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A STUDY ON CONFORMAL CHANGES IN SPECIAL FINSLER SPACES

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by

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Febraury-2018

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Dedicated to

Beloved mother

DECLARATION

I hereby declare that the thesis entitled "A Study on Conformal changes in special Finsler spaces", submitted to the faculty of Science and Technology, Kuvempu University for the award of Doctor of Philosophy degree in Mathematics is the result of the research work carried out by me in the Department of Mathematics, Kuvempu University under the guidance of Dr. S.K. Narasimhamurthy, Professor and Chairman, Department of P.G. Studies and Research in Mathematics, Kuvempu University, Jnana Sahyadri.

I further declare that this thesis or part thereof has not been previously formed the basis of the award of any degree, associationship etc., of any other university or institution.

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Certificate

*This is to certify that the thesis entitled **A Study on Conformal Changes in Special Finsler Spaces,** submitted to the Faculty of Science and Technology, Kuvempu University for the award of **Doctor of Philosophy** degree in Mathematics by **Thippeswamy K.R.** is the result of bonafide research work carried out by him under my guidance in the Department of P.G. Studies and Research in Mathematics, Kuvempu University, Jnana Sahyadri.*

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Finally, with great pleasure I take this opportunity to express my deep sense of gratitude and heart-felt thanks to several individuals and institutions from whom I received impetus, motivation and invaluable help during the course of my Ph.D work.

Thippeswamy K.R

Contents

PREFACE

Preface

The study of spaces endowed with generalized metrics was initiated by P.Finsler in 1918. It is usualy considerd as a generalization of the Riemannian geometry. In fact B. Riemannian his lecture in 1854 already suggested a possibility of studying more general geometry than Riemannian geometry. But he said the geometrical meanings of quantities appearing in such a genaralized space will not be clear and it can not produce any contribution to the geometry consequently all people had neglected for about 60 years to study such a geometry. Finsler started the study of such a geometry from the stand point of a geometrization of the variation caluclus.

Subsequently, due to investigations by J. Synge, V. Wagner, L. Berwald, E. Cartan, H. Rund, M. Matsumato and others, Finsler geometry becomes a separate branch of differential geometry. In modern implementation classical Finsler geometry represents a geometry of vectors fibre bundles over manifolds.

Finsler geometry is a Riemannian geometry without the quadratic restriction. It also asserts itself in the applications, most notably in theory of relativity, control theory and mathematical biology.

Conformal change is one of the important transformation which preserves the angle. The theory of conformal changes in Riemanian geometry has been deeply studied locally and intrinsically. As regards to Finsler geometry an almost complete local theory of conformal changes has been established.

M.S. Knebelman first defined the conformal theory of Finsler metrics, such that two metric functions L and $\overline{L}$ are conformal if the length of an arbitary vector in the space with the metric L is proportional to the length in the space with the metric $\overline{L}$.

The detail study of the conformal theory was carried out by the following authors: M. Hashinguchi (1976) given a special change named C-change, which is non homothetic conformal change, satisfying C-condition. C.Shibata and M. Azuma (1993) have investigated C-Conformal change invariant tensor of Finsler metric. The author S. Kikuchi (1998) give the condition for a finsler space to be conformally flat. H. Izumi gave the condition for a Finsler space to be h-conformally flat. S.H. Abed (2006) introduce the conformally β-change.

The whole work represented in the thesis has been partitioned into six chapters.

The **First chapter** includes basic concepts of Finsler space and notations. It also includes the basic concepts of Special Finsler spaces, Conformal change, Conformal β-change, Nonholonomic frame, Weekly Berwald space, Randers change, Killing vector field, Finslerian subspaces.

The **Second chapter** deals with the Conformal change of douglas space with (α, β)-metrics. The conformal theory of Finsler metrics based on the theory of Finsler spaces by M. Matsomoto, M. Hashiguchi in 1976 studied the conformal change of a Finsler metric namely $\overline{L}(x, y) = e^{\sigma}(x)L(x, y)$[1]. The concept of Douglas space has been introducing by M. Matsumoto and S. Bacso as a generalization of Berward spaces from stand point of view of geodesic equation. Finsler space is said to be of Douglas space if $D^{ij} = G^i y^j - G^j y^i$ are homogeneous polynomial of degree three in y^i. It is remarkable that a Finsler space is a Douglas space if and only if the Douglas tensor vanishes identically. In this chapter, we proved the following results:

1. A Finsler space with second approximate matsumato metric (α,β)-metric $L=\alpha+\beta+\frac{\beta^2}{\alpha}+\frac{\beta^3}{\alpha^2}$ is a Douglas space if and only if

$$i)\qquad \alpha^2\neq 0(mod\beta),\ b^2\neq\frac{1}{k}:b_{i;j}\ is\ written\ in\ the\ form$$

$$ii)\qquad \alpha^2=0(mod\beta):n=2\ and\ b_{i;j}\ is\ written\ in\ the\ form$$

$$where\ \alpha^2=\beta\delta,\ \delta=d_i(x)y^i,\ v_o=v_i(x)y^i.$$

2. $\alpha^2\neq 0(mod\beta)$, then the Douglas space with second approximate matsumato metric $L=\alpha+\beta+\frac{\beta^2}{\alpha}+\frac{\beta^3}{\alpha^2}$ is conformally transformed to a Douglas space if and only if transformation is homothetic.

3. A Finsler space $\overline{F}^n(n>2)$ which is obtained by a β-conformal change of Finsler space F^n with an special (α,β)-metric $L=\alpha-\frac{\beta^2}{\alpha}+\beta$ and $L=\alpha+\epsilon\beta+k\frac{\beta^2}{\alpha}(b^2\neq 1)$ of Douglas type, is also Douglas space.

4. A Finsler space $\overline{F}^n$ $(n>2)$ which is obtained by conformal Kropina change of a Kropina space F^n with (α,β)-metric $L=\frac{\alpha^2}{\alpha-\beta}$ is of Douglas type if and only if $M_{ij}(x)=n(b_j\sigma_i-b_i\sigma_j)$ is satisfied.

5. A Finsler space $\overline{F}^n$ $(n>2)$ which is obtained by conformal Kropina change of a Kropina space F^n with (α,β)-metric $L=\alpha-\frac{\beta^2}{\alpha}+\beta(b^2\neq 0)$ and $L=\sqrt{2\alpha\beta}$ is of Douglas type if and only if $s_{ij}=\frac{1}{b^2}(b_is_j-b_js_i)$ is satisfied.

The **Third chapter** deals with the Weakly Berwald Finsler spaces and scalar flag curvature. In 2004, R.Yoshikawa, Okubo and M.Matsumoto obtained the necessary and sufficient conditions for some (α,β)-metric spaces to be Weakly-Berwald spaces. C.Ninwei worked on a class of (α, β)-metrics to be Weakly-Berwald[26]. In 2009, X.Cheng has worked on (α, β)-metrics of scalar flag curvature with constant S-curvature[13]. In this chapter, we proved the following results:

1. The two (α, β)-metrics, $F = \frac{(\alpha+\beta)^2}{\alpha}$ and $F = \sqrt{c_1\alpha^2 + c_2\alpha\beta + c_3\beta^2}$ (where c_1, c_2 and c_3 are constants) are of non-Randers type if $\Phi \neq 0$(i.e, non-Riemannian).

2. Finsler space with (α, β)-metrics, $F = \frac{(\alpha+\beta)^2}{\alpha}$ and $F = \sqrt{c_1\alpha^2 + c_2\alpha\beta + c_3\beta^2}$ (where c_1, c_2 and c_3 are constants) are of scalar flag curvature $\mathbf{K} = \mathbf{K}(x, y)$. Then these metrics are weak Berwald metrics if and only if such metrics are Berwald metrics and $\mathbf{K} = 0$. In this case, Finsler metrics must be locally Minkowskian.

3. On an n-dimensional manifold M, special (α, β)- metric $F = \frac{\alpha^2}{\alpha-\beta} + \beta$ is weakly-Berwald metric if and only if $r_{ij} = 0$, $s_i = 0$.

4. On an $n-$dimensional manifold M, $F = \frac{\alpha^2}{\alpha-\beta} + \beta$ is weakly-Berwald and holds the following conditions:

   (a) F is of isotropic S-curvature, $S = (n + 1)cF$;

   (b) F is of isotropic mean Berwald curvature, $E = \frac{n+1}{2}cF^{-1}h$;

   (c) β is killing 1-form with $b = constant$ with respect to α, that is , $r_{ij} = 0$, $s_i = 0$;

   (d) S=0;

   (e) F is weakly-Berwald metric i.e., E=0.

Fourth chapter deals with the Nonholonomic frames In 1982, P.R. Holland studies a unified formalism that uses a nonholonomic frame on space-time arising from consideration of a charged particle moving in an external electromagnetic field([6][7]). In 1987 R.S. Ingarden was first to point out that the Lorentz force law can be written in this case as geodesic equation on a Finsler space called Randers space[8]. In 1995, the author R.G.Bail a gauge transformation is viewed as a nonholonomic frame on the tangent bundle of a four dimensional base manifold([9][10]). The geometry that follows from these considerations gives a unified approach to gravitation and gauge sym metries. In the above mentioned papers, the common Finsler idea used by the physicists R.G. Beil and P.R. Holland is the existence of a nonholonomic frame on the vertical subbundle V TM of the tangent bundle of a base manifold M. This nonholonomic frame relates a semi-Riemannian metric (the Minkovski or the Lorentz metric) with an induced Finsler metric. In 2001, P.L. Antonelli and I. Bucataru has been determined such a nonholonomic frame for two important classes of Finsler spaces that are dual in the sense of Randers and Kropina spaces([11][12]).Recently, Ioan Bucataru and Radu Miron has studied Finsler-Lagrange geometry and applications to dynamical systems[13].

Considering the above concepts, we found out the following results :

1. Consider a Finsler space $L^2 = (\frac{\beta^4}{(\beta-\alpha)^2})(\frac{\alpha^2}{\beta}) = \frac{\alpha^2\beta^4}{\beta(\beta-\alpha)^2}$ i.e. product of Infinite series metric and Kropina metric for which the condition $\rho_{-1}\beta + \rho_{-2}\alpha^2$ is true. Then

$$V_j^i = X_k^i Y_j^k$$

is a Finslerian nonholonomic frame with X_k^i and Y_j^k are given by

$$\begin{aligned} X_j^i &= \sqrt{\frac{-\beta^4}{(\alpha-\beta)^3}}\delta_j^i - \frac{1}{\beta^6(4\alpha-\beta)(4\alpha^2b^2-\alpha\beta b^2-3\beta^2)} \times \\ &\left\{\alpha(\alpha-\beta)^8\left[\sqrt{\frac{-\beta}{(\alpha-\beta)^3}} \pm \frac{1}{3}\sqrt{\frac{-9\beta^4}{(\alpha-\beta)^3} + \frac{3\beta^2(4\alpha-\beta)(4\alpha^2b^2-\alpha\beta b^2-3\beta^2)}{(\alpha-\beta)^4}}\right]\right\} \\ &\left(-\frac{\beta^3(4\alpha-\beta)}{(\alpha-\beta)^4}b^i + \frac{3\beta y^i}{\alpha(\alpha-\beta)^4}\right)\cdot\left(-\frac{\beta^3(4\alpha-\beta)}{(\alpha-\beta)^4}b_j + \frac{3\beta^4 y_j}{\alpha(\alpha-\beta)^4}\right). \end{aligned}$$

$$Y_j^i = \delta_j^i - \frac{1}{C^2}\left(1 \pm \sqrt{1 + \frac{C^2\alpha^2\beta}{(2\beta+\alpha)(\alpha^2+\alpha\beta+\beta^2)}}\right)b^i b_j;$$

$$C^2 = -\frac{\beta^4 b^2}{(\alpha-\beta)^3} + \frac{(4\alpha^2 b^2 - \alpha\beta b^2 - 3\beta^2)^2\beta^2}{3\alpha(\alpha-\beta)^4}.$$

2. Consider a Finsler space $(L^3 = c_1\alpha^2\beta + c_2\beta^3)(\frac{\alpha^2}{\beta}) = \frac{\alpha^2(c_1\alpha^2\beta+c_2\beta^3)}{\beta}$ i.e. product of Cube root metric and Kropina metric for which the condition $\rho_{-1}\beta + \rho_{-2}\alpha^2$ is true. Then

$$V_j^i = X_k^i Y_j^k$$

is a Finslerian nonholonomic frame with X_k^i and Y_j^k are given by

$$\begin{aligned} X_j^i &= \sqrt{\frac{2(c_1\alpha^2 + c_2\beta^2)}{\beta^2}}\delta_j^i \\ &- \frac{1}{4}\left[\left\{\frac{\left(\sqrt{2c_1\alpha^2 + c_2\beta^2} \pm \sqrt{\frac{2c_1^2\alpha^2 + c_1c_2\beta^2 + c_2\beta^2(c_2b^2+2c_1)}{c_1}}\right)}{c_2\beta^2(c_2b^2+2c_1)}\right\}\right. \\ &\left.(2c_2\beta b_i + 4c_1 y_i)(2c_2\beta b_j + 4c_1 y_j)\right] \end{aligned}$$

$$Y_j^i = \delta_j^i - \frac{1}{C^2}\left(1 \pm \sqrt{1 - \frac{C^2 c_1}{c_2(c_1\alpha^2 - c_2^2\beta^2)}}\right) b^i b_j$$

$$C^2 = (2c_1\alpha^2 c_2\beta^2)b^2 + \frac{b^2(c_2b^2+2c_1)^2}{c_1}.$$

3. Consider a Finsler space $L = \left(\alpha - \beta + \frac{\beta^2}{\alpha}\right)^2$, for which the condition $\rho_{-1}\beta + \rho_{-2}\alpha^2$ is true. Then

$$V_j^i = X_k^i Y_j^k$$

is a Finslerian nonholonomic frame with X_k^i and Y_j^k are given by

$$X_k^i = \sqrt{\frac{\alpha^2-\alpha\beta+\beta^2)(\alpha^2-\beta)^2}{\alpha^4}}\delta_k^i - \frac{\alpha^4}{(b^2\alpha^2-\beta^2)}$$
$$\left(\sqrt{(\alpha^2-\alpha\beta+\beta^2)(\alpha^2-\beta^2)} \pm \sqrt{(\alpha^2-\alpha\beta+\beta^2)(\alpha^2-\beta^2) - \frac{\alpha^3-3\alpha\beta^2+4\beta^3}{\beta}}\right)$$
$$\left(b_i - \frac{\beta}{\alpha^2}y^i\right)\left(b_k - \frac{\beta}{\alpha^2}y^k\right)$$

$$Y_j^k = \delta_j^k - \frac{1}{C^2}(1 \pm \frac{\sqrt{1+\frac{\alpha^2\beta c^2}{\alpha^3+3\alpha\beta(-\alpha+\beta)-2\beta^3}}^k}{b} b_j,$$

$$C^2 = \frac{b^2(\alpha^6+\alpha\beta+\beta^2)(\alpha^2-\beta^2)}{\alpha^4} - \frac{(\alpha^3-3\alpha\beta^2+4\beta^3)(b^2\alpha^2-\beta^2)^2}{\alpha^6\beta}.$$

In the **Fifth chapter**, we study L-Dually Randers Change of Matsumato metric

The (α,β)-metrics form an important class of Finsler metrics appearing iteratively in formulating Physics, Mechanics and Seismology, Biology, Control Theory, etc, see for instance. This class of metrics is were first introduced by Matsumoto [8]. An (α,β)-metric is a Finsler metric of the form $F := \alpha\phi\frac{\beta}{\alpha}$ where $\phi = \phi(s)$ is a C^∞ on $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. The Randers and Matsumoto metrics are special and significant (α,β)-metrics which constitute a majority of actual research. The Matsumoto and Randers metrics defined by $\phi(s) = \frac{1}{1-s}$ and $\phi(s) = 1+s$, respectively.

On the basis of the above work, we obtained the following results:

1. Let (M, F) be a Randers change of Matsumoto space and $b = (a_{ij}b^ib^j)^{\frac{1}{2}}$ the Riemannian length of b_i. Then if $b^2 = 1$, the L-dual of (M, F) is the space having the fundamental function:

$$H(x,p) = \frac{\frac{\beta^{*2}}{2}\left[1 - \left[\frac{-q}{2} + \sqrt{\frac{q^2}{4}+\frac{p^3}{27}}\right]^{\frac{1}{3}} - \left[\frac{-q}{2} - \sqrt{\frac{q^2}{4}+\frac{p^3}{27}}\right]^{\frac{1}{3}} + \frac{A}{3}\right]^2}{\left[2 + \left[\frac{-q}{2} + \sqrt{\frac{q^2}{4}+\frac{p^3}{27}}\right]^{\frac{1}{3}} + \left[\frac{-q}{2} - \sqrt{\frac{q^2}{4}+\frac{p^3}{27}}\right]^{\frac{1}{3}} - \frac{A}{3}\right]^2}$$

2. Let (M, F) be a Randers change of Matsumoto space and $b = (a_{ij}b^ib^j)^{\frac{1}{2}}$ the Riemannian length of b_i. Then if $b^2 \neq 1$, the L-dual of (M, F) is the space having the fundamental function:

$$H(x,p) = \frac{\frac{\beta^{*2}}{2}\left[1 - \frac{-H_2+\sqrt{H_2^2-4H_1H_3}}{2H_1} + \frac{A_1}{4}\right]^4}{\left[(b^2-2)\left[\frac{-H_2+\sqrt{H_2^2-4H_1H_3}}{2H_1} + \frac{A_1}{4}\right]^2 + 3\left[\frac{-H_2+\sqrt{H_2^2-4H_1H_3}}{2H_1} + \frac{A_1}{4}\right] + (b^2-1)\right]^2}.$$

The **Last chapter** deals with Conformal Change of Finsler Subspaces In 1976, M. Hashiguchi studied the conformal change of Finsler metrics, namely, $\bar{L} = e^{\sigma(x)}L$[1]. In particular, he also dealt with the special conformal transformation named C-conformal transformation. This change has been studied by H. Izumi, V. K. Kropina. In 2008, S. Abed introduced the transformation $\bar{L} = e^{\sigma(x)}L + \beta$, thus generalizing the conformal, Randers and generalized Randers changes. Moreover, he established the relationships between some important tensors associated with (M, L) and the corresponding tensors associated with $(M, \bar{L})$. He also studied some invariant and σ-invariant properties and obtained a relationship between the Cartan connection associated with (M, L) and the transformed Cartan connection associated with $(M, \bar{L})$.

Considering the above concepts, we found out the following results :

1. If a vector field $v^i(x)$ is Killing in F^n and $\bar{F}^n$, then

$$C_{rit}v^tD^r_j + C_{rjt}v^tD^r_i + v_rD^r_{ij} - M^l_{ij}v_{l|0} = 0.$$

2. If a vector field $v^i(x)$ is Killing in F^n and $\bar{F}^n$, then the vector $v_i(x, y)$ is orthogonal to the vector $D^i(x, y)$.

3. Let $b_i(x)$ be parallel with respect to $C\Gamma$ on F^n. Then the subspace F^m is totally geodesic, if and only if the subspace $\bar{F^m}$ is totally geodesic.

4. Let $b_i(x)$ be parallel with respect to $C\Gamma$ on F^n. Then the subspace F^m is totally h-autoparallel, if and only if the subspace $\bar{F^m}$ is totally h-autoparallel.

Finally, the thesis ends with a short list of bibliography.

