(\sqrt{2}-1\right)\ 2^{\frac{-3}{2q}} \qquad \text{(2-fold degenerate)} \\ \gamma_2 &= \left(1-\sqrt{2}\right)\ 2^{\frac{-3}{2q}} \qquad \text{(2-fold degenerate)} \\ \gamma_3 &= \left(2\left(1-\sqrt{2}\right)\right)^{\frac{-1}{q}} \qquad \text{(2-fold degenerate)} \end{aligned} \tag{6.13}$$

On evaluating the expression $\tilde{D}_q^T(\rho_{\mathrm{H4}}||\rho_B) = \frac{\sum_i \gamma_i^q - 1}{1-q}$ for CSTRE, it can be seen that $\tilde{D}_q^T(\rho_{\mathrm{H4}}||\rho_B)$ remains non-negative for each values of q, thus being unable to detect the bound entanglement in ρ_{H4}.

In order to evaluate the CSTRE $\tilde{D}_q^T(\rho_{\mathrm{H4}}||\rho_A)$ with respect to the first subsystem ρ_A, it is noticed that

$$\rho_A = \mathrm{Tr}_2\, \rho_{\mathrm{H4}} = \begin{pmatrix} 0.2928 & 0 & 0 & 0 \\ 0 & 0.2071 & 0 & 0 \\ 0 & 0 & 0.2071 & 0 \\ 0 & 0 & 0 & 0.2928 \end{pmatrix} \tag{6.14}$$

The eigenvalues of the sandwiched matrix $\Gamma = (\rho_A \otimes I_4)^{\frac{1-q}{2q}}\ \rho_{\mathrm{H4}}\ (\rho_A \otimes I_4)^{\frac{1-q}{2q}}$ are given by

$$\begin{aligned}\gamma_1 &= \frac{1}{2}\ \left(1-\frac{1}{\sqrt{2}}\right)^{\frac{1}{q}} \qquad \text{(4-fold degenerate)}\\ \gamma_2 &= \left(\frac{1}{\sqrt{2}}-\frac{1}{2}\right)^{\frac{1}{q}} \qquad \text{(4-fold degenerate)}\end{aligned} \tag{6.15}$$

On explicit evaluation of the CSTRE $\tilde{D}_q^T(\rho_{\mathrm{H4}}||\rho_A) = \frac{\sum_i \gamma_i^q - 1}{1-q}$, it is observed that $\tilde{D}_q^T(\rho_{\mathrm{H4}}||\rho_A)$ also remains non-negative for each values of q quite similar to its second subsystem counterpart $\tilde{D}_q^T(\rho_{\mathrm{H4}}||\rho_B)$. Thus in the case of ρ_{H4}, irrespective of the marginal under consideration, the CSTRE is unable to detect the bound entanglement.

4. **The two qutrit bound entangled state**: The two qutrit bound entangled state [23] is defined by

$$\rho_a = \frac{1}{1+8a}\begin{pmatrix} a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ a & 0 & 0 & 0 & 0 & a & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{(1+a)}{2} & 0 & \frac{\sqrt{1-a^2}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ a & 0 & 0 & 0 & a & 0 & \frac{\sqrt{1-a^2}}{2} & 0 & \frac{(1+a)}{2} \end{pmatrix}, \quad 0 \le a \le 1$$

The distinct non-zero eigenvalues of the state ρ_a are given by

$$\begin{aligned}\lambda_1 &= \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \frac{a}{1+8a} \quad \text{and}\\ \lambda_{6/7} &= \frac{1 + a\,(11+24a) \pm (1+8a)\sqrt{7a^2-4a+1}}{2\,(1+8a)^2}\end{aligned}$$

The first qutrit marginal $\rho_A = \mathrm{Tr}_2\, \rho_a$ of the state ρ_a is seen to be diagonal with diagonal elements (eigenvalues ρ_A)

$$\eta_1 = \eta_2 = \frac{3a}{1+8a}, \quad \eta_3 = \frac{1+2a}{1+8a}$$

One can evaluate the eigenvalues γ_i of sandwiched matrix $\Gamma = (\rho_A \otimes I_3)^{\frac{1-q}{2q}}\, \rho_a\, (\rho_A \otimes I_3)^{\frac{1-q}{2q}}$. The variation of $\tilde{D}_q^T(\rho_a||\rho_A) = \frac{\sum_i \gamma_i^q - 1}{1-q}$ as a function of x for different values of q is shown in the Fig.6.3.It is observed that $\tilde{D}_q^T(\rho_a||\rho_A)$ remains non-negative for all values of $q \geq 1$. Thus in the case of ρ_a by considering the first qutrit marginal the CSTRE is unable to detect the bound entanglement.