NOTATIONS & SYMBOLS

Notations and Symbols:

$$
\begin{aligned}
\partial_i &= \frac{\partial}{\partial x^i}, \\
\dot{\partial}_i &= \frac{\partial}{\partial y^i}, \\
M^n &- \; n-dimensional\ manifold, \\
L &- \; Finsler\ metric, \\
g_{ij} &- \; Metric\ tensor, \\
F^n &- \; Finsler\ space, \\
TM &- \; Tangent\ bundle, \\
T_P M &- \; Tangent\ space, \\
h_{ij} &- \; Angular\ metric\ tensor, \\
l_i &- \; Normalized\ element\ of\ support, \\
c_{ijk} &- \; Cartan's\ tensor, \\
\gamma^i_{jk} &- \; Christoffel\ symbol, \\
N^i_j &- \; Non\ linear\ connection, \\
{}_{|} &- \; h-covariant\ derivative\ w.r.t\ Cartan's\ connection, \\
| &- \; v-covariant\ derivative\ w.r.t\ Cartan's\ connection, \\
: &- \; Covariant\ derivative\ w.r.t\ Berwald's\ connection,
\end{aligned}
$$

H^i_{hjk} – *Berwald's curvature tensor*,

S^i_{hjk} – *Cartan's first curvature tensor*,

P^i_{hjk} – *Cartan's second curvature tensor*,

R^i_{hjk} – *Cartan's third curvature tensor*,

G^i_{hjk} – *hv − curvature tensor*,

R^i_{jk} – *h − torsion tensor field*,

α – *Riemannian metric*,

β – *Differential 1 − form*,

W^i_j – *Projective deviation tensor*,

W^i_{jhk} – *Weyl's projective curvature tensor*,

N^i – *Normal vector*,

H_α – *Normal curvature tensor*,

$H_{\alpha\beta}$ – *Second fundamental h − tensor*,

A_{ij} – *Symmetric tensor field*,

P_{ijk} – *Torsion tensor field*,

$K(P, y)$ – *Flag curvature*,

$M_{\alpha\beta}$ – *Second fundamental v − tensor*,

B^i_α – *Projection factor*,

D^i_{hjk} – *Douglas tensor*,

η^{ij} – *Minkowskian metric.*

CHAPTER-1

BASIC CONCEPTS AND PRELIMINARIES

Content of this chapter

Chapter 1

BASIC CONCEPTS AND PRELIMINARIES

1.1 Introduction to Finsler Geometry

The study of spaces endowed with generalized metrics was initiated by P.Finsler in 1918. It is usualy considerd as a generalization of the Riemannian geometry. In fact B. Riemannian his lecture in 1854 already suggested a possibility of studying more general geometry than Riemannian geometry. But he said the geometrical meanings of quantities appearing in such a genaralized space will not be clear and it can not produce any contribution to the geometry consequently all people had neglected for about 60 years to study such a geometry. Finsler started the study of such a geometry from the stand point of a geometrization of the variation caluclus. Subsequently, due to investigations by J. Synge, V. Wagner, L. Berwald, E. Cartan, H. Rund, M. Matsumato and others, Finsler geometry becomes a separate branch of differential geometry. In modern implementation classical Finsler geometry represents a geometry of vectors fibre bundles over manifolds. Finsler geometry is a Riemannian geometry without the quadratic restriction. It also asserts itself in the applications, most notably in theory of relativity, control theory and mathematical biology.

However, it was L. Berwald [84] who first introduced a connection what is now called Berwald connection and some non-Riemannian quantities in his connection and successfully extended the notion of Riemann curvature to Finsler spaces. Since then, Finsler geometry has been developed gradually. From this point of view, Berwald is the founder of differential geometry of Finsler spaces [131]. Further, E. Cartan laid the foundation for Finsler geometry in 1933 by introducing a metrical connection in a view point that a Finsler space is locally Euclidean and since then, important contributions to Finsler geometry have resulted one after another on the analogy of Riemann geometry. Further, Varga, Busemann, Rund and so on have made great contribution to Finsler geometry.

The theory of connections in fiber bundles, which had been developed in 1960s to treat connections in Finsler geometry from more generalized and systematical standpoints. Further, recently there have been an extensive study and advancement of theoretical physics and engineering which need geometric interpretation of structures appearing in these subjects and have stimulated the study of various generalizations of Riemannian geometry.

1.2 Evolution of Finsler geometry

The evolution of Finsler geometry can be divided into four periods.

The first period of the history of Finsler geometry began in 1924, three geometricians were almost simultaneously concerned with such a generalized space. J.H. Taylor and J.L. Synge introduced a special parallelism. But L. Berwald(1883-1942) was the real originator of Finsler geometry. In 1928, Taylor gave the name Finsler space to the space with such generalized metric.

The second period of the history of Finsler geometry began in 1934, Cartan showed that it was indeed possible to define connections and a covariant derivative so that Ricci lemma is preserved. On this basis Cartan developed a theory of curvature and practically all subsequent investigations concerning the geometry of Finsler spaces were dominated by this approach. Several mathematicians expressed the opinion that the theory had thus attained its final form. Further, E. Cartan laid the foundations for Finsler geometry in 1933 by introducing a metrical connection in a viewpoint that a Finsler space is locally Euclidean and since then, important contributions to Finsler geometry have resulted one after another on the analogy of Riemannian geometry.

The third period of the history of Finsler geometry began in 1951 by H. Rund. He introduced a new process of parallelism from the stand point of the so called Minkowski geometry. The $P^i_{hk}(x, \dot{x})$ were first introduced by Rund, but these quantities bear a close relationship to similar coefficients introduced by Cartan. Cartan introduced parallelism from the stand point of Euclidean geometry. The theory of E. Cartan which treats Finsler spaces from an entirely different point of view has played the most predominant role in the development of the subject and in order to do full justice to the methods of Cartan.

The fourth period of the history of Finsler geometry began in 1963, by H. Akbar. He devoloped the Modern theory of Finsler spaces based on the geometry of connections of fiber bundles. The reason of modernization is to establish a global definition of connections in Finsler spaces and to re-examine Cartan's system of axiomes. M. Matsumato came up with his book Foundation of Finsler geometry and special Finsler spaces which drew the attention towards the special Finsler spaces.

1.3 Scope and Applications of Finsler geometry

Finsler geometry was firrst applied in gravitational theory and this application lead to corrections to observational results predicted by general relativity. . The main application of Finsler geometry is the geometrization of electromagnetism and gravitation. A Finslerian approach to this geometrization was first introduced by Randers, but in his work Finsler geometry was not mentioned, although it was used. Randers metric produces a geodesic equation identical with Lorentz equation for a charged particle. But the metric depends on $\frac{q}{m}$ and defines a different space for each type of particle. In 1934, Cartan showed that it was indeed possible to define connection coefficients and a covariant derivative so that Ricci lemma is satisfied. This development is closely related to the present application of Finsler geometry in physics, namely, to geometrize both electromagnetism and gravity simultaneously.

Finsler geometry uses families of Minkowski norms, instead of families of inner products, to describe geometry. This situation is entirely analogous to how Banach spaces relate to Hilbert spaces. There has been a steady modernisation of the field during the past decade. Within the last two years, several areas of Finsler geometry have experienced accelerated growth. These include Finsler spaces of constant curvature, as well as applications of Finsler methods to industrial and medical sectors. Theoretical topics tentatively include Cartan spaces, classification of Berwald spaces, Finsler spaces of constant flag curvature, Finsler-Einstein metrics Finslerian volumes and measures, geodesic flows, Kobayashi metrics, Lagrange spaces, metric geometry, projective invariants, rigidity theorems.

Finsler geometry describes the geometry of a manifold M through tensor fields which live on the tangent bundle TM of the manifold and not on the manifold itself. Rather then considering only the points of M, the tangent bundle consists of the points and all directions of M. The mathematical structure offered by the tangent bundle will be crucial for the review of standard Finsler spaces and especially for our extension of this framework to physical Finsler spacetimes and the analysis of physics on Finsler spacetimes.

The applications of Finsler geometry in physics fall basically into two subjects. On the one hand it appears as an effective geometric description of point particle mechanics, point particle limits of field theories, like ray theory in media, and as a geometric description of fluid mechanics. On the other hand there are attempts to use Finsler geometry as the geometry of spacetime which describes gravity. We will mention the two most prominent Finsler length measures in this context, which include not only a metric, but also a vector field as building ingredients; we will encounter some work where Finsler geometry is used as a phenomenological tool to describe dark matter, dark energy, as well as quantum gravity effects and we discuss approaches to find field equations determining Finsler geometries dynamically. The applications of Finsler geometry as spacetime geometry give rise to a number of questions concerning the equations used to determine the geometry of spacetime, the existence of the geometric objects appearing, the description of observers and the coupling of matter fields. Finsler spacetimes provide a clear notion of causality which is encoded into the geometry, a precise definition of observers and their measurements, field theories coupled to the geometry and gravitational dynamics which determine the geometry of spacetime from its matter field content. The latter is constructed in such a way that, in case the Finsler geometry is identical to metric geometry, one recovers all the standard field theories known from general relativity. In this sense Finsler spacetimes become viable nonmetric geometric backgrounds for physics.

1.4 Geometry of Finsler spaces

The term Finsler space evokes in most mathematicians the picture of an impenetrable forest whose entire vegetation consists of tensors. Finsler spaces were discovered by Riemann in his lecture ber die Hypothesen, welche der Geometrie zu Grunde liegen in 1854. The goal which Riemann set for himself was the definition and discussion of the most general finite-dimensional space in which every curve has a length derived from an infinitesimal length or line element. In modern terminology Riemann's approach is this. Let a differentiate manifold M of a certain class be given. In any local coordinate system $(x_1, ..., x_n) = (x)$ a length $F(x, dx)$ must be assigned to a given line element $(x, dx) = (x_i, ..., x_n; dx, ..., dx_n)$ with origin x. If $x(t)$ is a smooth curve in M then $\int F(x, \dot{x})dt$ is its length. In order to insure that the length of a curve is positive and independent of the sense in which the curve is traversed, Riemann requires $F(x, dx) > 0$ for $dx \neq 0$ and $F(x, dx) = F(x, -dx)$. Next Riemann assumes that the length of the line element remains unchanged except for terms of second order, if all points undergo the same infinitesimal change. This amounts to the condition $F(x, kdx) = kF(x, dx)$ for $k > 0$. Nowadays we rather justify this condition by requiring that a change of the parametrization of the curve does not change its length. Riemann then turns immediately to the special case where

$$F(x, dx) = [\Sigma_{g_{ik}(x)} dx_i dx_k]^{\frac{1}{2}} \tag{1.4.1}$$

that is, to those spaces which are now called Riemann spaces. The general case is passed over with the following remarks the next simplest case would comprise the manifolds, in which the line element can be expressed as the fourth root of a bi-quadratic differential form. The investigation of these more general types would not require any essentially different principles, but it would be time consuming and contribute comparatively little new to the theory of space, because the results cannot be interpreted geometrically.

Here is one of the few instances where Riemann's feeling was wrong. Nevertheless the passage had a great influence: the general case was for a long time entirely neglected, and when it was taken up the principles of Riemannian geometry were applied. The results thus obtained are not different enough to enrich geometry materially, moreover they frequently do not lead themselves to a have geometric interpretation.

1.5 Review of Research and Development

The development of research in Finsler spaces which is carried out by the following national and internationl mathematicians. M.S. Knebelmen (1929)-Conformal geom- etry of generalized metric spaces. Y. Ichijyo and M. Hashinguchi (1940)- On locally flat generalised $(\alpha, \beta) - metric$. H. Rund (1959)- The differential geometry of Finsler spaces. B.N.Prasad B.N Gupta and D.D. Singh (1961) Conformal transformation in Finsler spaces with $(\alpha, \beta) - metric$. M. Hashinguchi (1976)-On Conformal transformation of Finsler metric. H. Izumi- (1977) Conformal transformations of Finsler spaces. I.Tensor. P.N. Pandey (1978)- Groups of conformal transformations of conformally related Finsler manifolds. H. Izumi (1980)- Conformal transformations of Finsler spaces. II.Tensor. R. Miron and M. Hashiguchi (1981)- Conformal Finsler connections. U.P. Singh and A.K. Singh (1985)- On Conformal transformations of Kropina metric. M. Matsumato(1986)-Foundations of Finsler geometry and special Finsler spaces. R. Miron (1988)- The geometry of Cartan spaces. M. Matsumato. C. Shibata and M. Azuma (1993)-C-Conformal invarient tensors of Finsler metrics, H.G. Nagaraja, C.S. Bagewadi and H. Izumi (1995)-On infinitastimal h-Conformal motion of Finsler metric, F. Ikeda (1997)-Criteria for Conformal flatness of Finsler spaces, Yong-Duck Lee (1997)-Conformal transformations of difference tensors of

Finsler space with an $(\alpha, \beta)-metric$, Y.D. Lee,(1997)- Conformal transformations of difference tensor of Finsler space with an $(\alpha, \beta)-metric$. M. Matsumoto (1999)- Conformally closed Finsler spaces. B.N. Prasad and A.K. Diwedi (1999)-Conformal change of three-dimentional Finsler space. S.I. Hojo M. Matsumoto and K. Okubo (2000)- Theory of Conformally Berwald Finsler spaces and its applications to $(\alpha, \beta)-metric$. D. Bao, S.S. Chern and Z. Shen(2000)-An intoduction to Riemann-Finsler geometry. S.K. Narasimhamurthy and C.S. Bagewadi (2004)-C-Conformal special Finsler space admitting a parallel vector field. X.Cheng and Z. Shen (2005)-Sub-manifolds of h-Conformally flat Finsler space. S.H.Abed (2006)- conformal β-change in Finsler spaces. S.K. Narasimhamurthy, C.S. Bagewadi and H.G. Nagaraja (2007)-On Infinitesimal C-Conformal motion of special Finsler space. S.H.Abed (2008)- Cartan connection associated with β-conformal change in Finsler geometry. Abed and A. Soleiman (2008)- A global theory of conformal Finsler geometry. M.K. Gupta and P.N. Pandey (2009)-Hypersurfaces of conformally and h-Conformally related Finsler spaces. N.L. Youssef, S.H. Abed S.G. Elgendi (2009)- Generalised β-conformal change of Finsler metrics. S.K. Narasimhamurthy, S.T. Aveesh,and Pradeep kumar (2009)-On-Curvature Tensor of C_3 -Like Conformal Finsler space. S.K. Narasimhamurthy and G.N. Latha kumari (2010)- On a hypersurface of a spe- cial Finsler space with a metric $L = \alpha + \beta + \frac{\beta^2}{\alpha}$. A. Tayebi and B. Najafi (2012)- On m^{th}-root Finsler metrics. A. Tayebi, E. Peyghan and A. Shahbazinia (2012)-On generalized $m^t h$-root Finsler metrics. S.K. Narasimhamurthy H. Anjan kumar and Ajith (2012)-The study of Cartan space with Randers metric. S.K. Narasimhamurthy and Vasantha. D (2012)-Some contributions to Finsler spaces with $(\alpha, \beta) - metric$. S.K. Narasimhamurthy and Ajith (2013)-A study on Conformal Finsler spaces.

1.6 Finsler Space Structure

Definition 1. *A Finsler metric on M is a function $L : TM \rightarrow [0, \infty)$ with the following properties:*

i) $L \, is \, C^{\infty} \; on \; TM_0,$

ii) $L \; is \; positively \; 1 - homogeneous \; on \; the fibers \; of \; tangent \; bundle \; TM, \; and$

iii) $The \; Hessian \; of \; F^2 \; with \; element \; g_{ij}(x, y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2, \; is \; regular \; on \; TM_0,$

$i.e., \; det(g_{ij} \neq 0).$

The pair (M^n, L) is then called a Finsler space. L is called fundamental function and g_{ij} is called fundamental tensor.

Let $C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$ be Cartan tensor. Consider the Finsler space $F^n = (M^n.L)$ equipped with an (α, β)-metric $L(\alpha, \beta)$. Let γ^i_{jk} denote the Christoffel symbols in the Riemannian space (M^n, α). Denote by $b_{i;j}$, the covariant derivative of the vector field b^i with respect to Riemannian connection γ^i_{jk}, i.e., $b_{i;j} = \frac{\partial b_i}{\partial x^j} - b_k\gamma^i_{jk}$.

Example 1. Let M^n be a real n-dimensional differentiable manifold endowed with a Riemannian metric g and a differentiable 1-form ω. $g_{ij}(x)$ and $\omega_j(x)$ be the components of g and ω with respect to the local chart (U, ϕ, R^n) and let L be a real function defined on $\phi(U) \times R^n$ by

$$L(x^i, y^i) = \omega_i(x)y^i + \frac{1}{2}(g_{ij}(x)y^iy^j)^{\frac{1}{2}}.$$

Clearly L is a global function on TM given locally by the above expression.

Moreover, L satisfies the homogeneity property and on an open submanifold A of TM satisfies

$$Rank(\dot{\partial}_i\dot{\partial}_j L^2/2) = n.$$

Thus, L is the fundamental function of Finsler space $F^n = (TM, L)$ and this space is called *Randers space*.

1.7 Cartan tensor and the generalized Christoffel symbols

From the metric tensor, we construct a tensor C_{ijk} by differentiating partially with respect to y^k. The tensor C_{ijk} is defined by

$$\begin{aligned} C_{ijk} &= \frac{1}{2}\dot{\partial}_k g_{ij} \\ &= \frac{1}{4}\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k L^2. \end{aligned} \tag{1.7.1}$$

This tensor is known as $(h)hv$-*torsion tensor* or *Cartan tensor*. It is positively homogeneous of degree-1 in y^i and symmetric in all its indices. By Euler's theorem on homogeneous functions, we get

$$\begin{aligned} (a)\ C_{ijk}y^i &= C_{jki}y^i = C_{kij}y^i = 0, \\ (b)\ C^i_{jk}y^j &= C^i_{kj}y^j = 0, \end{aligned} \tag{1.7.2}$$

where C^i_{jk} is the associate tensor of C_{ijk}, defined as

$$C^i_{jk} = g^{ih}C_{jhk}. \tag{1.7.3}$$

which is also positively homogeneous of degree -1 in y^i and symmetric in its lower indices. The *torsion vector* C^i is defined by $C^i = C^i_{jk}g^{jk}$.

The *angular metric tensor* h_{ij} is defined as

$$h_{ij} = g_{ij} - l_i l_j = L(\dot{\partial}_i\dot{\partial}_j L), \tag{1.7.4}$$

where $l_i = \dot{\partial}_i L$. The tensor h_{ij} is positively homogeneous of degree 0 and satisfies $h_{ij}y^i = 0$.

Let us define the *generalized Christoffel symbols* of the first kind and the second kind as in Riemannian geometry:

$$\left.\begin{aligned} &(a)\ \gamma^i_{jk} = g^{ih}\gamma_{jhk}, \\ &(b)\ \gamma_{jhk} = \frac{1}{2}\left\{\partial_k g_{jh} + \partial_j g_{kh} - \partial_h g_{jk}\right\}. \end{aligned}\right\} \tag{1.7.5}$$

Example 2. Let M^n be a real n-dimensional differentiable manifold endowed with a Riemannian metric g and a differentiable 1-form ω. Let H be a closed subset of $\phi(U)\times R^n$ consisting of all points (x^i, y^i) such that $\omega_i(x) = 0$. Let $L(x^i, y^i) = \frac{g_{ij}(x)y^iy^j}{\omega_i(x)y^i}$ be the real valued function defined on the open set $U^* = \phi(U)\times R^n - \{H\}$. Let B denote the union of all open sets $\phi^{-1}(U^*)$. It is clear that L satisfies the homogeneity property on B and satisfies $Rank(\dot{\partial}_i\dot{\partial}_j L^2/2) = n$ on an open submanifold A of B. Then the pair $F^n = (TM, L)$ is a Finsler space called *Kropina space*.

Example 3. Let M^n be a real n-dimensional differentiable manifold endowed with a Riemannian metric g and a differentiable 1-form ω, $g_{ij}(x)$ and $\omega_j(x)$ be the components of g and ω with respect to the local chart (U, ϕ, R^n) and let L be a real function defined on $\phi(U)\times R^n$ by

$$L(x^i, y^i) = \omega_i(x)y^i + \frac{1}{2}(g_{ij}(x)y^iy^j)^{\frac{1}{2}}. \tag{1.7.6}$$

Clearly L is a global function on TM given locally by the above expression. Moreover, L satisfies the homogeneity property and on an open submanifold A of TM satisfies

$$Rank(\dot{\partial}_i\dot{\partial}_j L^2/2) = n. \tag{1.7.7}$$

Thus, L is the fundamental function of Finsler space $F^n = (TM, L)$ and this space is called *Randers space*.

1.8 Finsler connections

A *Finsler connection* $F\Gamma$ *is defined by a triad* $(F^i_{jk}, N^i_j, V^i_{jk})$, *where*

(1) (F^i_{jk}) are the connection coefficients of *h-connection* which obey the transformation law

$$\bar{F}^l_{mp} = F^i_{jk}\frac{\partial\bar{x}^l}{\partial x^i}\frac{\partial x^j}{\partial\bar{x}^m}\frac{\partial x^k}{\partial\bar{x}^p} + \frac{\partial\bar{x}^l}{\partial x^i}\frac{\partial^2 x^i}{\partial\bar{x}^m\partial\bar{x}^p}. \tag{1.8.1}$$

(2) N^i_j are the connection coefficients of *non-linear connection* which obey the transformation law

$$\bar{N}^l_m = N^i_j\frac{\partial\bar{x}^l}{\partial x^i}\frac{\partial x^j}{\partial\bar{x}^m} + \frac{\partial\bar{x}^l}{\partial x^i}\frac{\partial^2 x^i}{\partial\bar{x}^m\partial\bar{x}^p}\bar{y}^p. \tag{1.8.2}$$

(3) V^i_{jk} are the connection coefficients of $v-$*connection* which are components of a tensor field of (1,2) type and obey the transformation law

$$\bar{V}^l_{mn} = V^i_{jk}\frac{\partial\bar{x}^l}{\partial x^i}\frac{\partial x^j}{\partial\bar{x}^m}\frac{\partial x^k}{\partial\bar{x}^n}. \tag{1.8.3}$$

For a given connection, the $h-$ and $v-$*covariant derivatives* of any tensor T^i_j are given by

$$T^i_{j|k} = \delta_k T^i_j + T^r_j F^i_{rk} - T^i_r F^r_{jk}, \tag{1.8.4}$$

and

$$T^i_j|_k = \dot{\partial}_k T^i_j + T^r_j V^i_{rk} - T^i_r V^r_{jk}, \tag{1.8.5}$$

where $\delta_k = \partial_k - N^h_k\dot{\partial}_h$.

The connection formulae for covariant derivatives of a contravariant vector X^i are given by

$$\begin{aligned}
X^i_{|j|k} - X^i_{|k|j} &= X^r R^i_{rjk} - X^i_{|r}T^r_{jk} - X^i|_r R^r_{jk},\\
X^i_{|j}|_k - X^i|_{k|j} &= X^r P^i_{rjk} - X^i_{|r}V^r_{jk} - X^i|_r P^r_{jk},\\
X^i|_j|_k - X^i|_k|_j &= X^r S^i_{rjk} - X^i|_r S^r_{jk}.
\end{aligned} \tag{1.8.6}$$

These are called the *Ricci identities* for the Finsler connection. Here we get three kinds of curvature tensors:

$h-$curvature $R^2 = (R^i_{hjk})$,

$hv-$curvature $P^2 = (P^i_{hjk})$,

$h-$curvature $S^2 = (S^i_{hjk})$,

and five kinds of torsion tensors: (v)$h-$torsion $R^1 = (R^i_{jk})$,

(v)$hv-$torsion $P^1 = (P^i_{jk})$,

(v)$v-$torsion $S^1 = (S^i_{jk})$,

(h)$h-$torsion $T = (T^i_{jk})$,

(h)$hv-$torsion $V = (V^i_{jk})$.

It is to be noted that the v-connection (V^i_{jk}) plays also a role of torsion tensor and

$$T^i_{jk} = \Gamma^{*i}_{jk} - F^i_{kj}, \qquad P^i_{jk} = \dot{\partial}_k N^i_j - F^i_{kj}, \qquad S^i_{jk} = V^i_{jk} - V^i_{kj}.$$

The *deflection tensor* $D = (D^i_j)$ of a Finsler connection is given by

$$D^i_j = y^i_{|j} = -N^i_j + y^r F^i_{rj}. \tag{1.8.7}$$

In the theory of Finsler spaces two Finsler connections $B\Gamma$ and $C\Gamma$ have been playing dominant role from the time of their introduction.

1.8.1 Berwald connection

The *Berwald connection* $B\Gamma = (G^i_{jk}, G^i_j, 0)$ is uniquely determined from the metric function L by the system of axioms:

(B1) L-metrical: $L_{|i} = 0$,

(B2) $(h)h$-torsion $T = 0$,

(B3) Deflection $D = 0$,

(B4) $(v)hv$-torsion $P^1 = 0$,

(B5) v-connection is flat: $V^i_{jk} = 0$.

Berwald connection parameters G^i_{jk} are defined by

$$G^i_{jk} = \dot{\partial}_j \dot{\partial}_k G^i, \tag{1.8.8}$$

where

$$G^i = \frac{1}{2}\gamma^i_{jk} y^j y^k. \tag{1.8.9}$$

Since the christoffel symbols of the first kind as well as of the second kind are positively homogeneous of degree 0 in y^i, G^i are positively homogeneous of degree 2 in y^i. In view of (1.8.8) the parameters G^i_{jk} are positively homogeneous of degree 0 in y^i.

Berwald's covariant derivatives of an arbitrary tensor T^i_j with respect to $B\Gamma$ are given by

$$\mathfrak{B}_k T^i_j = \partial_k T^i_j - (\dot{\partial}_h T^i_j) G^h_k + T^r_j G^i_{rk} - T^i_r G^r_{jk} \tag{1.8.10}$$

and

$$T^i_{j.k} = \dot{\partial}_k T^i_j, \tag{1.8.11}$$

where h- and v-covariant derivatives with respect to $B\Gamma$ are denoted by $\mathfrak{B}_k$ and . respectively and $G^i_j = \dot{\partial}_j G^i$.

Transvecting (1.8.8) by y^j and using Euler's theorem on homogeneous functions, we have

$$\text{(a) } G^i_{jk} y^j = G^i_k$$
$$\text{(b) } G^i_j y^j = 2G^i.$$

In view of the homogeneity of G^i_{jk} in y^i, G^i_j are positively homogeneous of degree 1 in y^i.

Berwald connection parameters G^i_{jk} do not form the components of a tensor but its partial derivatives with respect to y^k constitute the components of a tensor. This tensor is denoted as G^i_{hjk}. Thus

$$G^i_{hjk} = \dot{\partial}_h \dot{\partial}_k G^i_j. \tag{1.8.12}$$

This tensor is symmetric in all its lower indices and positively homogeneous of degree -1 in y^i. In view of Euler's theorem, we get

$$G^i_{hjk} y^h = G^i_{jhk} y^h = G^i_{jkh} y^h = 0. \tag{1.8.13}$$

The Berwald h-coveriant derivatives of y^i, y_i and L vanish identically, i.e.

$$a)\mathfrak{B}_k y^i = 0, \quad b)\mathfrak{B}_k y_i = 0, \quad c)\mathfrak{B}_k L = 0.$$

But the Berwald covariant derivative of the metric tensor g_{ij} does not vanish, in general, i.e. $\mathfrak{B}g_{ij} \neq 0$. It is given by

$$\mathfrak{B}_{\mathfrak{k}} g_{ij} = y_r G^r_{ijk}$$

. The h-covariant differentiation with respect to $B\Gamma$ and partial differentiation with respect to y^j commute according as

$$\dot{\partial}_j(\mathfrak{B}_k X^i) - \mathfrak{B}_k(\dot{\partial}_j X^i) = X^r G^i_{rkj}. \tag{1.8.14}$$

The Ricci commutation formulae for Berwald connection are given as follows:

$$\mathfrak{B}_k \mathfrak{B}_h X^i - \mathfrak{B}_h \mathfrak{B}_k X^i = X^r H^i_{rhk} - \left(\dot{\partial}_r\right) H^r_{hk}, \tag{1.8.15}$$

where

$$H^i_{hjk} = \partial_k G^i_{hj} - \partial_j G^i_{hk} + G^r_{hj} G^i_{rk} + G^r_{hk} G^i_{rj} + G^i_{rhk} G^r_j - G^i_{rhj} G^r_k$$

The tensor H^i_{jkh} is Berwald curvature tensor. This tensor is skew-symmetric in its last two lower indices and positively homogeneous of degree zero in y^i. The tensors H^i_{kh} and

H^i_{jkh} are related by

$$H^i_{kh} = H^i_{jkh} y^j. \tag{1.8.16}$$

Transvection of H^i_{jk} by y^j gives the deviation tensor H^i_k, i.e.

$$H^i_k = H^i_{jk} y^j. \tag{1.8.17}$$

Tensors H^i_{jkh} , H^i_{jk} and H^i_j are also related by

$$(a) H^i_{jkh} = \dot{\partial}_j H^i_{kh}, \qquad (b) H^i_{jk} = \frac{1}{3}(\dot{\partial}_j H^i_k - \dot{\partial}_k H^i_j)$$

Contraction of the indices i and j in H^i_{jkh} , H^i_{jk} and H^i_j yields

$$(a) H_{kh} = H^i_{ikh}, \qquad (b) H_k = H_k - H^i_{ik}, \qquad (c) H = \frac{1}{n-1} H^i_i.$$

Since contractions of the indices do not change the homogeneity in y^i, the tensor H_{rk}, the vector H_h and the scalar H are homogeneous of degree 0,1 and 2 in y^i respectively. The tensors H^i_{jkh} , H^i_{jk} and H_{jk} and H_j also satisfy the following:

$$\begin{aligned}
a) &\quad H^i_j y^i = 0, \\
b) &\quad y^i H^i_{jk} = 0, \\
c) &\quad y^i H^i_j = 0, \\
d) &\quad H_{jh} - H_{hj} = H^i_{hji}, \\
e) &\quad H^i_{jkh} + H^i_{hjk} + H^i_{khj} = 0 \quad \textit{and} \\
f) &\quad \mathfrak{B}_m H^i_{hjk} + \mathfrak{B}_k H^i_{hmj} + \mathfrak{B}_j H^i_{hkm} \\
+ &\quad H^l_{jk} G^i_{lhm} + H^l_{mj} G^l_{lhk} + H^l_{km} G^i_{lhj} = 0.
\end{aligned}$$

The equations (e) and (f) are known as *Bianchi identities* for *Berwald curvature tensor.*

1.8.2 Cartan connection

The *Cartan connection* $C\Gamma = (\Gamma^{*i}_{jk}, G^i_j, C^i_{jk})$ is uniquely determined from the metric function L by the system of axioms:

(C1) h-metrical: $g_{ij|k} = 0$,

(C2) $(h)h$-torsion $T = 0$,

(C3) Deflection $D = 0$,

(C4) v-metrical: $g_{ij}|_k = 0$

(C5) $(v)hv$-torsion $S^1 = 0$,.

The last two axioms (C4) and (C5) give

$$C^i_{jk} = \frac{1}{2} g^i r \dot{\partial}_r g_{jk}.$$

This shows that the v-connection of $C\Gamma$ and Cartan tensor are identical.

h-connection coefficients Γ^{*i}_{jk} are given by

$$\Gamma^{*i}_{jk} = \frac{1}{2} g^{ir} [\delta_k g_{jr} + \delta_j g_{kr} - \delta_r g_{jk}]. \tag{1.8.18}$$

The non-linear connection coefficients G^i_j of $C\Gamma$ are same as that of $B\Gamma$.

E.Cartan defined the covariant derivative of a vector field X^i by

$$X^i_{|k} = \partial_k X^i - (\dot{\partial}_r X^i)\Gamma^{*r}_{hk} y^h + X^r \Gamma^{*i}_{rk}, \tag{1.8.19}$$

where $\Gamma^{*r}_{hk} y^h = G^i_k$.

This type of covariant derivative introduced by Cartan is called as h-covariant derivative.

Ricci-commutation formula for such covariant derivative is given by

$$X^i_{|h|k} - X^i_{|k|h} = X^r K^i_{rhk} - (\dot{\partial}_r X^i) K^r_{shk} y^s, \tag{1.8.20}$$

where $K^i_{rhk} = (\partial_k \Gamma^{*i}_{rh}) - (\partial_h \Gamma^{*i}_{rk}) + (\dot{\partial}_m \Gamma^{*i}_{rk})\Gamma^{*m}_{ph} y^p - (\dot{\partial}_m \Gamma^{*i}_{rh})\Gamma^{*m}_{ph} y^p + \Gamma^{*s}_{rh}\Gamma^{*i}_{sk} - \Gamma^{*s}_{rk}\Gamma^{*i}_{sh}$.

The tensor K^i_{jkh} is called Cartan curvature tensor or h-curvature tensor. This tensor is skew-symmetric in last two lower indices and positively homogeneous of degree zero in y^i. Cartan curvature tensor K^i_{jkh} is connected with Berwald curvature tensor H^i_{jkh} and tensor H^i_{kh} by

$$H^i_{jkh} = K^i_{jkh} + y^r \dot{\partial}_j K^i_{rkh}, \tag{1.8.21}$$

and

$$K^i_{jkh} y^j = H^i_{kh}. \tag{1.8.22}$$

The commutation formula for the operators of partial differentiation with respect to y^k and h-covariant differentiation is given by

$$\dot{\partial}_k (X^i_{|h}) - (\dot{\partial}_k X^i)_{|h} = X^r \dot{\partial}_k \Gamma^{*i}_{rh} - (\dot{\partial}_r X^i)(\dot{\partial}_k \Gamma^{*r}_{sh}) y^s. \tag{1.8.23}$$

The above definition for covariant differentiation due to Matsumoto are similar to that of E. Cartan with only difference that $T^i_j|_k$ of Matsumoto coincides with $\frac{1}{L} T^I_j|_k$ of Cartan, though the notations are same.

It is easy to verify that

$$a) y^i_{|j} = 0, \quad b) L_{|j} = 0, \quad c) l^i_{|j} = 0,$$

and

$$a) y^i|_j = \delta^i_j, \quad b) L|_j = l_j, \quad c) l^i|_j = \frac{1}{L} h^i_j,$$

where $l_j = g_{ij} l^i$ and $h^i_j = g^{ir} h_{rj}$.

The Cartan connection coefficients and the Berwald connection coefficients are related by

$$G^i_{jk} = \Gamma^{*i}_{jk} + C^i_{jk|0}. \tag{1.8.24}$$

The covariant derivatives of a tensor field T^i_j with respect to symmetric connection coefficients $\Gamma^i_{jk}(x,\xi)$ is given by

$$T^i_{j;k} = \partial_k T^i_j + (\dot{\partial}_h T^i_j)\partial_k \xi^h + T^h j \Gamma^i_{hk} - T^i_h \Gamma^h_{jk}. \tag{1.8.25}$$

The commutation formula for covariant derivative is given by

$$X^i_{;hk} - X^i_{;kh} = \widetilde{K}^i_{jhk}(x,\xi)X^j, \tag{1.8.26}$$

where

$$\widetilde{K}^i_{jhk}(x,\xi) = (\partial_k \Gamma^i_{jh} + (\dot{\partial}_l \Gamma^i_{jh})\partial_k \xi^i) - (\partial_h \Gamma^i_{jk} + (\dot{\partial}_l \Gamma^i_{jk})\partial_h \xi^i) + \Gamma^i_{mk}\Gamma^m_{jh} - \Gamma^i_{mh}\Gamma^m jk.$$

This tensor is called relative curvature tensor, since it depends on partial derivatives of the field $\xi^m(x^k)$ with respect to x^h. The relative curvature tensor $\widetilde{K}^i_{jhk}(x,\xi)$ satisfies the following:

$$(a)\widetilde{K}^i_{jhk} = -\widetilde{K}^i_{jkh}, \qquad (b)\widetilde{K}^i_{jhk;m} + \widetilde{K}^i_{jmh;k} + \widetilde{K}^i_{jkm;h} = 0.$$

The associate tensor of the relative curvature tensor is defined as

$$g_{jm}\widetilde{K}^m_{ikh} = \widetilde{K}_{ijkh}.$$

The tensor satisfies

$$\widetilde{K}_{jihk}\dot{x}^i = -\widetilde{K}_{ijhk}\dot{x}^i.$$

1.9 Special Finsler spaces

1.9.1 Riemannian space

A Finsler space $F^n = (M, L)$ is said to be a *Riemannian space* if its fundamental function $L(x,y)$ is written as

$$L^2(x,y) = g_{ij}(x)y^i y^j. \tag{1.9.1}$$

Among Finsler spaces, the class of all Riemannian spaces is characterized by $C_{ijk} = 0$.

1.9.2 Berwald space

A Finsler space is called a *Berwald space* if the Berwald connection coefficient are linear, i.e, G^h_{ij} are functions of position only.

A Finsler space is Berwald space if and only if

for $B\Gamma : G^h_{ijk} = 0$.

for $C\Gamma : C_{hij|k} = 0$.

Berwald himself called a Berwald space an *affinely connected space*. It is clear that the class of Berwald space is included in the class of Landsberg spaces.

1.9.3 Landsberg Space

A Finsler space is called a *Landsberg space* if the Berwald connection is $h-$metrical, i.e.,$\mathfrak{B}_k g_{ij} = 0$.

A Landsberg space is characterized by one of the following conditions:

$$(1) \qquad P^h_{ijk} = 0,$$

$$(2) \qquad P^h_{ij} (= C^h_{ij|0}) = 0,$$

$$(3) \qquad C^h_{ij|k} = C^h_{ik|j} = 0.$$

1.9.4 Locally Minkowskian space

A Finsler space $F^n = (M, L)$ is called *locally Minkowskian space* if there exists a coordinate system (x^i) in which L is a function of y^i only.

A Finsler space is locally Minkowskian space if and only if

for $B\Gamma : H^h_{ijk} = 0 = G^h_{ijk}$,

for $C\Gamma : R^i_{ijk} = 0 = C_{hij|k}$.

1.10 Finsler spaces with (α, β)-metric

There is a class of Finsler metrics defined by a Riemannian metric and a 1-form on a manifold, with some curvature properties called (α, β)-metrics and these metrics are computable.

The concept of (α, β)-metric $L(\alpha, \beta)$ was introduced by M. Matsumoto in 1972 and has been studied by many Finsler geometers. Physicists are also interested in these metrics. They seek for some non-Riemannian models for space time. For example, by using (α, β)-metrics, G. S. Asanov introduced Finsleroid-Finsler spaces and formulated pseudo-Finsleroid gravitational field equations.

Definition 2. *The Finsler space $F^n = (M^n, L)$ is said to have an (α, β)-metric if L is a positively homogeneous function of degree one in two variables $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)y^i$, where α is a Riemannian metric and β is differential 1-form.*

An (α, β)-metric is expressed in the following form

$$L = \alpha\phi(s), \quad s = \beta/\alpha, \tag{1.10.1}$$

where $\phi = \phi(s)$ is a C^∞ positive function on an open interval $(-b_0, b_0)$. The norm $\parallel \beta_x \parallel_\alpha$ of β with respect to α is defined by

$$\parallel \beta_x \parallel_\alpha = sup_{y \in T_x M}\{\beta(x, y), \alpha(x, y)\} = \sqrt{a^{ij}(x)b_i(x)b_j(x)}.$$

In order to define L, β must satisfy the condition $\parallel \beta_x \parallel_\alpha < b_0$ for all $x \in M$.

The derivative of the (α, β)-metric with respect to α and β are given by

$$\begin{aligned} L_\alpha &= \partial L/\partial\alpha, \\ L_\beta &= \partial L/\partial\beta, \\ L_{\alpha\alpha} &= \partial L_\alpha/\partial\alpha, \\ L_{\beta\beta} &= \partial L_\beta/\partial\beta, \\ L_{\alpha\beta} &= \partial L_\alpha/\partial\beta. \end{aligned}$$

Then the normalized element of support $l_i = \dot{\partial}_i L$ is given by

$$l_i = \alpha^{-1} L_\alpha Y_i + L_\beta b_i, \tag{1.10.2}$$

where $Y_i = a_{ij} y^j$. The angular metric tensor $h_{ij} = L\dot{\partial}_i\dot{\partial}_j L$ is given by

$$h_{ij} = pa_{ij} + q_0 b_i b_j + q_1(b_i Y_j + b_j Y_i) + q_2 Y_i Y_j, \tag{1.10.3}$$

where

$$\begin{aligned} p &= LL_\alpha\alpha^{-1}, \\ q_0 &= LL_{\beta\beta}, \\ q_1 &= LL_{\alpha\beta}\alpha^{-1}, \\ q_2 &= L\alpha^{-2}(L_{\alpha\alpha} - L_\alpha\alpha^{-1}). \end{aligned}$$

The fundamental tensor $g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2$ is given by

$$g_{ij} = pa_{ij} + p_0 b_i b_j + p_1(b_i Y_j + b_j Y_i) + p_2 Y_i Y_j,$$

where

$$\begin{aligned} p_0 &= q_0 + L_\beta^2, \\ p_1 &= q_1 + L^{-1} p L_\beta, \\ p_2 &= q_2 + p^2 L^{-2}. \end{aligned}$$

Moreover, the reciprocal tensor g^{ij} of g_{ij} is given by

$$g^{ij} = p^{-1}a^{ij} + S_0 b^i b^j + S_1(b^i y^j + b^j y^i) + S_2 y^i y^j, \tag{1.10.4}$$

where

$$b^i = a^{ij}b_j, \qquad S_0 = (pp_0 + (p_0p_2 - p_1^2)\alpha^2)/\zeta,$$

$$S_1 = (pp_1 + (p_0p_2 - p_1^2)\beta)/\zeta p,$$

$$S_2 = (pp_2 + (p_0p_2 - p_1^2)b^2)/\zeta p, \qquad b^2 = a_{ij}b^i b^j,$$

$$\zeta = p(p + p_0b^2 + p_1\beta) + (p_0p_2 - p_1^2)(\alpha^2 b^2 - \beta^2).$$

The hv-torsion tensor $C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}$ is given by

$$2pC_{ijk} = p_1(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + \gamma_1 m_i m_j m_k, \tag{1.10.5}$$

where

$$\gamma_1 = p\frac{\partial p_0}{\partial \beta} - 3p_1q_0, \qquad m_i = b_i - \alpha^{-2}\beta Y_i.$$

The positive homogeneity of $L = L(\alpha, \beta)$ gives

$$L_\alpha\alpha + L_\beta\beta = L, \quad L_{\alpha\alpha}\alpha + L_{\alpha\beta}\beta = 0,$$

$$L_{\beta\alpha}\alpha + L_{\beta\beta}\beta = 0, \quad L_{\alpha\alpha\alpha}\alpha + L_{\alpha\alpha\beta}\beta = -L_{\alpha\alpha},$$

$$L_\alpha = \partial L/\partial\alpha, \quad L_\beta = \partial L/\partial\beta, \quad L_{\alpha\alpha} = \partial^2 L/\partial\alpha\partial\alpha,$$

$$L_{\alpha\beta} = L_{\beta\alpha} = \partial^2 L/\partial\alpha\partial\beta, \quad L_{\alpha\alpha\alpha} = \partial^3 L/\partial\alpha\partial\alpha\partial\alpha.$$

Some important (α, β)-metrics are listed below.