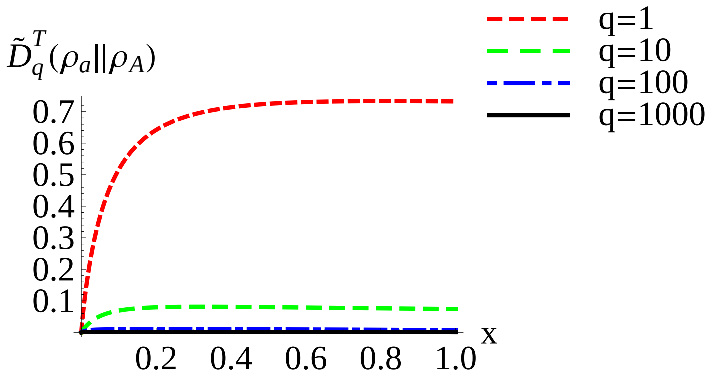

FIGURE 6.3: The CSTRE $\tilde{D}_q^T(\rho_a||\rho_A)$ as a function of x for different value of q in the state ρ_a.

To check whether the CSTRE detects the bound entanglement by considering second qutrit marginal, the marginal ρ_B of the state ρ_a is evaluated and it is given by

$$\rho_B = \mathrm{Tr}_1\, \rho_a = \begin{pmatrix} \frac{1+5a}{2+16a} & 0 & \frac{\sqrt{1-a^2}}{2+16a} \\ 0 & \frac{3a}{1+8a} & 0 \\ \frac{\sqrt{1-a^2}}{2+16a} & 0 & \frac{1+5a}{2+16a} \end{pmatrix} \tag{6.16}$$

The eigenvalues of ρ_B are found to be

$$\lambda_1 = \frac{3a}{1+8a}, \quad \lambda_{2/3} = \frac{1 + a(13+40a) \pm (1+8a)\sqrt{(1-a^2)}}{2(1+8a)^2}$$

The unitary matrix that diagonalizes the marginal ρ_B is given by

$$U_B = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{pmatrix}$$

One can evaluate the eigenvalues γ_i of sandwiched matrix $\Gamma = (I_3 \otimes \rho_B)^{\frac{1-q}{2q}}\, \rho_a\, (I_3 \otimes \rho_B)^{\frac{1-q}{2q}}$ through the eigenvalues of the unitarily equivalent matrix $\Gamma_U = (I_3 \otimes U_B)\Gamma(I_3 \otimes U_B)^\dagger$. The variation of $\tilde{D}_q^T(\rho_a||\rho_B) = \frac{\sum_i \gamma_i^q - 1}{1-q}$ as a function of x for different values of q is shown in the Fig.6.4. It is observed that $\tilde{D}_q^T(\rho_a||\rho_B)$ remains non-negative for all values of $q \geq 1$. Thus in the case of ρ_a, the CSTRE is unable to detect the bound entanglement.

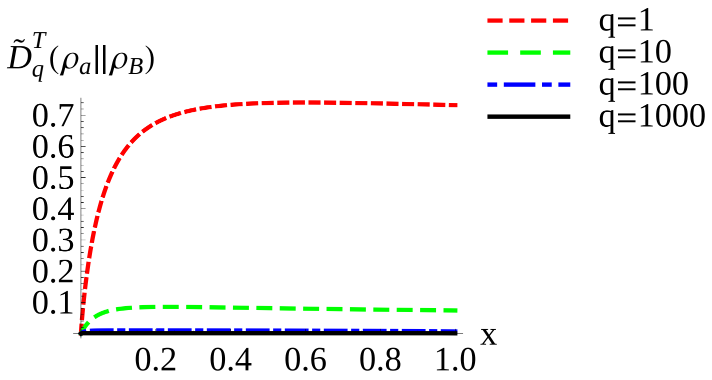

FIGURE 6.4: The CSTRE $\tilde{D}_q^T(\rho_a||\rho_B)$ as a function of x for different value of q in the state ρ_a.

5. **Another two-qutrit bound entangled state** [79]: A two qutrit state ρ_{AB} defined by [79]

$$\rho_{AB} = \sum_{i=1}^{4} \lambda_i \left|\varphi_i\right\rangle \left\langle\varphi_i\right|,$$

with

$$\lambda_1 = \frac{3257}{6884}, \quad \lambda_2 = \lambda_3 = \frac{450}{1721}, \quad \lambda_3 = \frac{27}{6884}$$

and

$$
\begin{aligned}
|\varphi_1\rangle &= \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right), \quad |\varphi_2\rangle = \frac{a}{12}\left(|01\rangle + |10\rangle\right) + \frac{1}{60}\left(|02\rangle\right) - \frac{3}{10}\left(|21\rangle\right), \\
|\varphi_3\rangle &= \frac{a}{12}\left(|00\rangle - |11\rangle\right) + \frac{1}{60}\left(|12\rangle\right) + \frac{3}{10}\left(|20\rangle\right), \\
|\varphi_4\rangle &= \frac{1}{\sqrt{3}}\left(-|01\rangle + |10\rangle + |22\rangle\right), \quad a = \sqrt{\frac{131}{2}}
\end{aligned}
\tag{6.17}
$$

is shown to be a bound entangled state [79].

The eigenvalues of the state ρ_{AB} are given by

$$\lambda_1 = \frac{3257}{6884}, \quad \lambda_2 = \lambda_3 = \frac{450}{1721}, \quad \lambda_4 = \frac{27}{6884}$$

The first qutrit marginal $\rho_A = \mathrm{Tr}_B\, \rho_{AB}$ of ρ_{AB} is diagonal with eigenvalues

$$\eta_1 = \eta_2 = \frac{6551}{13768}, \quad \eta_3 = \frac{333}{6884}$$

On obtaining the eigenvalues γ_i of the sandwiched matrix $\Gamma = (\rho_A \otimes I_3)^{\frac{1-q}{2q}}\, \rho_{AB}\, (\rho_A \otimes I_3)^{\frac{1-q}{2q}}$, the expression for CSTRE given by $\tilde{D}_q^T(\rho_{AB}||\rho_A) = \frac{\sum_i \gamma_i^q - 1}{1-q}$ can be evaluated. It is observed that $\tilde{D}_q^T(\rho_{AB}||\rho_A)$ remains non-negative for each values of q. Thus in the case of ρ_{AB}, by considering the first qutrit marginal the CSTRE is unable to detect bound entanglement.

The second qutrit marginal $\rho_B = \mathrm{Tr}_A\, \rho_{AB}$ of ρ_{AB} is diagonal with eigenvalues

$$\eta_1 = \eta_2 = \frac{3437}{6884}, \quad \eta_3 = \frac{10}{6884}$$

Explicit evaluation of the eigenvalues γ_i of the sandwiched matrix $\Gamma = (I_3 \otimes \rho_B)^{\frac{1-q}{2q}}\, \rho_{AB}\, (I_3 \otimes \rho_B)^{\frac{1-q}{2q}}$, lead to the evaluation of CSTRE $\tilde{D}_q^T(\rho_{AB}||\rho_B) = \frac{\sum_i \gamma_i^q - 1}{1-q}$ and it is seen that $\tilde{D}_q^T(\rho_{AB}||\rho_B) \geq 0$ for all $q \geq 1$. Thus in this case also, the CSTRE is unable to detect bound entanglement in ρ_{AB}.

6. $4 \otimes 4$ **bound entangled state of Bennati et.al.** [80]: A $4 \otimes 4$ bound entangled state $\rho_{4\otimes 4}$ is explicitly given by [80]

$$\rho_{4\otimes 4} = \frac{1}{24}\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\
0 & 3 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 3 & 0 \\
1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

There exist only one distinct, 6-fold degenerate non-zero eigenvalue $\lambda = \frac{1}{6}$ for the state $\rho_{4\otimes 4}$.