$L(\alpha,\beta)=\alpha+\beta,$	$\phi(s)=1+s,$
$L(\alpha,\beta)=\alpha^2/\beta,$	$\phi(s)=1/s,$
$L(\alpha,\beta)=\alpha^{(m+1)}/\beta^m,$	$\phi(s)=1/s^m,$
$L(\alpha,\beta)=\alpha^2/(\alpha-\beta),$	$\phi(s)=\frac{1}{1-s},$
$L(\alpha,\beta)=\alpha+\beta+\beta^2/\alpha,$	$\phi(s)=1+s+s^2,$
$L(\alpha,\beta)=\alpha+\beta+\beta^2/\alpha+\beta^3/\alpha^2,$	$\phi(s)=1+s+s^2+s^3,$
$L(\alpha,\beta)=\alpha\Sigma_{k=0}^{\infty}(\beta/\alpha)^k,$	$\phi(s)=\Sigma_{k=0}^{\infty}s^k,$
$L(\alpha,\beta)=\beta^{(m+1)}/\alpha^m,$	$\phi(s)=s^{m+1},$
$L(\alpha,\beta)=c_1\alpha+c_2\beta+\beta^2/\alpha,\ \ c_2\neq 0,$	$\phi(s)=c_1+c_2s+s^2,$
$L(\alpha,\beta)=c_1\alpha+c_2\beta+\alpha^2/\beta,\ \ c_1\neq 0,$	$\phi(s)=c_1+c_2s+1/s,$
$L(\alpha,\beta)=\alpha+\beta^2/\alpha,$	$\phi(s)=1+s^2,$
$L(\alpha,\beta)=\alpha+\beta^{(m+1)}/\alpha^m,$	$\phi(s)=1+s^{m+1},$
$L(\alpha,\beta)=(\alpha+\beta)^2/\alpha,$	$\phi(s)=1+2s+s^2,$
$L(\alpha,\beta)=(c_1\alpha^2+c_2\alpha\beta+c_3\beta^2)/(\alpha+\beta),$	$\phi(s)=\left(\frac{c_1+c_2s+c_3s^2}{1+s}\right),$
$L^2(\alpha,\beta)=2\alpha\beta,$	$\phi(s)=\sqrt{2s},$
$L^2(\alpha,\beta)=c_1\alpha^2+c_2\alpha\beta+c_3\beta^2,$	$\phi(s)=\sqrt{c_1+c_2s+c_3s^2},$
$L^3(\alpha,\beta)=c_1\alpha^2\beta+c_2\beta^3,$	$\phi(s)=\sqrt[3]{c_1s+c_2s^3}.$

1.11 Conformal change

Let $F^n=(M^n,L(x,y))$ and $\overline{F}^n=(M^n,\overline{L}(x,y))$ be two Finsler spaces on a same underlying manifold M^n. If the angle in F^n is equal to that in $\overline{F}^n$ for any tangent vectors, then F^n is called conformal to $\overline{F}^n$ and the change $L\longrightarrow\overline{L}=e^{\sigma(x)}L$ of the metric is called a conformal change, where $\sigma(x)$ is conformal factor is a function of x alone.

1.11.1 β-change

Let $F^n=(M^n,L)$ be a Finsler space associated with the another Finsler space ${}^*F^n=(M^n,{}^*L)$, where ${}^*L(x,y)$ is given by the transformation

$$L(x,y)\rightarrow{}^*L(x,y)=L(x,y)+\beta(x,y), \tag{1.11.1}$$

where $\beta(x,y) = b_i(x)y^i$ is a 1-form, L is Finslerian. This transformation is called β-change.

1.11.2 Conformal β-change

Let $F^n = (M^n, L)$ be a Finsler space associated with the another Finsler space ${}^*F^n = (M^n, {}^*L)$, where ${}^*L(x,y)$ is given by the transformation

$$L(x,y) \rightarrow {}^*L(x,y) = e^{\sigma(x)}L(x,y) + \beta(x,y), \tag{1.11.2}$$

where $\sigma = \sigma(x)$ is a function of x and $\beta(x,y) = b_i(x)y^i$ is a 1-form, L is Finslerian. This transformation is called conformal β-change.

1.11.3 Conformal Kropina change

Let $F^n = (M^n, L)$ and $\overline{F}^n = (M^n, \overline{L})$ be two Finsler space on the same underlying manifold M^n. If we have a function $\sigma(x)$ in each coordinate neighbourhoods of M^n such that $\overline{L}(\overline{\alpha}, \overline{\beta}) = e^{\sigma}[\frac{L^2(\alpha,\beta)}{\beta}]$, then F^n is called conformal Kropina to $\overline{F}^n$ and the change $L \rightarrow \overline{L}$ of metric is called conformal Kropina change of (α, β)-metric. A conformal change of (α, β)-metric is expressed as $(\alpha, \beta) \longrightarrow (\overline{\alpha}, \overline{\beta})$.

1.12 Conformal Vector Fields

A vector Field V on a Finsler manifold (M, F) is called a Conformal vector field with a Conformal factor $C = C(x)$ if the 1-parameter transformation φ_t generated by V is a conformal transformation on (M, F), that is

$$F(\varphi_t(x), (\varphi_t) * (y)) = e^2 c(x) t F(x,y). \tag{1.12.1}$$

In particular,V is called a homothetic vector field with dilation c if c is constant and V is called a Killing vector field if $c = 0$.

1.13 Nonholonomic Finsler frames

Definition 3. Let U be an open set of TM and

$$V_i : u \in U \mapsto V_i(u) \in V_u TM, i \in \{1, ..., n\} \tag{1.13.1}$$

be a vertical frame over U. If $V_i(u) = V_i^j(u)(\partial/\partial y^j)$, then $V_i^j(u)$ are the entries of a invertible matrix for all $u \in U$. Denote by $\widetilde{V}_k^j(u)$ the inverse of this matrix. This means that

$$V_j^i \widetilde{V}_k^j = \delta_k^i, \quad \widetilde{V}_j^i V_k^j = \delta_k^i :$$

We call V_j^i a nonholonomic frame.

Consider $a_{ij}(x)$, the components of a Riemannian metric on the base manifold M, $a(x, y) > 0$ and $b(x, y) > 0$ two functions on TM and $B(x, y) = B_i(x, y)dx^i$ a vertical 1-form on TM. Then

$$g_{ij}(x, y) = a(x, y)a_{ij}(x) + b(x, y)B_i(x)B_j(x) \tag{1.13.2}$$

is a generalized Lagrange metric, called the Beil metric. We say also that the metric tensor g_{ij} is a *Beil deformation* of the Riemannian metric a_{ij}. It has been studied and applied by R. Miron and R.K. Tavakol in general relativity for $a(x, y) = exp(2\sigma(x, y))$ and $b = 0$[137]. The case $a(x, y) = 1$ with various choices of b and B_i was introduced and studied by R.G. Beil for constructing a new unified field theory [25].

In this thesis such a nonholonomic frame has been determined for two important classes of Finsler spaces that are dual in the sense of Matsumoto and special Finsler spaces.

1.14 Locally Dually flat Finsler spaces

Definition 4. A Finsler metric $F = F(x, y)$ on a manifold M is said to be locally dually flat if at any point there is a standard coordinate system (x^i, y^i) in TM which satisfies

$$(F^2)_{x^k y^l} y^k = 2(F^2)_{x^l}.$$

It is known that a Riemannian metric $F = \sqrt{g_{ij}(x)y^i y^j}$ is locally dually flat if and only if in an adapted coordinate system,

$$g_{ij} = \frac{\partial^2 \Psi}{\partial x^i \partial x^j}(x).$$

where $\Psi = \Psi(x)$ is a C^∞ function.

An (α, β)-metric $F = \alpha\phi(s)$, where $s = \frac{\beta}{\alpha}$ is dually flat on an open subset $U \subset R^n$ if and only if

$$3\alpha^2 a_{ml} G^m_\alpha + Q(3s_{l0} - r_{l0})\alpha^3 - \alpha^2[\frac{\partial(y^m G^m_\alpha)}{\partial y^l} + \alpha Q \frac{\partial(y^m G^m_\alpha)}{\partial y^l}] + Q\alpha(r_{00} + 2b_m G^m_\alpha) y_l$$
$$+2Q(y^m G^m_\alpha) + \Phi[\alpha r_{00} + 2(b_{m\alpha} - s y_m) G^m_\alpha](\alpha b_l - s y_l) = 0,$$

where $r_{i0} = r_{ij} y^j$, $s_{i0} = s_{ij} y^j$, $y_i = a_{ij} y^j$.

1.15 Subspace of Finsler Spaces

We consider an m-dimensional Finsler subspace F^m of Finsler space F^n may be parametrically represented by the equation

$$x^i = x^i(u^\alpha),$$

where α=$1, ..., m$ and u^α are the Gaussian coordinates of F^m.

Suppose that the matrix of the projection factor $B^i_\alpha = \partial x^i / \partial u^\alpha$ is of rank m. The element of the support X^i of F^n is taken to be tangential to F^m. i.e., $X^i = B^i_\alpha(u) X^\alpha$.

Thus X^α is the element of support F^m at a point u^α. The metric tensor $g_{\alpha\beta}$ and

Cartan's C-tensor $C_{\alpha\beta\gamma}$ of F^m are given by

$$a) \qquad g_{\alpha\beta} = g_{ij} B^{ij}_{\alpha\beta},$$

$$b) \qquad C_{\alpha\beta\gamma} = C_{ijk} B^{ijk}_{\alpha\beta\gamma},$$

where $B^{ij\ldots}_{\alpha\beta\ldots} = B^i_\alpha B^j_\beta \ldots$.

Since the rank of the matrix $((B^i_\alpha))$ is m, it follows that there exists a field of $(n-m)$ linearly independent vectors $N^i_{(\mu)}$ normal to F^m and they are given by the relation

$$g_{ij} N^i_{(\mu)} B^j_\alpha = 0, \qquad (\mu = m+1, \ldots, n).$$

These vectors are normalized by means of relations:

$$a) \qquad g_{ij} N^i_{(\mu)} N^j_{(\gamma)} = \delta_{\mu\gamma},$$

$$b) \qquad N^i_{(\mu)} = g^{ij} N_{j(\mu)}.$$

If $(B^\alpha_i, N_{i(\mu)})$ is the inverse matrix $(B^i_\alpha, N^i_{(\mu)})$, we have

$$B^i_\alpha B^\beta_i = \delta^\beta_\alpha, \quad B^i_\alpha N_{i(\mu)} = 0, \quad N^i_{(\mu)} B^\alpha_i = 0, \quad N^i_\mu N_{i(\mu)} = 1,$$

and further

$$B^i_\alpha B^\alpha_j + N^i_\mu N_{j(\mu)} = \delta^i_j.$$

Making use of the inverse matrix $(g^{\alpha\beta})$ of $(g_{\alpha\beta})$, we get

$$B^\alpha_i = g^{\alpha\beta} g_{ij} B^j_\beta, \qquad N_{i(\mu)} = g_{ij} N^j_{(\mu)}.$$

CHAPTER-2

CONFORMAL CHANGE OF DOUGLAS SPACE WITH SPECIAL (α, β)-METRICS

Content of this chapter

Publications based on this Chapter;

- M.Y.Kumbar, **Thippeswamy K.R** and Narasimhamurthy S.K., *"On β-Conformal Change of Douglas type with Finsler Special (α, β)-metric,", Proce. International Conference On DGAFM-2016. Page No:202-213.*

- **Thippeswamy K.R** *and Narasimhamurthy S.K., "Conformal Change of Douglas Special Finsler space with Second approximate Matsumato metric", International Journal of Advanced Research, 5(4), 1290-1295(2017).*

- **Thippeswamy K.R** *and Narasimhamurthy S.K., "Conformal Kropina change of a Finsler Special (α, β)-metric of Douglas type,", Communicated.*

Chapter 2

CONFORMAL CHANGE OF DOUGLAS SPACE WITH SPECIAL (α, β)-METRICS

2.1 Introduction

The conformal theory of Finsler metrics based on the theory of Finsler spaces by M. Matsomoto, M. Hashiguchi ([70], [92]) in 1976 studied the conformal change of a Finsler metric namely $\overline{L}(x, y) = e^{\sigma}(x)L(x, y)$. The concept of Douglas space ([89], [95], [110], [111], [112]) has been introducing by M. Matsumoto and S. Bacso as a generalization of Berward spaces from stand point of view of geodesic equation. Finsler space is said to be of Douglas space if $D^{ij} = G^i y^j - G^j y^i$ are homogeneous polynomial of degree three in y^i. it is remarkable that a Finsler space is a Douglas space if and only if the Douglas tensor vanishes identically.

Further many authors including S. K. Narasimhamurthy ([104], [105], [106]) has derived the condition for Douglas spaces of Finsler spaces with different (α, β)-metrics. Recently B. N. Prasad [[113][114]] gave the condition that Finsler space with (α, β)-metric of Douglas type is conformally transformed to a Douglas space with (α, β)-metric.

2.2 Preliminaries

Let $\alpha(x,y) = \sqrt{a_{ij}y^i y^j}$ be Riemannian metric and $\beta(x,y) = b_i(x)y^i$ be a differential one-form in an n-dimentional differetiable manifold M^n. If the Finsler metric function $L(\alpha, \beta)$ is positively homogeneous of degree one in α and β in M^n, then $F^n = (M^n, L(\alpha, \beta))$ is called a Finsler space with (α, β)-metric[94].

The space $R^n = (M^n, \alpha)$ is called a Riemannian spce associated with F^n and Christoffel symbol of R^n are indicated by $\gamma^i_{jk}(x)$ by ∇.

$$\frac{d^2x^i}{dt^2}y^j - \frac{d^2x^j}{dt^2}y^i + 2(G^i y^j - G^j y^i) = 0, \quad y^i = \frac{dx^i}{dt},$$

in a parameter t. The function $G^i(x,y)$ is given by

$$2G^i(x,y) = g^{ij}(y^r \dot{\partial}_j \partial_r F - \partial_j F) = \gamma^i_{jk} y^j y^k,$$

where $\partial_i = \frac{\partial}{\partial x^i}$, $\quad F = \frac{L^2}{2}$ and $g^{ij}(x,y)$ be the inverse of we use the following symbols :

$$r_{ij} = \frac{1}{2}(\nabla_j b_i + \nabla_i b_j), \;\; s_{ij} = \frac{1}{2}(\nabla_j b_i + \nabla_i b_j), \;\; s^i_j \;\; = \;\; a^{ir} s_{rj}, \quad s_j = b_r s^r_j. \quad (2.2.1)$$

It is to be noted that

$$s_{ij} \;\; = \;\; \frac{1}{2}(\dot{\partial}_j b_i - \dot{\partial}_i b_j).$$

Throughout the paper the symbols ∂_i and $\dot{\partial}_i$ stand for $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial y^i}$ respectively. We are concerned with the Berwald connection BΓ=(G^i_{jk}, G^i_j) which given by

$$2G^i(x,y) = g^{ij}(y^r \dot{\partial}_j \partial_r F - \partial_j F),$$

$$where \; F = \frac{L^2}{2}, \;\; G^i_j = \dot{\partial}_j G^i \quad and \quad G^i_{jk} = \dot{\partial}_k G^i_j.$$

The Finsler space F^n is said to be of Douglas type (Douglas space)[95] if $D^{ij} = G^i y^j - G^j y^i$ are homogeneous polynomial of degree r in y^i, by $hp(r)$.

For a Finsler space F^n with (α,β)-metric ([80],[94]), we have

$$2G^i = \gamma^i_{00} + 2B^i. \tag{2.2.2}$$

where

$$\begin{aligned}
B^i &= \frac{E}{\alpha}y^i + \frac{\alpha L_\beta}{L_\alpha}s^i_0 - \frac{\alpha L_{\alpha\alpha}}{L_\alpha}C^*(\frac{y^i}{\alpha} - \frac{\alpha}{\beta}b^i), \ E = \frac{\beta L_\beta}{L}C^* \\
C^* &= \frac{\alpha\beta(r_{00}L_\alpha - 2\alpha s_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})}, \qquad (2.2.3)\\
\gamma^2 &= b^2\alpha^2 - \beta^2
\end{aligned}$$

and the scripts α and β in L denote the partial differentiation with respect to α and β respectively. Since $\gamma^i_{00} = \gamma^i_{jk}(x)y^i y^j$ is homogeneous polynomial degree two in (y^i), we have [95]:

Lemma 2.2.1. *A Finsler space F^n with an (α,β)-metric is a Douglas space if and only if $B^{ij} = B^i y^j - B^j y^i$ are hp(3). Equation (2.3.3) gives*

$$B^{ij} = \frac{\alpha L_\beta}{L_\alpha}(s^i_0 y^j - s^j_0 y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha}C^*(b^i y^j - b^j y^i). \tag{2.2.4}$$

2.3 β-Conformal change of Douglas type with Finsler (α,β)-metric

This section is devoted to determine the condition for the Finsler space $\overline{F}^n$ which is obtained by β-conformal change of Finsler space F^n with the (α,β)-metric of Douglas type, to be also of Douglas type and vice versa.

For a β-conformal change $\overline{L} = e^\sigma L + \beta$, the associated normalized supporting element is given by:

$$\overline{L_i}(x,y) = e^{\sigma(x)}L_i(x,y) + b_i(x). \tag{2.3.1}$$

Consequently, if we write $L_{ij} = \dot{\partial}_j L_i$, $L_{ijk} = \dot{\partial}_k L_{ij}, \cdots$ etc., we get

$$\overline{L}_{ij}(x, y) = e^{\sigma(x)} L_{ij}(x, y),$$
$$\overline{L}_{ijk}(x, y) = e^{\sigma(x)} L_{ijk}(x, y).$$

We may put

$$\overline{G}^i = G^i + D^i. \tag{2.3.2}$$

Then $\overline{G}^i_j = G^i_j + D^i_j$ and $\overline{G}^i_{jk} + G^i_{jk} + D^i_{jk}$, where $D^i_j = \dot{\partial}_j D^i$ and $D^i_{jk} = \dot{\partial}_k D^i_j$.

The tensors D^i, D^i_j and D^i_{jk}, are positively homogeneous in y^i of degree two, one and zero respectively. S. H. Abed[1] determined the explicit expression for the required D^r as

$$D^r = \frac{1}{2}\left\{Le^{-\sigma}F^r_0 + \frac{L}{\overline{L}}(E_{00} - 2Le^{-\sigma}F_{\beta 0})L^r - L^2\sigma^r + \frac{L}{\overline{L}}(2Le^{\sigma}\sigma_0 + L^2\sigma_\beta)L^r\right\}, \tag{2.3.3}$$

where $\sigma_\beta = \sigma_i b^i$, $\quad F^r_0 = g^{ir}F_{i0}$, $\quad F_{\beta 0} = F_{i0}b^i$ and $L^r = \frac{y^r}{L}$.