Both the subsystem density matrices ρ_A, ρ_B of $\rho_{4\otimes 4}$ are maximally mixed and are given by $I_4/4$. On evaluating the expression for CSTRE $\tilde{D}_q^T(\rho_{4\otimes 4}||\rho_B) = \frac{\sum_i \gamma_i^q - 1}{1-q}$ where γ_i are the eigenvalues of the sandwiched matrix $\Gamma = (I_4 \otimes \rho_B)^{\frac{1-q}{2q}}\ \rho_{4\otimes 4}\ (I_4 \otimes \rho_B)^{\frac{1-q}{2q}}$, it is observed that $\tilde{D}_q^T(\rho_{4\otimes 4}||\rho_B)$ remains non-negative for all values of $q \geq 1$. Thus in the case of $\rho_{4\otimes 4}$ the CSTRE is unable to detect the entanglement.

6.3 Summary

The features of the CSTRE criterion which can characterize entanglement in those states where the knowledge of the local and global spectra is unable to do so is outlined in this chapter. The non-spectral feature of the CSTRE criterion is illustrated using a pair of isospectral states one of which is entangled and the other is separable. While the von-Neumann conditional entropy and AR q-conditional entropy are unable to characterize entanglement in the entangled isospectral state, the conditional sandwiched relative entropy is shown to be negative for the state for all values of $q > 1$. Using the result that only non-spectral witnesses are able to identify the hidden entanglement in bound entangled states, an attempt is done to check whether CSTRE criterion can detect entanglement in bound entangled states. Several well-known bound entangled states are subjected to the CSTRE criterion but all the states considered showed non-negative CSTRE thus not revealing their hidden entanglement through CSTRE criterion. It is therefore concluded that the amount of non-spectrality in the CSTRE criterion is insufficient to identify entanglement in bound entangled states. There is also a possibility that at least some bound entangled state, not examined here, may reveal its entanglement through CSTRE criterion and that remains an open problem.

* * * * * * * * * * * * *

Chapter 7

Conclusion

This concluding chapter summarizes the results in the thesis and gives a lead to the possible open problems.

7.1 Summary of the Thesis

Chapter 1 The Introductory chapter of the thesis reviewed the important concepts essential to understand the research work detailed in the subsequent chapters of the thesis. The concept of Quantum entanglement through its presently accepted definition is outlined. This includes the definition of entangled pure as well as mixed states of a bipartite system. This is followed by the characterization of entangled states through the negativity of conditional entropies. An overview of the literature on entropic characterization of separability/entanglement and the context of the present work is also given. The outline of the contents of thesis is given towards the end of the Introductory Chapter.

Chapter 2 This Chapter begins with an introduction to the non-commuting generalization of Tsallis relative entropy analogous to the non-commuting (sandwiched) version of Rényi relative entropy [56, 57]. A conditional version of the sandwiched Tsallis relative entropy is defined (see Eq. 2.5) and its commuting counterpart is recognized to be the Abe-Rajagopal q-conditional entropy [62]. It is shown that the negativity of CSTRE is sufficient to establish that the state is entangled [63]. Using this result, separability range in all possible bipartitions of symmetric one-parameter family involving 3-, 4-qubit W-, GHZ [49, 50] and W$\bar{\text{W}}$ states is evaluated using CSTRE criterion [62, 63]. It is observed that for all the states considered, the 1 : 2, 1 : 3 CSTRE separability ranges match with the corresponding PPT separability ranges but in the 2 : 2 partition of the symmetric one parameter family with 4-qubit W, GHZ, W$\bar{\text{W}}$ states, PPT criterion provides a stricter separability range than the CSTRE criterion. For the one-parameter family involving GHZ states, 1 : 2, 1 : 3 separability ranges obtained using AR-, CSTRE- and PPT criteria match with one another. The equivalence of the separability ranges due to AR- and CSTRE-criterion is attributed to the maximally mixed and hence commuting nature of the single qubit marginal for the family involving GHZ states. Similar situation, that is, equivalence of AR-, CSTRE- separability ranges owing to the maximally mixed single qubit marginal occur for the 1 : 3 partition of the family involving 4 qubit W$\bar{\text{W}}$ states [63]. The symmetric one parameter family of states involving 3-qubit W$\bar{\text{W}}$ are found to behave differently by not having a maximally mixed single qubit marginal and the CSTRE providing a stricter 1 : 2 separability range compared to the one due to AR-criterion. For the one-parameter family of states involving W states also, the 1 : 2, 1 : 3 CSTRE separability ranges are stricter than that due to AR-criterion and here too the the single qubit marginal is not maximally mixed hence not commuting with the global state. Thus, it is concluded that whenever the subsystem density matrix under consideration is not maximally mixed thus not commuting with the global state, CSTRE criterion fares better than its commuting counterpart, the AR-criterion.