Thus, we have the following:

Theorem 2.3.1. *The tensor D^i of (3.2.1) arising from β-conformal change is given by (2.4.3).*

Now from (3.2.1) and (2.4.3), we have

$$\begin{aligned}\overline{G}^i y^j - \overline{G}^j y^i &= G^i y^j - G^j y^i + (2Le^{\sigma}(F^i_0 y^j - F^j_0 y^i) - L^2(\sigma^i y^j - \sigma^j y^i)),\\ &= G^i y^j - G^j y^i + K^{ij},\end{aligned}$$

where $K^{ij} = 2Le^{\sigma}(F^i_0 y^j - F^j_0 y^i) - L^2(\sigma^i y^j - \sigma^j y^i)$. Suppose F^n is a Douglas space, that is, $G^i y^j - G^j y^i$ is $hp(3)$. Thus we state:

Theorem 2.3.2. *Let F^n be a Douglas space and $\overline{F}^n$ a Finsler space which is obtained by β-conformal change, $\overline{F}^n$ is also a Douglas space if and only if K^{ij} are hp(3).*

From (2.3.4), $\overline{B}^{ij} = \overline{B}^i y^j - \overline{B}^j y^i$ in $\overline{F}^n$ are written as

$$\begin{aligned}
\overline{B}^{ij} &= \frac{\alpha L_\beta}{L_\alpha}(s^i_0 y^j - s^j_0 y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha} C^*(b^i y^j - b^j y^i) \\
&+ e^{-\sigma}\left[\frac{\alpha}{L_\alpha}(s^i_0 y^j - s^j_0 y^i) - \frac{\alpha^4 s_0 L_{\alpha\alpha}}{L_\alpha(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})}(b^i y^j - b^j y^i)\right] \\
&+ \left[-\frac{(e^{-\sigma}\alpha^4 L_{\alpha\alpha} + \alpha^4 L_{\alpha\alpha} L_\beta)}{2L_\alpha(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})}(b^2\sigma_0 - \rho\beta) + \frac{\alpha^3 L_{\alpha\alpha}}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})}(\rho\alpha^2 - \sigma_0\beta)\right. \\
&\left. + \frac{\alpha e^{-\sigma} + \alpha L_\beta}{2L_\alpha}\sigma_0\right](b^i y^j - b^j y^i) - \frac{\alpha\beta(L_\beta + e^{-\sigma})}{2L_\alpha}(\sigma^i y^j - \sigma^j y^i), \qquad (2.3.4)
\end{aligned}$$

$$\overline{B}^{ij} = B^{ij} + C^{ij} \qquad (2.3.5)$$

where

$$\begin{aligned}
C^{ij} &= e^{-\sigma}\left[\frac{\alpha}{L_\alpha}(s^i_0 y^j - s^j_0 y^i) - \frac{\alpha^4 s_0 L_{\alpha\alpha}}{L_\alpha(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})}(b^i y^j - b^j y^i)\right] \\
&+ \left[-\frac{(e^{-\sigma}\alpha^4 L_{\alpha\alpha} + \alpha^4 L_{\alpha\alpha} L_\beta)}{2L_\alpha(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})}(b^2\sigma_0 - \rho\beta) + \frac{\alpha^3 L_{\alpha\alpha}}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})}(\rho\alpha^2 - \sigma_0\beta)\right. \\
&\left. + \frac{\alpha e^{-\sigma} + \alpha L_\beta}{2L_\alpha}\sigma_0\right](b^i y^j - b^j y^i) - \frac{\alpha\beta(L_\beta + e^{-\sigma})}{2L_\alpha}(\sigma^i y^j - \sigma^j y^i). \qquad (2.3.6)
\end{aligned}$$

Suppose F^n is a Douglas space. The necessary and sufficient condition for $\overline{F}^n$ to be also a Douglas space is that C^{ij} is hp (3). Thus, we have the following:

Theorem 2.3.3. *Let $F^n = (M^n, L)$ be a Finsler space with an (α, β)-metric of Douglas type, then $\overline{F}^n = (M^n, \overline{L})$ which is obtained by a β-conformal change of F^n is also a Douglas space if and only if C^{ij} is $hp(3)$.*

2.4 β-Conformal change of Douglas type with Finsler (α, β)-metric $L = \alpha - \frac{\beta^2}{\alpha} + \beta$

Let $F^n = (M^n, L = \alpha + \beta)$ be Randers Space and $\overline{F}^n = (M^n, \overline{L} = \overline{\alpha} + \overline{\beta})$, so that $L_\alpha = 1 + \frac{\beta^2}{\alpha}$, $L_{\alpha\alpha} = \frac{-2\beta^2}{\alpha^3}$ and $L_\beta = \frac{-2\beta}{\alpha} + 1$. A Finsler space which is obtained by

β-conformal change of $F^n = (M^n, L)$, we have

$$\overline{B}^{ij} = B^{ij} + C^{ij},$$

where

$$C^{ij} = \alpha^2[\frac{e^{-\sigma}}{\alpha^2+\beta^2}(s^i_0 y^j - s^j_0 y^i) + \frac{(\alpha(1+e^{-\sigma})-2\beta)}{2(\alpha^2+\beta^2)}\{\sigma_0(b^i y^j - b^j y^i) - \beta(\sigma^i y^j - \sigma^j y^i)\}].$$

We know that a Finsler space with Randers metric is Douglas space if and only if $s_{ij} = 0$. Hence C^{ij} reduces to

$$C^{ij} = \frac{\alpha^2[(\alpha(1+e^{-\sigma})-2\beta)}{2(\alpha^2+\beta^2)}\{\sigma_0(b^i y^j - b^j y^i) - \beta(\sigma^i y^j - \sigma^j y^i)\}]. \qquad (2.4.1)$$

Since α is irrational function in y^i, from (2.5.1) it follows that C^{ij} is hp (3) if and only if

$$\sigma_0(b^i y^j - b^j y^i) - \beta(\sigma^i y^j - \sigma^j y^i) = 0, \quad C^{ij} = 0.$$

The first of the above equations may be written as

$$(\sigma_k\delta^j_h + \sigma_h\delta^j_k)b^i - (b_k\delta^j_h + b_h\delta^j_k)\sigma^i - (\sigma_k\delta^i_h + \sigma_h\delta^i_k)b^j + (b_k\delta^i_h + b_h\delta^i_k)\sigma^j = 0. \qquad (2.4.2)$$

Contracting (2.5.2) by j and h, we get $\sigma_k b^i - b_k\sigma^i = 0$, i.e., $b_i\sigma_j - b_j\sigma_i = 0$ which gives $\sigma_i = (\rho/b^2)b^i$.

Conversely if $\sigma_i = (\rho/b^2)b_i$, then $\sigma_0 = (\rho/b^2)\beta$ and (2.5.1) gives $C^{ij} = 0$.

Hence (2.4.5) gives $\overline{B}^{ij} = B^{ij}$. Thus, we state the following:

Theorem 2.4.1. *Let F^n be a Finsler space with Randers metric of Douglas type, then β-conformal change of Rander space is also Douglas space if and only if $\sigma_i = (\rho/b^2)b_i$.*

Theorem 2.4.2. *Let F^n be a Douglas space with Special (α, β)-metric $L = \alpha - \frac{\beta^2}{\alpha} + \beta$, then β-conformal change of Finsler space is also Douglas space.*

Let $\overline{F}^n = (M^n, \overline{L})$ a Finsler space which is obtained by a β-Conformal change of $F^n = (M^n, L = \alpha - \frac{\beta^2}{\alpha} + \beta)$, (2.3.4) gives

$$B^{ij} = \frac{\alpha^2(\alpha - 2\beta)}{\alpha^2 + \beta^2}(s^i_0 y^j - s^j_0 y^i) \tag{2.4.3}$$

$$+\frac{\alpha^2\{r_{00}(\alpha^2 + \beta^2) - 2s_0\alpha^2(\alpha - 2\beta)\}}{(\alpha^2 + \beta^2)\{\alpha^2(1 - 2b^2) + 3\beta^2\}}(b^i y^j - b^j y^i).$$

Suppose that $\overline{F}^n$ be a Douglas space that is B^{ij} be hp(3) separating (2.5.3) in to rational and irrational terms of y^i, we have

$$\{\alpha^2(1 - 2b^2) + 3\beta^2\}\{(\alpha^2 + \beta^2)\overline{B}^{ij} + 2\alpha^2\beta(s^i_0 y^j - s^j_0 y^i)\}\times$$

$$\alpha^2\{r_{00}(\alpha^2 + \beta^2) + 4s_0\alpha^2 k\beta\}(b^i y^j - b^j y^i)$$

$$-\alpha[2s_0\alpha^4(b^i y^j - b^j y^i) + \alpha^2\{\alpha^2(1 - 2b^2) + 3\beta^2\}(s^i_0 y^j - s^j_0 y^i)] = 0.$$

which yeild two equations as follows

$$\{\alpha^2(1 - 2b^2) + 3\beta^2\}\{(\alpha^2 + \beta^2)B^{ij} + 2\alpha^2\beta(s^i_0 y^j - s^j_0 y^i)\} \tag{2.4.4}$$

$$+\alpha^2\{r_{00}(\alpha^2 + \beta^2) + 4s_0\alpha^2\beta\}(b^i y^j - b^j y^i) = 0.$$

$$[2s_0\alpha^2(b^i y^j - b^j y^i) - \{\alpha^2(1 - 2b^2) - 3\beta^2\}(s^i_0 y^j - s^j_0 y^i)] = 0. \tag{2.4.5}$$

Transvecting (2.5.5) by $b_i y_j$, we obtain

$$[2s_0\alpha^2(b^i y^j - b^j y^i) - \{\alpha^2(1 - 2b^2) - 3k\beta^2\}(s^i_0 y^j - s^j_0 y^i)] = 0.$$

which implies $s_0(\alpha^2(\beta^2 - \alpha^2)) = 0$. Therefore we get $s_i = 0$. Hence (2.5.5) is reduced to $s^i_0 y^j - s^j_0 y^i = 0$, and transvection by y_i gives $s^i_0 = 0$. Consequently $s_{ij} = 0$. On the other hand, substituting (2.5.5) in (2.5.4), we have

$$\{\alpha^2(1-2b^2)-3\beta^2\}B^{ij}+\alpha^2\{r_{00}(b^iy^j-b^jy^i)\}=0, \tag{2.4.6}$$

only the terms $3\beta^2B^{ij}$ of (2.5.6) seemingly do not contain α^2. Hence we must have $hp(3)v_3^{ij}$ satisfying

$$3\beta^2B^{ij}=\alpha^2v_3^{ij}. \tag{2.4.7}$$

For the sake of brevity we suppose $\alpha^2\neq 0(mod\beta)$. Then (2.5.6) is reduced to $B^{ij}=\alpha^2v^{ij}$, where v^{ij} are $hp(1)$. Thus (2.5.7) leads to

$$\{\alpha^2(1-2b^2)-3\beta^2\}v^{ij}-\{r_{00}(b^iy^j-b^jy^i)\}=0. \tag{2.4.8}$$

transvecting (2.5.8) by b_iy_j, we get

$$\{\alpha^2(1-2b^2)-3\beta^2\}b_iv^{ij}y_j-r_{00}(b^2\alpha^2-\beta^2)=0,$$

which imply

$$\alpha^2\{(1-2b^2)-3\beta^2\}b_iv^{ij}y_j-b^2r_{00}=\beta^2(3kb_iv^{ij}y_j-r_{00}).$$

Therefore there exists a function $f_1(x)$ satisfying

$$(1-2b^2)b_iv^{ij}y_j-b^2r_{00}=f_1(x)\beta^2, 3b_iv^{ij}y_j-r_{00}=f_1(x)\alpha^2.$$

Eliminating $b_iv^{ij}y_j$ from above the equations, we obtain

$$r_{00}=f_1(x)\frac{(1-2\beta^2)\alpha^2-3\beta^2}{b^2-1}. \tag{2.4.9}$$

From (2.5.9) and $s_{ij}=0$,

$$b_{i;j}=f_2(x)\{(1-b^2)a_ij-3b_ib_j\}, \tag{2.4.10}$$

where $f_2(x)=\frac{f_1(x)}{b^2-1}$.

Conversely, if (2.5.10) is satisfied, then $s_{ij} = 0$ and

$$r_{00} = f_2(x)\{(1 - 2\beta^2)\alpha^2 - 3\beta^2\},$$

from which B^{ij} of (2.5.3) are hp(3). Thus we have the following

Theorem 2.4.3. *A Finsler space $\overline{F}^n (n > 2)$ which is obtained by β-conformal change of Finsler space F^n with an special (α, β)-metric $L = \alpha - \frac{\beta^2}{\alpha} + \beta$ of Douglas type is also Douglas space.*

2.5 β-Conformal change of Douglas type with Finsler (α, β)-metric $L = \alpha + \varepsilon\beta + k\frac{\beta^2}{\alpha}$

Here we consider Finsler $(\alpha, \beta) - metric$, $L = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}$, where ϵ, k are non zero constants, so that $L_\alpha = 1 - \frac{k\beta^2}{\alpha^2}$, $L_{\alpha\alpha} = \frac{2k\beta^2}{\alpha^3}$ and $L_\beta = \frac{2k\beta}{\alpha}$. Hence from (2.4.6) the value of C^{ij} given by

$$C^{ij} = \alpha^2[\frac{e^{-\sigma}}{\alpha^2 - \beta^2}(s^i_0 y^j - s^j_0 y^i) + \frac{(\alpha(\epsilon + e^{-\sigma}) + 2k\beta)}{2(\alpha^2 - \beta^2)}\{\sigma_0(b^i y^j - b^j y^i) - \beta(\sigma^i y^j - \sigma^j y^i)\}].$$

We know that a Finsler space with (α, β)-metric is Douglas space if and only if $s_{ij} = 0$. Hence C^{ij} reduces to

$$C^{ij} = \alpha^2[\frac{(\alpha(1 + e^{-\sigma}) - 2\beta)}{2(\alpha^2 + \beta^2)}\{\sigma_0(b^i y^j - b^j y^i) - \beta(\sigma^i y^j - \sigma^j y^i)\}]. \tag{2.5.1}$$

Since α is irrational function in y^i, from (2.6.1) it follows that C^{ij} is hp (3) if and only if

$$\sigma_0(b^i y^j - b^j y^i) - \beta(\sigma^i y^j - \sigma^j y^i) = 0, \quad C^{ij} = 0.$$

The first of the above equations may be written as

$$(\sigma_k\delta^j_h + \sigma_h\delta^j_k)b^i - (b_k\delta^j_h + b_h\delta^j_k)\sigma^i - (\sigma_k\delta^i_h + \sigma_h\delta^i_k)b^j + (b_k\delta^i_h + b_h\delta^i_k)\sigma^j = 0. \tag{2.5.2}$$

Contracting (2.6.2) by j and h, we get $\sigma_k b^i - b_k \sigma^i = 0$, i.e., $b_i \sigma_j - b_j \sigma_i = 0$ which gives $\sigma_i = (\rho/b^2) b^i$.

Conversely if $\sigma_i = (\rho/b^2) b_i$, then $\sigma_0 = (\rho/b^2)\beta$ and (2.6.1) gives $C^{ij} = 0$. Hence (2.4.5) gives $\overline{B}^{ij} = B^{ij}$. Thus, we state the following:

Theorem 2.5.1. *Let F^n be a Finsler space with Randers metric of Douglas type, then β-conformal change of Randers space is also Douglas space if and only if $\sigma_i = (\rho/b^2) b_i$.*

Theorem 2.5.2. *Let F^n be a Douglas space with Special (α, β)-metric $L = \alpha + \varepsilon\beta + k\frac{\beta^2}{\alpha}$, where ϵ, k are non zero constants, then β-conformal change of Finsler space is also Douglas space.*

From (2.3.4) gives

$$B^{ij} = \frac{\alpha^2(\alpha\epsilon + 2k\beta)}{\alpha^2 - k\beta^2}(s^i_0 y^j - s^j_0 y^i) \tag{2.5.3}$$

$$+\frac{\alpha^2 k\{r_{00}(\alpha^2 - k\beta^2) - 2\alpha^2 s_0 \alpha^2(\alpha\epsilon + 2k\beta)\}}{(\alpha^2 - k\beta^2)\{\alpha^2(1 + 2kb^2) - 3k\beta^2\}}(b^i y^j - b^j y^i).$$

Suppose that $\overline{F}^n$ be a Douglas space that is B^{ij} be hp(3) separating (2.6.3) in to rational and irrational terms of y^i, we have

$$\{\alpha^2(1 + 2kb^2) - 3k\beta^2\}\{(\alpha^2 - k\beta^2)B^{ij} - 2\alpha^2 k\beta(s^i_0 y^j - s^j_0 y^i)\}$$

$$\alpha^2 k\{r_{00}(\alpha^2 - k\beta^2) - 4s_0\alpha^2 k\beta\}(b^i y^j - b^j y^i)$$

$$+\alpha[2s_0\alpha^4(b^i y^j - b^j y^i) - \alpha^2\{\alpha^2(1 + 2kb^2) - 3k\beta^2\}(s^i_0 y^j - s^j_0 y^i)] = 0.$$

which yeild two equations as follows

$$\{\alpha^2(1 + 2kb^2) - 3k\beta^2\}\{(\alpha^2 - k\beta^2)B^{ij} - 2\alpha^2 k\beta(s^i_0 y^j - s^j_0 y^i)\} \tag{2.5.4}$$

$$-\alpha^2k\{r_{00}(\alpha^2-k\beta^2)-4s_0\alpha^2k\beta\}(b^iy^j-b^jy^i)=0.$$

$$[2s_0\alpha^2(b^iy^j-b^jy^i)-\{\alpha^2(1+2kb^2)-3k\beta^2\}(s^i_0y^j-s^j_0y^i)]=0. \tag{2.5.5}$$

Transvecting (2.6.5) by b_iy_j, we obtain

$$[2s_0\alpha^2(b^iy^j-b^jy^i)-\{\alpha^2(1+2kb^2)-3k\beta^2\}(s^i_0y^j-s^j_0y^i)]=0.$$

which implies $s_0(\alpha^2(\beta^2-\alpha^2))=0$. Therefore we get $s_i=0$. Hence (2.6.5) is reduced to $s^i_0y^j-s^j_0y^i=0$, and transvection by y_i gives $s^i_0=0$. Consequently $s_{ij}=0$. On the other hand, substituting (2.6.5) in (2.6.4),

we have

$$\{\alpha^2(1+2kb^2)-3k\beta^2\}B^{ij}-\alpha^2\{r_{00}(b^iy^j-b^jy^i)\}=0. \tag{2.5.6}$$

only the terms $3k\beta^2B^{ij}$ of (2.6.6) seemingly do not contain α^2. Hence we must have $hp(3)v^{ij}_3$ satisfying

$$3k\beta^2\overline{B}^{ij}=\alpha^2v^{ij}_3. \tag{2.5.7}$$

For the sake of brevity we suppose $\alpha^2\neq 0(mpd\beta)$. Then (2.6.6) is reduced to

$B^{ij}=\alpha^2v^{ij}$, where v^{ij} are $hp(1)$. Thus (2.6.6) leads to

$$\{\alpha^2(1+2kb^2)-3k\beta^2\}v^{ij}-\{r_{00}(b^iy^j-b^jy^i)\}=0. \tag{2.5.8}$$

Transvecting (2.6.8) by b_iy_j, we get

$$\{\alpha^2(1+2kb^2)-3k\beta^2\}b_iv^{ij}y_j-r_{00}(b^2\alpha^2-\beta^2)=0.$$

which imply

$$\alpha^2\{(1+2kb^2)-3k\beta^2\}b_iv^{ij}y_j-b^2r_{00}=\beta^2(3kb_iv^{ij}y_j-r_{00}).$$

Therefore there exists a function $f_1(x)$ satisfying

$$(1+2kb^2)b_iv^{ij}y_j - b^2r_{00} = f_1(x)\beta^2, 3kb_iv^{ij}y_j - r_{00} = f_1(x)\alpha^2.$$

Eliminating $b_iv^{ij}y_j$ from above the equations, we obtain

$$r_{00} = f_1(x)\frac{(1+2k\beta^2)\alpha^2 - 3k\beta^2}{b^2-1}. \tag{2.5.9}$$

From (2.6.9) and $s_{ij} = 0$.

$$b_{i;j} = f_2(x)\{(1+2kb^2)a_{ij} - 3b_ib_j\}, \tag{2.5.10}$$

where $f_2(x) = \frac{f_1(x)}{b^2-1}$.

Conversely if (2.6.10) is satisfied, then $s_{ij} = 0$ and

$$r_{00} = f_2(x)\{(1+2k\beta^2)\alpha^2 - 3k\beta^2\},$$

from which B^{ij} of (2.6.3) are hp(3). Thus we have the following

Theorem 2.5.3. *A Finsler space $\overline{F}^n(n>2)$ which is obtained by a β-conformal change of Finsler space F^n with an special (α,β)-metric $L = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}(b^2 \neq 1)$ of Douglas type, is also Douglas space.*

2.6 Finsler space with Second Approximate Matsumato metric of Berwald type

In the present section, we find the condition that a Finsler space F^n with a Second Approximate Matsumato metric

$$L = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2} \tag{2.6.1}$$

A Finsler space is called Berwald space if the Berwald connection $B\Gamma = (G^i_{jk}, G^i_j, 0)$ is linear. In [145], the function G^i of a Finsler space with an (α, β)-metric are given by $2G^i = \gamma^i_{00} + 2B^i$, then we have $G^i_j = \gamma^i_{0j} + B^i_j$ and $G^i_{jk} = \gamma^i_{jk} + B^i_{jk}$, where $B^i_{jk} = \dot{\partial}_k B^i_j$ and $B^i_j = \dot{\partial}_j B^i$. Thus a Finsler space with an (α, β)-metric is a Berwald space iff $G^i_{jk} = G^i_{jk}(x)$ equivalently $B^i_{jk} = B^i_{jk}(x)$. Moreover on account of [94] B^i_j is determined by

$$L_\alpha B^t_{ji} y^j y_t + \alpha L_\beta (B^t_{ji} b_t - b_{j;i}) y^j = 0 \tag{2.6.2}$$

where $y_k = a_{ik} y^i$. For the special (α, β)-metric (2.7.1) we have,

$$L_\alpha = 1 - \frac{\beta^2}{\alpha^2} - \frac{2\beta^3}{\alpha^3}, \quad L_\beta = 1 + \frac{2\beta}{\alpha} + \frac{3\beta^2}{\alpha^2}, \quad L_{\alpha\alpha} = \frac{2\beta^2}{\alpha^3} + \frac{6\beta^3}{\alpha^4}, \quad L_{\beta\beta} = \frac{2}{\alpha} + \frac{6\beta}{\alpha^2}. \tag{2.6.3}$$

Substituting (2.7.3) in (2.7.2) equation, we have

$$(\alpha^3 - \alpha\beta^2 - 2\beta^3) B^t_{ji} y^j y_t + \alpha^2 (\alpha^2 + 2\alpha\beta + 3\beta^2)(B^t_{ji} b_t - b_{j;i}) y^j = 0. \tag{2.6.4}$$

Assume that F^n is a Berwald space, i.e., $B^i_{jk} = B^i_{jk}(x)$. Separating (2.7.4) in rational and irrational terms of y^i as

$$\begin{aligned} &(\alpha^3 - \alpha\beta^2 - 2\beta^3) B^t_{ji} y^j y_t + \alpha^4 (B^t_{ji} b_t - b_{j;i}) y^j + 2\alpha^3 \beta (B^t_{ji} b_t - b_{j;i}) y^j \\ &+ 3\alpha^2 \beta^2 (B^t_{ji} b_t - b_{j;i}) y^j = 0, \end{aligned} \tag{2.6.5}$$

which yields two equations

$$(\alpha^3 - \alpha\beta^2 - 2\beta^3) B^t_{ji} y^j y_t + \alpha^4 (B^t_{ji} b_t - b_{j;i}) y^j + 2\alpha^3 \beta (B^t_{ji} b_t - b_{j;i}) y^j, \tag{2.6.6}$$

and

$$(B^t_{ji} b_t - b_{j;i}) y^j = 0. \tag{2.6.7}$$

Substituting (2.7.7) in (2.7.6), we have

$$(\alpha^3 - \alpha\beta^2 - 2\beta^3) B^t_{ji} y^j y_t = 0. \tag{2.6.8}$$

Case(i): *If $B^t_{ji} y^j y_t = 0$, we have*

$$B^t_{ji} a_{th} + B^t_{hi} a_{tj} = 0 \text{ and } B^t_{ji} b_t - b_{j;i} = 0. \tag{2.6.9}$$

Thus we obtain $B^t_{ji} = 0$ by Christoffel process in the first equation of (2.7.9) and from second of (2.7.9), we have $b_{i;j} = 0$.

Case(ii): *If $(\alpha^3 - \alpha\beta^2 - 2\beta^3) = 0$,*

$\Rightarrow \alpha$ is a one form, which is a contradiction.

Conversly, if $b_{i;j} = 0$, then $B^t_{ji} = 0$ are uniquely determined from (2.7.4).

Hence, we conclude the following:

Theorem 2.6.1. *A Finsler space with a special (α, β)-metric $L = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$ is a Berwald space iff $b_{i;j} = 0$.*

2.7 Finsler space with Second Approximate Matsumato metric of Douglas type

In this section, we find the condition for a Finsler space F^n with a Second Approximate Matsumato metric $L = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$, to be Douglas type.

For a Finsler space F^n with a Second Approximate Matsumato metric $L = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$, then equation (2.3.4), becomes

$$\{\alpha^3(1+2b^2) + \beta^2(-3\alpha - 2 - 6\beta) + 6b^2\alpha^2\beta\}\{(\alpha^3 - \alpha\beta^2 - 2\beta^3)B^{ij} - \beta(2\alpha + 3\beta) \times (s^i_0 y^j - s^j_0 y^i)\} - \alpha^2(\alpha + 3\beta)\{r_{00}(\alpha^3 - \alpha\beta^2 - 2\beta^3) - 2\alpha^2\beta^2(2\alpha - 3\beta)\} \times (b^i y^j - b^j y^i) = 0. \tag{2.7.1}$$

Suppose that F^n is a Douglas space, that is, B^{ij} are hp(3). Arranging the rational and irrational terms, equation (2.8.1) can be written as

$$\{\alpha^3(1+2b^2)+\beta^2(-3\alpha-2-6\beta)+6b^2\alpha^2\beta\}\{(\alpha^3-\alpha\beta^2-2\beta^3)B^{ij}-\beta(2\alpha+3\beta)$$
$$(s_0^iy^j-s_0^jy^i)\}-\alpha^2(\alpha+3\beta)\{r_{00}(\alpha^3-\alpha\beta^2-2\beta^3)-2\alpha^2\beta^2(2\alpha-3\beta)\}(b^iy^j-b^jy^i)$$

$$+\alpha^2[2s_0\alpha^4(\alpha+3\beta)(b^iy^j-b^jy^i)-\alpha^2(\alpha+3\beta)\{\alpha^3(1+2b^2)+\beta^2(-3\alpha-2-6\beta)$$
$$+6b^2\alpha^2\beta\}(s_0^iy^j-s_0^jy^i)]=0. \tag{2.7.2}$$

Separating rational and irrational terms of y^i in (2.8.2) we have the following two equations

$$\{\alpha^3(1+2b^2)+\beta^2(-3\alpha-2-6\beta)+6b^2\alpha^2\beta\}\{(\alpha^3-\alpha\beta^2-2\beta^3)B^{ij}-\beta(2\alpha+3\beta)$$
$$(s_0^iy^j-s_0^jy^i)\}-\alpha^2(\alpha+3\beta)\{r_{00}(\alpha^3-\alpha\beta^2-2\beta^3)-2\alpha^2\beta^2(2\alpha-3\beta)\}\times$$
$$(b^iy^j-b^jy^i), \tag{2.7.3}$$

and

$$2s_0\alpha^2(\alpha+3\beta)(b^iy^j-b^jy^i)-(\alpha+3\beta)\{\alpha^3(1+2b^2)$$
$$+\beta^2(-3\alpha-2-6\beta)+6b^2\alpha^2\beta\}(s_0^iy^j-s_0^jy^i)=0. \tag{2.7.4}$$

Substituting (2.8.4) in (2.8.3), we have

$$\{\alpha^3(1+2b^2)+\beta^2(-3\alpha-2-6\beta)+6b^2\alpha^2\beta\}\{(\alpha^3-\alpha\beta^2-2\beta^3)B^{ij}$$
$$-\alpha^2(\alpha+3\beta)r_{00}(\alpha^3-\alpha\beta^2-2\beta^3)(b^iy^j-b^jy^i)=0. \tag{2.7.5}$$

Only the term $4\beta^5B^{ij}$ of (2.8.5) does not contain α^2. Hence we must have hp(6) v_6^{ij} satisfying

$$4\beta^5B^{ij}=\alpha^2v_6^{ij}. \tag{2.7.6}$$

Now we study the following two cases:

Case(i): $\alpha^2\neq 0(mod\beta)$

In this case, (2.8.6) is reduced to $B^{ij} = \alpha^2 v^{ij}$ are hp(1). Thus (2.8.5) gives

$$\alpha^3(1 + 2b^2) - \beta^2(-3\alpha - 2 - 6\beta)B^{ij} - r_{00}(b^i y^j - b^j y^i) = 0. \qquad (2.7.7)$$

Transvecting this by $b_i y_j$, where $y_j = a_{jk}y^k$, we have

$$\alpha^3(1 + 2b^2)v^{ij}b_i y_j - b^2 r_{00} = \beta^2(r_{00} - 8v^{ij}b_i y_j). \qquad (2.7.8)$$

Since $\alpha^2 \neq 0(mod\beta)$ there exist a function $h(x)$ satisfying

$$(1 + 2b^2)v^{ij}b_i y_j - b^2 r_{00} = h(x), \beta^2(r_{00} - 8v^{ij}b_i y_j) = h(x)\alpha^2.$$

Eliminating $v^{ij}b_i y_j$ from the above two equations, we obtain

$$(1 + b^2)r_{00} = h(x)\{(1 + 2b^2)\alpha^2 - 8\beta^2\}, \qquad (2.7.9)$$

from (2.8.9), we get

$$b_{i:j} = k\{(1 + 2b^2)a_{ij} - 3b_i b_j\}, \qquad (2.7.10)$$

where $k = \frac{h(x)}{(1+b^2)}$. Hence, b^i is a gradient vector.

Conversely, if (2.8.10) holds, then $s_{ij} = 0$ and we get (2.8.9). Therefore, (2.8.3) is written as follows:

$$B^{ij} = k\{\alpha^2(b^i y^j - b^j y^i)\},$$

which are hp(3), that is, F^n is a Douglas space.

Case(ii): $\alpha^2 = 0(mod\beta)$.

Consider the following lemma,

Lemma 2.7.1. *[145] If $\alpha^2 = 0(mod\beta)$, that is, $a_{ij}(x)y^i y^j$ contains $b_i y^i$ as a factor, then the dimention n is equal to 2 and b^2 vanishes. In this case we have 1-form $\delta = d_i(x)y^i$ satisfying $\alpha^2 = \beta\delta$ and $d_i b^i = 2$.*

The equation (2.8.6) is reduced to $B^{ij} = \delta w_2^{ij}$, *where* w_2^{ij} *are* $hp(2)$.

Hence, the equation (2.8.4) leads to

$$2s_0\delta(b^i y^j - b^j y^i) - (\delta - 3\beta)(s_0^i y^j - s_0^j y^i) = 0. \tag{2.7.11}$$

Transvecting the above equation by $y_i b_j$, *we have* $s_0 = 0$. *Substituting* $s_0 = 0$ *in the above equation, we have* $s_{ij} = 0$. *Therefore, (2.8.7) reduces to*

$$(\delta - 3\beta)w_2^{ij} b_i y_j - r_{00}\beta^2 = 0,$$

which is written as

$$\delta w_2^{ij} b_i y_j = \beta(\beta r_{00} - 3w_2^{ij} b_i y_j).$$

Therefore, there exists an $hp(2)$, $\lambda = \lambda_{ij}(x)y^i y^j$ *such that*

$$w_2^{ij} b_i y_j = \beta\lambda, \beta r_{00} + 3w_2^{ij} b_i y_j = \delta\lambda.$$

Eliminating $w_2^{ij} b_i y_j$ *from the above equations, we get*

$$\beta r_{00} = 3\beta\lambda - \delta\lambda = \lambda(3\beta - \delta), \tag{2.7.12}$$

which implies there exists an hp(1), $v_0 = v_i(x)y^i$ *such that*

$$r_{00} = v_0(3\beta - \delta) = v_0\beta. \tag{2.7.13}$$

From r_{00} *given by (2.8.13) and* $s_{ij} = 0$, *we get*

$$b_{i:j} = \frac{1}{2}\{v_i(3b_j + d_j) + v_j(3b_i + d_i)\}. \tag{2.7.14}$$

Hence b_i *is gradient vector.*

Conversely, if (2.8.14) holds, then $s_{ij} = 0$, *which implies* $r_{00} = v_0(3\beta + \delta)$. *Therefore, (2.8.3) is written as follows:*

$$B^{ij} = v_0\delta(b^i y^j - b^j y^i),$$

which are hp(3). Therefore, F^n is a Douglas space.

Thus, we have

Theorem 2.7.2. *A Finsler space with (α, β)-metric $L = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$ is a Douglas space if and only if*

either

$$i) \quad \alpha^2 \neq 0(mod\beta),\ b^2 \neq \frac{1}{k} : b_{i;j} \text{ is written in the form } (2.8.10), or$$

$$ii) \quad \alpha^2 = 0(mod\beta) : n = 2 \text{ and } b_{i;j} \text{ is written in the form } (2.8.14)$$

$$\text{where } \alpha^2 = \beta\delta,\ \delta = d_i(x)y^i,\ v_o = v_i(x)y^i.$$

2.8 Conformal Kropina change of Finsler spaces with (α, β)-metric of Douglas type

Since, $\overline{L} = e^\sigma L(\alpha, \beta)$, is equivalent to $\overline{L} = L(e^\sigma \alpha, e^\sigma \beta)$ by homogeneity. Therefore, a conformal change of (α, β)-metric is expressed as $(\alpha, \beta) \rightarrow (\overline{\alpha}, \overline{\beta})$, where $\overline{\alpha} = e^\sigma \alpha$, $\overline{\beta} = e^\sigma \beta$, we have

$$\overline{y}^i = y^i,\ \overline{y}_i = e^{-2\sigma} y_i,\ \overline{a}_{ij} = e^{2\sigma} a_{ij}, \quad \overline{b}_i = e^\sigma b_i$$

$$, \quad \overline{a}_{ij} = e^{-2\sigma} a^{ij}, \quad \overline{b}^i = e^{-\sigma} b^i \text{ and } \overline{b}^2 = b^2. \tag{2.8.1}$$

Therefore we have

Proposition 2.8.1. *In a Finsler space with (α, β)-metric the length b of b_i with respect to the Riemannian metric α is invariant under any conformal change of metric.*

From (2.9.1), it follows that the conformal change of Christoffel symbols is given by

$$\overline{\gamma}^i_{jk} = \gamma^i_{jk} + \delta^i_j \sigma_k + \delta^i_k \sigma_j - \sigma^i a_{jk}, \tag{2.8.2}$$

where $\sigma_j = \partial_j \sigma$ and $\sigma^i = a^{ij}\sigma_j$.

From (2.3.1) (2.9.1) and (2.9.2), we have the following identities:

$$
\begin{aligned}
\overline{\nabla}_j \bar{b}_i &= e^{\sigma}(\nabla_j b_i + \rho a_{ij} - \sigma_i b_j),\\
\bar{r}_{ij} &= e^{\sigma}[r_{ij} + \rho a_{ij} - \frac{1}{2}(b_i\sigma_j + b_j\sigma_i)], \quad \bar{s}_{ij} = e^{\sigma}[s_{ij} + \frac{1}{2}(b_i\sigma_j - b_j\sigma_i)], \\
\bar{s}^i_j &= e^{-\sigma}[s^i_j + \frac{1}{2}(b^i\sigma_j - b_j\sigma^i)], \qquad \bar{s}_j = s_j + \frac{1}{2}(b^2\sigma_j - \rho b_j),
\end{aligned} \tag{2.8.3}
$$

where $\rho = \sigma_r b^r$.

From (2.9.2) and (2.9.3), we can easily obtain the following:

$$
\begin{aligned}
\bar{\gamma}^i_{00} &= \gamma^i_{00} + 2\sigma_0 y^i - \alpha^2\sigma_j, \qquad \bar{r}_{00} = e^{\sigma}(r_{00} + \rho\alpha^2 - \sigma_0\beta), \\
\bar{s}^i_0 &= e^{-\sigma}[s^i_0 + \frac{1}{2}(\sigma s_0 b^i - \beta\sigma^i)], \quad \bar{s}_0 = s_0 + \frac{1}{2}(\sigma_0 b^i - \rho\beta).
\end{aligned} \tag{2.8.4}
$$

To find the conformal Kropina change of B^{ij} given in (2.3.4), we find the conformal Kropina change of C^* given in (2.3.3).

Since $\overline{L}(\overline{\alpha}, \overline{\beta}) = e^{\sigma}[\frac{L^2(\alpha,\beta)}{\beta}]$, we have

$$
\overline{L}_{\overline{\alpha}} = \frac{2L}{\beta}, L_{\alpha}, \quad \overline{L}_{\overline{\alpha}\ \overline{\alpha}} = e^{-\sigma}\frac{2}{\beta}[LL_{\alpha\alpha} + (L_{\alpha})^2], \quad \overline{L}_{\overline{\beta}} = \frac{2\beta LL_{\beta} - L^2}{\beta^2}, \quad \overline{\gamma}^2 = e^{2\sigma}\gamma^2. \tag{2.8.5}
$$

from (2.3.3), (2.9.4), (2.9.5), we have

$$
\overline{C}^* = e^{\sigma}(C^* + D^*), \tag{2.8.6}
$$

where

$$
\begin{aligned}
D^* = {} & \frac{\alpha L(\beta^2 L_{\alpha} + \alpha\gamma^2 L_{\alpha\alpha})[\beta(\rho\alpha^2 - \sigma_0\beta)L_{\alpha} - \alpha\beta(b^2\sigma_0 - \rho\beta)L_{\beta} +}{2(\beta^2 L_{\alpha} + \alpha r^2 L_{\alpha\alpha})} \\
& \frac{\alpha L s_0 + \frac{1}{2}(b^2\sigma_0 - \rho\beta))] - \alpha^2\beta\gamma^2(L_{\alpha})^2(r_{00}L_{\alpha} - 2\alpha s_0 L_{\beta})}{}.
\end{aligned} \tag{2.8.7}
$$

Hence the conformal Kropina change B^{ij} is written in the form

$$
\overline{B}^{ij} = B^{ij} + C^{ij}, \tag{2.8.8}
$$

where

$$C^{ij} = \frac{\alpha L(2\beta L_\beta - L)\sigma_0(b^iy^j - b^jy^i) - \beta(\sigma^iy^j - \sigma^jy^i) - 2\alpha L^2(s^i_0y^j - s^j_0y^i)+}{4\beta L L\alpha}$$
$$\frac{4\alpha^2(L\alpha)^2C^* + [LL_{\alpha\alpha} + (L\alpha)^2]D^*(b^iy^j - b^jy^i)}{} \tag{2.8.9}$$

Suppose F^n is a Douglas space. The necessary and sufficient condition for $\overline{F}^n$ to be also a Douglas space is that C^{ij} is hp(3). Thus, we have the following:

Theorem 2.8.2. *A Douglas space with (α,β)-metric is transformed to a Douglas space with (α,β)-metric under conformal Kropina change if and only if C^{ij} defined in equation (2.9.9) is hp(3).*

In the following three sections we deal with conformal Kropina change of Finsler spaces with three special (α,β)-metric.