Chapter 3 In Chapter 3, using the relevant results of Chapter 2 and obtaining the 1 : 4, 1 : 5 separability ranges using CSTRE criterion, the $1 : N-1$ separability range in the symmetric one parameter family involving W, GHZ and W$\bar{\text{W}}$ states are obtained [63]. It is clearly established, through the stricter separability range obtained using CSTRE criterion, that CSTRE criterion is superior to AR-criterion in the case of the family containing W states. For the symmetric one parameter family of states containing GHZ states, the CSTRE- AR- and PPT criteria are found to be equivalent in discerning the

$1 : N-1$ separability range in this family with $N \geq 3$. The equivalence of AR-, CSTRE- and PPT criteria is also seen while obtaining the $1 : N-1$ separability range in the symmetric one parameter family of states involving N-qubit W$\bar{\text{W}}$ states with $N \geq 4$ [63]. The utility of CSTRE criterion in identifying bipartite separability/entanglement in one parameter families of qubit-qutrit and qutrit-qutrit states is also illustrated in this chapter [63]. It has been found that one can use the positivity of conditional version of sandwiched relative R*é*nyi entropy (CSRRE) as a sufficient criterion for separability in symmetric one parameter families of states containing W, GHZ and W$\bar{\text{W}}$ giving results equivalent to the ones obtained using the positivity of CSTRE [63].

Chapter 4 In chapter 4, $1 : N-1$ separability range in two non-symmetric one parameter families of noisy states involving W, GHZ states are obtained using CSTRE criterion [72]. The two non-symmetric one parameter family of noisy states considered are

1. N-qubit Pseudopure (PP) noisy states containing either W or GHZ states as the pure part of the noisy state [69]
2. N-qubit generalization of Werner states containing either W or GHZ states as the pure part of the noisy state [19]

For both families of states, when the pure part of the noisy mixture is an N-qubit W state, the CSTRE criterion is seen to provide stricter $1 : N-1$ separability range than that due to its commutative version, the AR-criterion [72]. When the pure part of the noisy mixture in both the families (PP and Werner-like) is an N-qubit GHZ state, the $1 : N-1$ separability ranges obtained using CSTRE and AR-criteria match with one another. For all the states considered namely PP family with W/GHZ states and Werner-like family with W/GHZ states, the $1 : N-1$ CSTRE separability range is seen to match with the corresponding PPT separability range. It is illustrated that the matching of AR- and CSTRE criterion for both the families involving GHZ states is due to the maximally mixed nature of the single qubit marginal. For all the states under consideration, the CSTRE separability range is shown to be necessary and sufficient for separability in the $1 : N - 1$ partition [72].

Chapter 5 Chapter 5 illustrates the utility of CSTRE criterion in identifying the $1 : N - 1$ separability range in noisy one parameter family of N-qudits with its pure part constituted by the N-qudit generalization of the GHZ state [74]. It is established that the $1 : N-1$ CSTRE separability range with the threshold value of x (above which the state is entangled) being a function of both d, N is equivalent to the one obtained using AR-criterion. The reason for this equivalence is shown to be due to the maximally mixed nature of the single qudit marginal of this so-called Werner-Popescu state [19, 20, 35]. In spite of the fact that the $1 : N - 1$ separability ranges in both AR- and CSTRE criteria are equivalent, the mode of convergence of the parameter x in the implicit plot of CSTRE (in the limit $q \to \infty$) is shown to be smoother and hence more stochastic than in the case of AR q conditional entropy [74].