2.8.1 Conformal Kropina change of Finsler spaces with Matsumato metric of Douglas type $L = \frac{\alpha^2}{\alpha-\beta}$

For an (α,β)-metric, we have $L = \frac{\alpha^2}{\alpha-\beta}$

$$L_\alpha = \frac{\alpha(\alpha-2\beta)}{(\alpha-\beta)^2},\ \ L_\beta = \frac{\alpha^2}{(\alpha-\beta)^2},\ \ L_{\alpha\alpha} = \frac{2b^2}{(\alpha-\beta)^3}.$$

Hence the values of C^*, D^* and C^{ij} given by equation (2.3.3), (2.9.7) and (2.9.9) respectively reduce to

$$C^* = -\frac{(\alpha-\beta)\{(-\alpha+2\beta)r_{00} + 2\alpha^2s_0\}}{2\beta(\alpha-3\beta+2\alpha b^2)}$$

$$D^* = \frac{(\alpha-\beta)\{\alpha\beta(\alpha-3\beta+2\alpha b^2)(\alpha^3b^2 - 3\alpha^2\beta b^2 - 2\alpha\beta^2 + 4\beta^3)\sigma_0}{4\beta(\alpha-3\beta+2\alpha b^2)(\alpha\beta^3 + 6b^2\alpha^2\beta^2 + b^2\alpha^4 - 4b^2\alpha^3\beta - 4\beta^4)}$$
$$\frac{+\alpha^3\beta^2(\alpha-\beta)(\alpha-3\beta+2\alpha b^2)\rho + 2\alpha^2(-6b^2\alpha^2\beta(\alpha-\beta) + \alpha^3\beta}{}$$
$$\frac{-6\alpha^2\beta^2 + 11\alpha\beta^3 + 2b^2\alpha^4 - 8\beta^4)s_0 - 2(\alpha-2\beta^3)(b\alpha-\beta)(b\alpha+\beta)r_{00}}{}$$

$$C^{ij} = \frac{\alpha\{[-\alpha\beta^3(\alpha-\beta)(12\beta^2-9\alpha\beta+2\alpha^2)(2\alpha\beta+\alpha-3\beta)\sigma_0}{4\beta^2(\alpha-2\beta)(\alpha-3\beta+2\alpha b^2)(\alpha\beta^3+6b^2\alpha^2\beta^2+b^2\alpha^4-4b^2\alpha^3\beta-4\beta^4)}$$
$$\underline{+\alpha^3\beta^2(\alpha-\beta)(\alpha^2+6\beta^2-4\alpha\beta)(\alpha-3\beta+2b^2\alpha)\rho}$$
$$\underline{+2(\alpha-2\beta)^3\beta^2(\alpha^2-4\alpha\beta+2\beta^2)r_{00}]}(b^iy^j-b^jy^i)$$
$$\underline{+[\alpha\beta^2(\alpha-3\beta)(\alpha-3\beta+2b^2\alpha)(b^2\alpha^4+6b^2\alpha^2\beta^2}$$
$$\underline{-4b^2\alpha^3\beta+\alpha\beta^3-4\beta^4)]}(\sigma^iy^j-\sigma^jy^i)\} \qquad (2.8.10)$$

The equation (2.9.10) can be written as

$$4\beta^2(\alpha-2\beta)(\alpha-3\beta+2\alpha b^2)(\alpha\beta^3+6b^2\alpha^2\beta^2+b^2\alpha^4-4b^2\alpha^3\beta-4\beta^4)C^{ij}$$
$$+\alpha\{[-\alpha\beta^3(\alpha-\beta)(12\beta^2-9\alpha\beta+2\alpha^2)(2\alpha\beta+\alpha-3\beta)\sigma_0+\alpha^3\beta^2(\alpha-\beta)\times$$
$$(\alpha^2+6\beta^2-4\alpha\beta)(\alpha-3\beta+2b^2\alpha)\rho+2(\alpha-2\beta)^3\beta^2(\alpha^2-4\alpha\beta+2\beta^2)r_{00}]\times$$
$$(b^iy^j-b^jy^i)+[\alpha\beta^2(\alpha-3\beta)(\alpha-3\beta+2b^2\alpha)(b^2\alpha^4+6b^2\alpha^2\beta^2-4b^2\alpha^3\beta+$$
$$\alpha\beta^3-4\beta^4)](\sigma^iy^j-\sigma^jy^i)\}=0. \qquad (2.8.11)$$

Since α is an irrational function in y^i, the equation (2.9.11) gives rise to two equations as follows:

$$(-96\beta^8-208b^2\alpha^3\beta^5-96b^4\alpha^3\beta^5+64b^2\alpha\beta^7+96b^2\alpha^2\beta^6+4\alpha^3\beta^5-36\alpha^2\beta^6+104\alpha\beta^7)C^{ij}$$
$$-\alpha\beta^4\{[-\alpha(22b^2\alpha^3-42b^2\alpha^2\beta-54\alpha^2\beta+75\alpha\beta^2+17\alpha^3-36\beta^3)\sigma_0-6\alpha^3\beta^5(-3\beta$$
$$+2b^2\alpha+6\alpha)\rho+(20\alpha^3-72\alpha^2\beta+112\alpha\beta^2-84\alpha^3+184\alpha^2\beta-208\alpha\beta^2-64\beta^3$$
$$+96\beta^3)r_{00}](b^iy^j-b^jy^i)+\alpha^2\beta^2[\alpha^2b^2(40\beta^4+\alpha^4+2b^2\alpha^4)-\alpha^3\beta^3(-36b^4+70b^2+1)$$
$$-10\alpha^2\beta^4+33\alpha\beta^5+\alpha^3\beta^3-36\beta^6](\sigma^iy^j-\sigma^jy^i)\}=0, \qquad (2.8.12)$$

and

$$4b^2\beta^2(28b^2\beta^2+32\beta^2-12b^2\alpha\beta-9\alpha\beta+\alpha^2+2b^2\alpha^2)C^{ij}-[2\beta^3(\alpha^3+2b^2\alpha^3-12b^2\beta^3)\sigma_0$$
$$-\alpha^3\beta^2(-\beta^2+\alpha^2-8\alpha\beta+2b^2\alpha^2-10b^2\alpha\beta+20b^2\beta^2)\rho-2\alpha^2(\alpha\beta^2-9\beta^3)r_{00}](b^iy^j-b^jy^i)$$
$$[+10b^2\alpha^4\beta^3-39b^2\alpha^3\beta^4-25-36b^4\alpha^3\beta^4+14b^4\alpha^4\beta^3-24b^2\beta^7](\sigma^iy^j-\sigma^jy^i)=0 \qquad (2.8.13)$$

Take $n > 2$, $\alpha^2 \neq 0(mod\beta)$. The terms β of (2.9.13) which seemingly do not contain α^2 are $\beta^6\sigma_0(b^i y^j - b^j y^i) + \beta^7(\sigma^i y^j - \sigma^j y^i)$. Hence we must have hp(0), $M^{ij}(x)$ such that the above expression is equal to $\alpha^2\beta^3 M^{ij}(x)$. Therefore we have

$$\sigma_0(b^i y^j - b^j y^i) + \beta(\sigma^i y^j - \sigma^j y^i) = \alpha^2 M^{ij}(x). \tag{2.8.14}$$

The equation (2.9.14) can be written as

$$[(\sigma_h\delta_k^j + \sigma_k\delta_h^j)b^i - (\sigma_k\delta_h^i + \sigma_h\delta_k^i)b^j] + [(b_h\delta_k^j + b_k\delta_h^j)\sigma^i - (b_h\delta_k^i + b_k\delta_h^i)\sigma^j]$$
$$= a_{hk}M^{ij}. \tag{2.8.15}$$

Contracting (2.9.15) by $j = h$, we get

$$n(b_k\sigma^i - b^i\sigma_k) = M_k^i$$

which implies

$$M_{ij}(x) = n(b_j\sigma_i - b_i\sigma_j). \tag{2.8.16}$$

Theorem 2.8.3. *A Finsler space $\overline{F}^n$ $(n > 2)$ which is obtained by conformal Kropina change of a Kropina space F^n with (α, β)-metric $L = \frac{\alpha^2}{\alpha - \beta}$ is of Douglas type if and only if (2.9.16) is satisfied.*

2.8.2 Conformal Kropina change of Finsler spaces with (α, β)-metric $L = \alpha - \frac{\beta^2}{\alpha} + \beta$ is of Douglas type

For an (α, β)-metric, we have $L = \alpha - \frac{\beta^2}{\alpha} + \beta$

$$L_\alpha = 1 + \frac{\beta^2}{\alpha^2}, \quad L_\beta = \frac{-2\beta}{\alpha} + 1, \quad L_{\alpha\alpha} = \frac{-2\beta^2}{\alpha^3}$$

Hence the values of C^*, D^* and C^{ij} given by equation (2.3.3), (2.9.7) and (2.9.9) respectively reduce to

$$C^* = \frac{\alpha(-r_{00}\alpha^2 - r_{00}\beta^2 - 4\alpha^2 s_0\beta + 2\alpha^3 s_0)}{2\beta(-\alpha^2 - 3\beta^2 + 2b^2\alpha^2)}$$

$$D^* = \frac{\begin{gathered}(-8\alpha^6\beta^4b^2-\alpha^9\beta b^2-6\alpha^5\beta^5b^4-16\alpha^4\beta^6b^2-6\alpha\beta^9+13\alpha^3\beta 67b^2+8\alpha^6\beta^4b^4+2\alpha^7\beta^3b^4\\ +2\alpha^9\beta b^4+6\alpha^2\beta^8-2\alpha^3\beta^7+2\alpha^7\beta^3+8\alpha^4\beta^6+2\alpha^6\beta^4+6\alpha^5\beta^5-8\alpha^7\beta^3b^2)\sigma_0\\ +(2\alpha^9\beta^2b^2+4\alpha^8\beta^3b^2-2\alpha^7\beta^4b^2+6\alpha^4\beta^7-4\alpha^6\beta^5b^2+2\alpha^5\beta^6b^2-\alpha^9\beta^2-4\alpha^6\beta^5\\ -2\alpha^7\beta^4-2\alpha^8\beta^3-3\alpha^3\beta^8+2\alpha^5\beta^6)\rho+(8\alpha^8\beta^2b^2-4\alpha^7\beta^3b^2-8\alpha^6\beta^4b^2+4\alpha^5\beta^5b^2\\ -16\alpha^5\beta^5+8\alpha^6\beta^4-8\alpha^3\beta^7+4\alpha^4\beta^6+4\alpha^8\beta^2-4\alpha^10b^2-8\alpha^7\beta^3+4\alpha^9\beta b^2-6\alpha^3\beta^7\\ -4\alpha^8\beta^2-2\alpha^9\beta-4\alpha^7\beta^3-8\alpha^6\beta^4+8\alpha^9\beta b^2+12\alpha^4\beta^6+4\alpha^5\beta^5-8\alpha^8\beta^2b^2\\ +16\alpha^7\beta^3b^2-4\alpha^6\beta^4b^2+8\alpha^5\beta^3b^2)s_0+(-6\alpha^3\beta^6+2\alpha^9\beta^2-2\alpha^7\beta^2-6\alpha^5\beta^4\\ +6\alpha^7\beta^2b^2+6\alpha^5\beta^4b^2+2\alpha^3\beta^6b^2-2\alpha\beta^8)r_{00}\end{gathered}}{4b(-\alpha^2-3\beta^2+2\alpha^2b^2)[b^2\alpha^3(\alpha^3-\alpha+1)+\alpha\beta^4(2\beta+1)-3\beta^6-b^2\alpha^2\beta^3(2\alpha-3\beta)]}. \tag{2.8.17}$$

and

$$\begin{aligned}&[\beta^4(-2\alpha+\beta^4-5\alpha^2\beta^2+2\alpha^4b^2)+\alpha^3(5\beta^5b^2-4\alpha^2\beta^3b^2-6\beta^5+12\alpha^4\beta b^4+6\alpha^3\beta^2b^4)]4\beta\\ &(\alpha^4-\beta^4+\alpha^3\beta+\alpha\beta^3)C^{ij}=[\beta^4(-2\alpha+\beta^4-5\alpha^2\beta^2+2\alpha^4b^2)+\alpha^3(5\beta^5b^2-4\alpha^2\beta^3b^2\\ &-6\beta^5+12\alpha^4\beta b^4+6\alpha^3\beta^2b^4+12\alpha^5b^4-12\alpha^3\beta^3b^2)(-2\alpha(3\alpha^2+2\alpha\beta+\frac{2\alpha^3}{\beta}+b^2+\frac{\alpha^4}{\beta^2})]\\ &\times(s^i_0y^j-s^j_0y^i)+[(\beta^4(-2\alpha+\beta^4-5\alpha^2\beta^2+2\alpha^4b^2)+\alpha^3(5\beta^5b^2-6\beta^5+12\alpha^4\beta b^4\\ &+6\alpha^3\beta^2b^4))4\beta(\alpha^4-\beta^4)(2\alpha^3)(-r_{00}\alpha^2-r_{00}\beta^2-4\alpha^2\beta s_0+2\alpha^3s_0)+\{(6\alpha^5\beta+3\alpha^4\beta^2\\ &+6\alpha^4)((\beta^7b^2+2\alpha^7b^4+2\alpha\beta^6b^2+3\alpha^2\beta^5b^2+2\alpha^3\beta^4b^2+\alpha^4\beta^3b^2+2\alpha^3\beta^4b^4+6\alpha^5\beta^2b^4\\ &+4\alpha^4\beta^3b^4+4\alpha^6\beta b^4)\sigma_0+(-3\beta^6\alpha^2-2\alpha\beta^7-2\alpha^3\beta^5-\alpha^4\beta^4-2\alpha^7\beta b^2-\beta^8-2\alpha^3\beta^5b^2\\ &-4\alpha^6\beta^2-24\alpha^4\beta^2b^2-16\alpha^5\beta b^2+12\alpha\beta^5+24\alpha^2\beta^4+16\alpha^3\beta^3+2\beta^6)r_{00}+(8\alpha^3\beta^4b^2\\ &+24\alpha^5\beta^2b^2-8\alpha^6\beta b^2+24\alpha^4\beta^3b^2-12\alpha^7b^2-8\alpha^3\beta^4-10\alpha^2\beta^5+18\alpha^4\beta^3\\ &+16\alpha^5\beta^2-2\beta^7)s_0)\}](b^iy^j-b^jy^i).\end{aligned} \tag{2.8.18}$$

Take $n>2$, $\alpha^2\neq 0(mod\beta)$. the terms in (2.9.18) which seemingly do not contain β are

$$72b^2\alpha^{11}s_0(b^iy^j-b^jy^i)-72b^4\alpha^{11}(s^i_0y^j-s^j_0y^i).$$

Hence we must have $hp(1)V^{ij}_{(1)}$ such that the above expression is equal to $72b^2\alpha^{11}\beta V^{ij}_{(1)}$. therefore we have

$$s_0(b^iy^j - b^jy^i) - b^2(s^i_0y^j - s^j_0y^i) = \beta V^{ij}_{(1)}, \tag{2.8.19}$$

by putting $V^{ij}_{(1)} = V^{ij}_{(x)}y^k$, the equation (2.9.19) can be written as

$$(s_h\delta^j_k + s_k\delta^j_h)b^i - (s_h\delta^i_k + s_k\delta^i_h)b^j - b^2[s^i_h\delta^j_k + s^i_k\delta^j_h - s^j_h\delta^i_k - s^j_k\delta^i_h]$$
$$= b_hV^{ij}_k + b_kV^{ij}_h. \tag{2.8.20}$$

Contracting (2.9.20) by $j = k$, we get

$$nb^is_h - nb^2s^i_h = b_hV^{ir}_r + b_rV^{ir}_h, \tag{2.8.21}$$

Next transvecting (2.9.20) by b_jb^h, we have

$$-b^2(b^2s^i_k - s^ib_k - s_kb^i) = b^2b_rV^{ir}_k + b_kb_rV^{ir}_sb^s \tag{2.8.22}$$

Transvecting (2.9.22) by b^k, we get

$$2b^4s^i = 2b^2b_rV^{ir}_sb^s,$$

which gives

$$b_rV^{ir}_sb^s = b^2s^i, \ \ \textit{provided}, \ b^2 \neq 0. \tag{2.8.23}$$

substituting the value of $b_rV^{ir}_sb^s$ from (2.9.23) in (2.9.22) we get

$$b_rV^{ir}_h = b^is_h - b^2s^i_h. \tag{2.8.24}$$

substituting the valuve of $b_rV^{ir}_h$ from (2.9.24) in (2.9.21), we get

$$b^2s^i_h = b^is_h - \frac{1}{(n-1)}V^{ir}_rb_h. \tag{2.8.25}$$

If we put $v^i = \frac{1}{(n-1)}V^{ir}_r$, then equation (2.9.25) gives $b^2 s^i_h = b^i s_h - v^i b_h$ which implies $b^2 s_{ij} = b_i s_j - v_i b_j$,where $v^i = a_{ij}v^j$.

Since s_{ij} is skew-symmetric tensor, we have $V_i = s_i$ easily. Hence

$$s_{ij} = \frac{1}{b^2}(b_i s_j - b_j s_i). \tag{2.8.26}$$

Theorem 2.8.4. *A Finsler space $\overline{F}^n$ $(n > 2)$ which is obtained by conformal Kropina change of a Kropina space F^n with (α,β)-metric $L = \alpha - \frac{\beta^2}{\alpha} + \beta (b^2 \neq 0)$ is of Douglas type if and only if (2.9.26) is satisfied.*

2.8.3 Conformal Kropina change of $L = \sqrt{2\alpha\beta}$

Consider, the (α,β)-metric $L = \sqrt{2\alpha\beta}$, we have

$$L_\alpha = \frac{\sqrt{2}\beta}{2\sqrt{\alpha\beta}}, \;\; L_\beta = \frac{\sqrt{2}\alpha}{2\sqrt{\alpha\beta}}, \;\; L_{\alpha\alpha} = -\frac{\sqrt{2}\beta^2}{4(\alpha\beta)^{3/2}}.$$

Hence the values of C^*, D^* and C^{ij} given by equation (2.3.3), (2.9.7) and (2.9.9) respectively reduce to

$$C^* = \frac{\sqrt{\alpha^3\beta^3}(-r_{00}\beta + 2\alpha^2 s_0)}{\beta\sqrt{\alpha\beta}(-3\beta^2 + b^2\alpha^2)}$$

$$D^* = \frac{\alpha\{\rho\alpha^2(-3\beta^2 + b^2\alpha^2) + \sigma_0\beta(3\beta^2 - \alpha b^2) - 4\alpha^2\beta s_0 + r_{00}(\alpha^2 b^2 - \beta^2)\}}{2\beta(-3\beta^2 + b^2\alpha^2)}$$

$$C^{ij} = -\frac{\alpha^2\{(-3\beta^2 + \alpha^2 b^2)2(s^i_0 y^j - s^j_0 y^i) + (\beta r_{00} - 2\alpha^2 s_0)(b^i y^j - b^j y^i)\}}{2\beta(-3\beta^2 + b^2\alpha^2)}. \tag{2.8.27}$$

Since $\{\frac{6\alpha^2\beta^2}{2\beta(-3\beta^2+b^2\alpha^2)}\}(s^i_0 y^j - s^j_0 y^i)$ and $-\{\frac{\alpha^2\beta r_{00}}{2\beta(-3\beta^2+b^2\alpha^2)}\}(b^i y^j - b^j y^i)$ are hp(3), these terms of (2.9.27) may be neglected in our discussion and we treat only of

$$V^{ij}_{(3)} = \frac{(\alpha^2)^2 s_0}{2\beta(-3\beta^2 + b^2\alpha^2)}(s^i_0 y^j - s^j_0 y^i) - \frac{(\alpha^2)^2 b^2}{2\beta(-3\beta^2 + b^2\alpha^2)}(b^i y^j - b^j y^i), \tag{2.8.28}$$

where $V^{ij}_{(3)}$ is hp(3).

The equation (2.9.28) can be written as

$$\beta(-3\beta^2+b^2\alpha^2)V^{ij}_{(3)}-(\alpha^2)^2s_0(b^iy^j-b^jy^i)+b^2(\alpha^2)^2(s^i_0y^j-s^j_0y^i)=0. \qquad (2.8.29)$$

Take $n>2$, $\alpha^2\neq 0(mod\beta)$ [95]. The terms of (2.9.29) which seemingly do not contain β are $b^2(\alpha^2)^2(s^i_0y^j-s^j_0y^i)-(\alpha^2)^2s_0(b^iy^j-b^jy^i)$. Hence we must have hp(1) $V^{ij}_{(1)}$ such that the above expression is equal to $\alpha^4\beta V^{ij}_{(1)}$. Thus

$$b^2(s^i_0y^j-s^j_0y^i)-s_0(b^iy^j-b^jy^i)=\beta V^{ij}_{(1)}. \qquad (2.8.30)$$

By putting $V^{ij}_{(1)}=V^{ij}_k(x)y^k$, the equation (2.9.30) is written as

$$b^2[s^i_h\delta^j_k+s^i_k\delta^j_h-s^j_h\delta^i_k-s^j_k\delta^i_h]-[(s_h\delta^j_k+s_k\delta^j_h)b^i-(s_h\delta^i_k+s_k\delta^i_h)b^j]$$
$$=b_hV^{ij}_k+b_kV^{ij}_h. \qquad (2.8.31)$$

Contracting (2.9.31) by $j=k$, we get

$$nb^2s^i_h-nb^is_h=b_hV^{ir}_r+b_rV^{ir}_h. \qquad (2.8.32)$$

Next, transvecting (2.9.31) by b_jb^h, we have

$$b^2(b^2s^i_k-s^ib_k-s_kb^i)=b^2b_rV^{ir}_k+b_kb_rV^{ir}_sb^s. \qquad (2.8.33)$$

Transvecting (2.9.33) by b^k, we get

$$-2b^4s^i=2b^2b_rV^{ir}_sb^s, \qquad (2.8.34)$$

which gives

$$b_rV^{ir}_sb^s=-b^2s^i, \qquad (2.8.35)$$

provided $b^2\neq 0$.

Putting the value of $b_rV^{ir}_sb^s$ from (2.9.35) in (2.9.33), we get

$$b_rV^{ir}_k=b^2s^i_k-s_kb^i. \qquad (2.8.36)$$

Substituting the value of $b_r V_h^{ir}$ from (2.9.36) in (2.9.32), we get

$$b^2 s_h^i = \frac{1}{(n-1)} V_r^{ir} b_h + b^i s_h. \tag{2.8.37}$$

If we put $v^i = \frac{1}{(n-1)} V^{ir}$, then equation (2.9.37) gives $b^2 s_h^i = v^i b_h + b^i s_h$ which implies $b^2 s_{ij} = v_i b_j + b_i s_j$, where $v_i = a_{ij} v^j$. Since s_{ij} is skew-symmetric tensor, we have $v_i = -s_i$ easily. Thus

$$s_{ij} = \frac{1}{b^2}(b_i s_j - b_j s_i). \tag{2.8.38}$$

Theorem 2.8.5. *A Finsler space $\overline{F}^n$ $(n > 2)$ which is obtained by conformal Kropina change of a Kropina space F^n with (α,β)-metric $L = \sqrt{2\alpha\beta}$ is of Douglas type if and only if (2.9.38) is satisfied.*

2.9 Conclusion

In Finsler geometry, Douglas curvature is an important projectively invariant, which is introduced by J. Douglas in 1927. It it also a non-Riemannian quantity, because all the Riemannian metrics have vanishing Douglas curvature inherently. Finsler metrics with vanishing Douglas curvature are called Douglas metrics. Roughly speaking, a Douglas metric is a Finsler metric which is locally projectively equivalent to a Riemannian metric.