Chapter 6 Chapter 6 illustrates a very peculiar feature of the CSTRE criterion which is not found in other entropic separability criteria using von-Neumann conditional entropy or q-conditional entropies. This feature is the non-spectral nature of the CSTRE criterion which indicates that negativity of CSTRE can detect entanglement that is not detected by criteria entirely dependent on the global and local spectra (eigenvalues of the given composite state and its subsystem density matrix) [62]. The non-spectral nature of the CSTRE criterion is established through examining two isospectral states [43] one of which is entangled and the other is separable. Though none of the spectral separability criteria including the entropic ones could detect which among the isospectral states is entangled, CSTRE is seen to be negative for all values of $q > 1$ when the state is entangled. Motivated by the non-spectral nature of the CSTRE criterion and observing the hypothesis [75] that only non-spectral witnesses can detect bound entanglement in the so-called bound entangled (or entangled states with Positive Partial Transpose) states [23, 24], a honest effort has been put to identify entanglement in the bound entangled states through CSTRE criterion. But the efforts towards that end has not yet been successful. A detailed account of the efforts, using CSTRE criterion, to detect bound entanglement in some well-known bound entangled states is given in Chapter 6.

7.2 Future Directions. . .

While the $1 : N-1$ separability range of one parameter families of N-qubit states using CSTRE criterion matched exactly with that obtained through PPT criterion, similar conclusion cannot be drawn about separability ranges of other bipartitions such as $2 : N-2$. For instance, the $2 : 2$ separability range of $\rho_4^{\rm W}(x)$ using CSTRE criterion is found to be [62] $(0,\ 0.2105)$ but PPT criterion yielded $(0,\ 0.0808)$ as the $2 : 2$ separability range of $\rho_4^{\rm W}(x)$. Similarly, for $\rho_4^{\rm GHZ}(x)$ the $2 : 2$ separability ranges obtained through CSTRE and PPT criterion are found to be $(0,\ 0.2105), (0,\ 0.0625)$ respectively. But an investigation into bipartitions such as $3 : N-3$ and so on may yield results which are either identical, weaker or stricter than that through PPT criterion. It is also to be noted here that PPT and CSTRE separability criteria are of entirely different origins and there is no reason to expect that the separability ranges obtained through them match with each other in all bipartitions of an N-qubit state. It is indeed surprising that the $1 : N-1$ separability ranges of one parameter families of states that are investigated here, obtained using PPT and CSTRE criterion, matched exactly with each other whereas the separability ranges in other bipartitions may not do so. Having seen that CSTRE criterion provides separability ranges either identical or weaker in comparison with PPT criterion, whether it can provide a separability range stricter than that through PPT criterion in at least some bipartitions of an N-qubit state still remains an open question.

It would be of interest to examine completely random multi-qubit states and analyze the separability ranges in different bipartitions, obtained through CSTRE and PPT criteria. While a numerical investigation on the set of all bipartite mixed states has revealed [38] that PPT criterion is superior to AR criterion, a comprehensive numerical survey on the PPT as well as CSTRE separability ranges in different bipartitions of random states is warranted in order to identify the hierarchy between CSTRE and PPT criteria. Such a numerical survey, on the same lines as in Ref. [38, 39], will also help in strengthening the results of the research work carried out in this thesis.

It is to be recalled here that the conditional version of sandwiched Rényi relative entropy is as helpful as its Tsallis entropy counterpart, i.e., the CSTRE in identifying whether a given mixed state is entangled or not (See Sec. 3.4.2). At this juncture it is of importance to notice that a conditional version of Rényi relative entropy is defined in Ref. [81] by maximizing over the marginal state ρ_B. While the results obtainable through such a maximization over ρ_B are of interest, in view of the fact that it is operationally difficult to identify ρ_B that maximizes the conditional entropy (either Rényi or Tsallis), the analysis here is restricted to the case of the actual marginal ρ_B of the bipartite state ρ_{AB}. It would be interesting to examine the consequences of maximization over marginals in the detection of entanglement using conditional generalized entropies (Rényi or Tsallis) and at present it remains an open problem.

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