In this chapter, we use to find the condition that conformal Kropina change of Finsler space with special (α,β)-metric of Douglas type yields a space of Douglas type. Further we find the necessary and sufficient condition under which a Kropina change becomes a projective change and mainly devoted to find the condition for the Finsler space with Second approximate Matsumato metric to be Berwald space, Douglas type and conformally Berwald and finally, we apply the conformal change of Finsler space with the metric of Douglas type and also derive the condition for a Finsler space $\overline{F}^n$ which is obtained by

β-Conformal change of Finsler space F^n with (α, β)-metric of Douglas type to be also of Douglas type. Finally we have shown that the Finsler space with Randers Special (α, β)-metric are also Douglas spaces under β-Conformal change.

CHAPTER-3

WEAKLY BERWALD FINSLER SPACES

Content of this chapter

Publication based on this Chapter;

- **Thippeswamy K.R** and Narasimhamurthy S.K., *" Two Kinds of Weakly Berwald Special (α, β)-metrics of Scalar flag curvature", International Journal of Mathematics Trends and Technology, vol 44(2017), ISSN:2231-5373.*

- **Thippeswamy K.R** *and Narasimhamurthy S.K., " On Weakly Berwald Finsler Special (α, β)-metric", Journal of Progressive Research in Mathematics, vol 12(2017), ISSN:2395-0218.*

Chapter 3

WEAKLY BERWALD FINSLER SPACES

3.1 Introduction

Curvatures are the central thought of Finsler geometry. For a Finsler manifold (M, F), the flag curvature is a function $\boldsymbol{K}(P, y)$ to the tangent planes $P \subset T_xM$ and non zero $y \,\varepsilon\, P$. A Finsler metric F is of scalar flag curvature if for any non-zero vector $y \,\varepsilon\, T_xM$, $\boldsymbol{K} = \boldsymbol{K}(x, y)$ is of independent P containing $y \,\varepsilon\, T_xM$ (hence $\boldsymbol{K} = \sigma(x)$ when F is Riemannian). It is of nearly isotropic flag curvature if

$$\boldsymbol{K} = \frac{3c_{x^m}y^m}{F} + \sigma, \tag{3.1.1}$$

where $c = c(x)$ and $\sigma = \sigma(x)$ are scalar functions on M. It is one of the important problems in Finsler geometry is to study and symbolize Finsler manifolds of almost isotropic flag curvature [64].

To find out about the geometric properties of a Finsler metric, one also considers non-Riemannian quantities. In Finsler geometry, there are a number of essential non-Riemannian quantities. The Cartan torsion $\boldsymbol{C}$, the Berwald curvature $\boldsymbol{B}$, the mean Landsberg curvature $\boldsymbol{J}$ and S-curvature $\boldsymbol{S}$, etc ([141] [76] [64]). These are geometric quantities which vanish for Riemannian metrics.

Among the non-Riemannian quantities, the S-curvature $\boldsymbol{S} = \boldsymbol{S}(x,y)$ is closely related to the flag curvature which built through Z. Shen for given comparison theorems on Finsler manifolds. An n-dimentional Finsler metric F is said to have isotropic S-curvature if

$$\boldsymbol{S} = (n+1)cF, \tag{3.1.2}$$

for some scalar function $c = c(x)$ on M. In [64], it is proved that if a Finsler metric F of scalar flag curvature is of isotropic S-curvature (3.1.2), then it has almost isotropic flag curvature (3.1.1).

The geodisc curves of a Finsler metric $F = F(x,y)$ on a smooth manifold M, are determined by $\ddot{c}^i + 2G^i(\dot{c}) = 0$, where the local functions $G^i = G^i(x,y)$ are called the spray coefficients. A Finsler metric F is known as a Berwald metric, if G^i are quadratic in $y \varepsilon T_xM$ for any $x \varepsilon M$. A Finsler metric F is stated to be isotropic Berwald metric if its Berwald curvature is in the following form

$$B^i_{jkl} = c\{F_{y^jy^k}\delta^i_l + F_{y^ky^l}\delta^i_j + F_{y^ly^j}\delta^i_k + F_{y^jy^ky^ly^i}\}, \tag{3.1.3}$$

where $c = c(x)$ is a scalar function on M [141].

3.2 Preliminaries

Let M be a n-dimentional C^∞ manifold. Denote by T_xM the tangent space at $x \varepsilon M$, by $TM = \bigcup_{x\varepsilon M} T_xM$ the tangent bundle of M, and by $TM_0 = TM \setminus \{0\}$ the slit tangent bundle on M. A Finsler metric on M is a function $F : TM \to [0,\infty)$ which has the following properties:

i) F *is* C^∞ *on* TM_0;

ii) F *is positively* $1-$*homogeneous on the fibers of tangent bundle* TM;

iii) *for each* $y \varepsilon T_xM$, *the following quadratic form* g_y *on* T_xM *is positive definite,*

$$g_y(u,v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}[F^2(y+su+tv)]|_s, t=0, \quad u,v\,\epsilon\, T_xM.$$

Let $x\,\epsilon\, M$ and $F_x := F|T_xM$. To measure the non-Euclidean feature of F_x, define C_y : $T_xM \bigotimes T_xM \bigotimes T_xM \to R$ by

$$C_y(u,v,w) := \frac{1}{2}\frac{d}{dt}[g_y + tw(u,v)]|_t = 0, \quad u,v,w\ \epsilon\, T_xM.$$

The family $C := \{C_y\}_{y\,\epsilon\, TM_0}$ is referred to as the Cartan torsion. It is nicely acknowledged that $C=0$ if and only if F is Riemannian. For $y\,\varepsilon\, T_xM_0$, define mean Cartan torsion I_y by using $I_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$. By Diecke theorem, F is Riemannian if and only if $I_y = 0$.

The horizantal covariant derivatives of I alongside geodiscs give upward shove to the mean Landsberg curvature $J_y(u) := J_i(y)u^i$, where $J_i := I_{i|s}y^s$. A Finsler metric is said to be weakly Landsbergian if $J=0$.

Given a Finsler manifold (M,F), then a global vector field G is induced by F on TM_o, which in a standard coordinate (x^i, y^i) for TM_0 is given by $G = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$, where

$$G^i := \frac{1}{4}g^{il}\left[\frac{\partial^2(F^2)}{\partial x^k \partial y^l}y^k - \frac{\partial(F^2)}{\partial \partial x^l}\right], y\,\epsilon\, T_xM.$$

Let G is called the spray assosiated to (M,F). In local coordinates, a curve $c(t)$ is geodesic if and only if its coordinates $c^i(t)$ satisfy $\ddot{c}^i + 2G^i(\ddot{c}) = 0$.

For a tangent vector $y\,\varepsilon\, T_xM_0$, define $B_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ and E_y : $T_xM \otimes T_xM \to R$ by $B_y(u,v,w) := B^i_{jkl}(y)u^jv^kw^l\frac{\partial}{\partial x^i}|_x$ and $E_y(u,v) := E_{jk}(y)u^jv^k$, where

$$B^i_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \quad E_{jk} := \frac{1}{2}B^m_{jkm}.$$

Let B and E are called the Berwald curvature and mean Berwald curvature, respectively. Then F is called a Berwald metric and weakly Berwald metric if $B=0$ and $E=0$, respectively.

A Finsler metric F is said to be isotropic mean Berwald metric if its mean Berwald curvature is in the following form

$$E_{ij} = \frac{n+1}{2F} c h_{ij}, \tag{3.2.1}$$

where $c = c(x)$ is a scalar function on M and h_{ij} is the angular metric [141].

Define $D_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ by $D_y(u, v, w) := D^i_{jkl}(y)u^i v^j w^k \frac{\partial}{\partial x^i}|_x$

where

$$D^i_{jkl} := B^i_{jkl} - \frac{2}{n+1}\{E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j + E_{jk,ly^i}\}.$$

We call $D := \{D_y\}_{y\,\varepsilon\, TM_0}$ the Douglas curvature. A Finsler metric with $D = 0$ is called a Douglas metric. The notion of Douglas metrics was proposed by Basco-Matsumato as a generalization of Berwald metrics [23].

For a Finsler metric F on an n-dimentional manifold M, the Busemann-Hausdorff volume form $dV_F = \sigma_F(x)dx^1...dx^n$ is defined by

$$\sigma_F(x) := \frac{Vol(B^n(1))}{Vol\{(y^i)\varepsilon R^n | F(y^i \frac{\partial}{\partial x^i}|_x) < 1\}}.$$

In general, the local scalar function $\sigma_F(x)$ can now not be expressed in terms of elementary functions, even F is locally expressed by using elementary functions. Let G^i denote the geodisc coefficients of F in the equal local coordinate system. The S-curvature can be defined by

$$S(Y) := \frac{\partial G^i}{\partial y^i}(x, y) - y^i \frac{\partial}{\partial x^i}\left[In\, \sigma_F(x)\right],$$

where $Y = y^i \frac{\partial}{\partial x^i}|_x \,\epsilon\, T_xM$. It is proved that $S = 0$ if F is a Berwald metric. There are many non-Berwald metrics satisfying $S = 0$. S said to be isotropic, if there is a scalar functions $c(x)$ on M such that $S = (n + 1)c(x)F$.

The Riemann curvature $R_y = R^i_k dx^k \otimes \partial x^i|_x : T_xM \to T_xM$ is a family of linear maps on tangent spaces, defined by

$$R^i_k = 2\frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^i \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

For a flag $P = span\{y, u\} \subset T_xM$ with flagpole y, the flag curvature $K = K(p, y)$ is defined by

$$K(P, y) := \frac{g_y(u, R_y(u))}{g_y(y, y)g_y(u, u) - g_y(y, u)^2}.$$

We say that a Finsler metric F is of scalar curvature if for any $y \epsilon T_xM$, the flag curvature $K = K(x, y)$ is a scalar function on the slit tangent bundle TM_0. In this case , for some scalar function K on TM_0 the Riemann curvature is in the following form

$$R^i_k = KF^2\{\delta^i_k - F^{-1}F_{y^k}y^i\}.$$

If K=constant, then F is said to be of constant flag curvature. A Finsler metric F is called isotropic flag curvature, if $K = K(x)$.

3.3 Weakly Berwald Finsler Spaces

Let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ be an $(\alpha, \beta) - metric$, where $\phi = \phi(s)$ is a C^∞ on $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on a manifold M. Let

$$r_{ij} := \frac{1}{2}\left[b_{i|j} + b_{j|i}\right], \quad s_{ij} := \frac{1}{2}\left[b_{i|j} - b_{j|i}\right].$$

$$r_j := b^i r_{ij}, \quad s_j := b^i s_{ij}.$$

Where $b_{i|j}$ denote the coefficients of the covariant derivative of β with respect to α. Let

$$r_{i0} := r_{ij}y^j, \quad s_{i0} := s_{ij}y^j, \quad r_0 := r_jy^j, \quad s_0 := s_jy^j.$$

Put

$$Q = \frac{\phi'}{\phi - s\phi'}, \quad \Theta = \frac{(\phi - s\phi')\phi' - s\phi\phi''}{2\phi((\phi - s\phi') + (b^2 - s^2)\phi'')}, \quad \Psi = \frac{\phi''}{2((\phi - s\phi') + (b^2 - s^2)\phi'')} \tag{3.3.1}$$

Then the S-curvature is given by

$$\begin{aligned} \boldsymbol{S} &= \left[Q' - 2\Psi Q s - 2(\Psi Q)'(b^2 - s^2) - 2(n+1)Q\Theta + 2\lambda\right] s_0 \\ &\quad +2(\Psi + \lambda)r_0 + \alpha^{-1}[(b^2 - s^2)\Psi' + (n+1)\Theta]r_{00}. \end{aligned} \tag{3.3.2}$$

Let us put

$$\begin{aligned} \Delta &= 1 + sQ + (b^2 - s^2)Q', \\ \Phi &= -(n\Delta + 1 + sQ)(Q - sQ') - (b^2 - s^2)(1 + sQ)Q''. \end{aligned}$$

In [142], Cheng-Shen characterize (α, β)-metrics with isotropic S-curvature.

Lemma 3.3.1. *Let $F = \alpha\phi(\beta/\alpha)$ be an (α, β)-metric on an n-manifold.Then, F is of isotropic S-curvature $S = (n+1)cF$, if and only if one of the following holds*

(i) β satisfies

$$r_{ij} = \varepsilon\{b^2 a_{ij} - b_i b_j\}, \quad s_j = 0, \tag{3.3.3}$$

where $\varepsilon = \varepsilon(x)$ is a scalar function, and $\phi = \phi(s)$ satisfies

$$\Phi = -2(n+1)k\frac{\phi\Delta^2}{b^2 - s^2}, \tag{3.3.4}$$

where k is a constant. In this case, $c = k\epsilon$.

(ii) β satisfies

$$r_{ij} = 0, \quad s_j = 0. \tag{3.3.5}$$

In this case, $c = 0$.

Let

$$\begin{aligned}\Psi_1 &:= \sqrt{b^2-s^2}\Delta^{\frac{1}{2}}\left[\frac{\sqrt{b^2-s^2}\Phi}{\Delta^{\frac{3}{2}}}\right]',\\ \Psi_2 &:= 2(n+1)(Q-sQ')+3\frac{\Phi}{\Delta},\end{aligned}$$

$$\theta := \frac{Q-sQ'}{2\Delta}. \tag{3.3.6}$$

Then the formula for the mean Cartan torsion of an (α,β)-metric is given by following

$$I_i = \frac{1}{2}\frac{\partial}{\partial y^i}\left[(n+1)\frac{\phi'}{\phi}-(n-2)\frac{s\phi''}{\phi-s\phi'}-\frac{3s\phi''-(b^2-s^2)\phi'''}{\phi-s\phi'+(b^2-s^2)\phi''}\right] \tag{3.3.7}$$

$$= -\frac{\Phi(\phi-s\phi')}{2\Delta\phi\alpha^2}(\alpha b_i - sy_i). \tag{3.3.8}$$

In [143], it is proved that the condition $\Phi = 0$ characterizes the Riemannian metrics among (α,β)-metrics. Hence, in the continue, we suppose that $\Phi \neq 0$.

Let $G^i = G^i(x,y)$ and $\bar{G}^i_\alpha = \bar{G}^i_\alpha(x,y)$ denote the coefficients of F and α respectively in the same coordinate system. By definition, we have

$$G^i = \bar{G}^i_\alpha + Py^i + Q^i, \tag{3.3.9}$$

where

$$\begin{aligned}P &:= \alpha^{-1}\Theta\left[-2Q\alpha s_0 + r_{00}\right]\\ Q^i &:= \alpha Q s^i_0 + \Psi\left[-2Q\alpha s_0 + r_{00}\right] b^i.\end{aligned}$$

Simplifying (3.3.9) yields the following

$$G^i = \bar{G}^i_\alpha + \alpha Q s^i_0 + \theta(-2\alpha Q s_0 + r_{00})\left[\frac{y^i}{\alpha}+\frac{Q'}{Q-sQ'}b^i\right]. \tag{3.3.10}$$

Clearly, if β is parallel with respect to α ($r_{ij} = 0$ *and* $s_{ij} = 0$), then $P = 0$ and $Q^i = 0$. In this case, $G^i = \bar{G}^i_\alpha$ are quadratic in y, and F is a Berwald metric.

For an (α, β)-metric $F = \alpha\phi(s)$, the mean Landsberg curvature is given by

$$J_i = -\frac{1}{2\Delta\alpha^4}\{\frac{2\alpha^2}{b^2 - s^2}[\frac{\Phi}{\Delta} + (n+1)(Q - sQ^{'}](s_0 + r_0)h_i + \frac{\alpha}{b^2 - s^2}[\Psi_1 + s\frac{\Phi}{\Delta}](r_{00} - 2Q\alpha s_0)h_i + \alpha[-\alpha Q^{'} s_0 h_i + \alpha Q(\alpha^2 s_i - y_i s_0) + \alpha^2 \Delta s_{i0} + [\alpha^2(r_{i0} - 2\alpha Q s_i) - (r_{00} - 2\alpha Q s_0)y_i]\frac{\Phi}{\Delta}\}. \quad (3.3.11)$$

Besides, they also obtained

$$\bar{J} = J_i b^i = -\frac{1}{2\Delta\alpha^2}\{\Psi_1(r_{00} - 2\alpha Q s_0) + \alpha\Psi_2(r_0 + s_0)\}. \quad (3.3.12)$$

The horizontal covariant derivatives $J_{i;m}$ and $J_{i|m}$ of J_i with respect to F and α respectively are given by

$$J_{i;m} = \frac{\partial J_i}{\partial x^m} - J_l\Gamma^l_{im} - \frac{\partial J_i}{\partial y^l}N^l_m, \; J_{i|m} = \frac{\partial J_i}{\partial x^m} - J_l\bar{\Gamma}^l_{im} - \frac{\partial J_i}{\partial y^l}\bar{N}^l_m,$$

where, $\Gamma^l_{ij} = \frac{\partial G^l}{\partial y^i \partial y^j}$, $N^l_j = \frac{\partial G^l}{\partial y^j}$ and $\bar{\Gamma}^l_{ij} = \frac{\partial \bar{G}^i}{\partial y^i \partial y^j}$, $\bar{N}^l_j = \frac{\partial \bar{G}^l}{\partial y^j}$.

Then we have

$$J_{i;m}y^m = \{J_{i|m} - J_l(\Gamma^l_{im} - \bar{\Gamma}^l_{im}) - \frac{\partial J_i}{\partial y^l}(N^l_m - \bar{N}^l_m)y^m$$

$$= J_{i|m}y^m - J_l(N^l_i - \bar{N}^l_i) - 2\frac{\partial J_i}{\partial y^l}(G^l - \bar{G}^l). \quad (3.3.13)$$

Let F be a Finsler metric of scalar flag curvature K. By Akbar-Zadeh's theorem it satisfies following

$$A_{ijk;s;m}y^s y^m + \boldsymbol{K}F^2 A_{ijk} + \frac{F^2}{3}\,[h_{ij}\boldsymbol{K}_k + h_{jk}\boldsymbol{K}_j + h_{ki}\boldsymbol{K}_j] = 0, \quad (3.3.14)$$

where $A_{ijk} = FC_{ijk}$ is the Cartan torsion and $\boldsymbol{K}_i = \frac{\partial K}{\partial y^i}$ [49]. Contracting (3.3.14) with g^{ij} yields

$$J_{i;m}y^m + \boldsymbol{K}F^2 I_i + \frac{n+1}{3}F^2\boldsymbol{K}_i = 0. \quad (3.3.15)$$

By (3.3.13) and (3.3.14), for an (α, β)-metric $F = \alpha\phi(s)$ of constant flag curvature $\boldsymbol{K}$, then

$$J_{i|m}y^m - J_l \frac{\partial(G^l - \bar{G}^l)}{\partial y^i} - 2\frac{\partial J_i}{\partial y^i}(G^l - \bar{G}^l) + \boldsymbol{K}\alpha^2\phi^2 I_i = 0. \qquad (3.3.16)$$

Contracting (3.3.16) with b^i implies that

$$\bar{J}_{|m} - J_i a^{ik} b_{k|m} y^m - J_l \frac{\partial(G^l - \bar{G}^l)}{\partial y^i} b^i - 2\frac{\partial \bar{J}}{\partial y^l}(G^l - \bar{G}^l) + \boldsymbol{K}\alpha^2\phi^2 I_i b^i = 0. \qquad (3.3.17)$$

There exists a relation between mean Berwald curvature E and the S-curvature S. Indeed, taking twice vertical covariant derivatives of the S-curvature gives rise the E-curvature. It is easy to see that, every Finsler metric of isotropic S-curvature (3.1.2) is of isotropic mean Berwald curvature (3.2.1). Now, is the equation $\boldsymbol{S} = (n+1)cF$ equivalent to the equation $\boldsymbol{E} = \frac{n+1}{2}cF^{-1}h$?.

Recently, Cheng-Shen prove that a Randers metric $F = \alpha+\beta$ is of isotropic S-curvature if and only if it is of isotropic E-curvature [141]. Then, Chun-Huan-Cheng [144] extend this equivalency to the Finsler metric $F = \alpha^{-m}(\alpha + \beta)^{m+1}$ for every real constant m, including Randers metric .

Now in this chapter we prove the following theorem.

Theorem 3.3.2. *Let $F = \frac{\alpha^2}{\alpha-\beta} + \beta$ be a special metric on a manifold M of dimention n. Then the following are equivalent*

(i) F is of isotropic S-curvature, $\boldsymbol{S} = (n+1)c(x)F$;

(ii) F is of isotropic mean Berwald curvature, $\boldsymbol{E} = \frac{n+1}{2}c(x)F^{-1}h$;

where $c = c(x)$ is a scalar function on the manifold M.

In this case, $\mathbf{S}$=*0. Then β is a Killing* 1*-form with constant length with respect to α, that is, $r_{00} = 0$.*

Proof: $(i) \to (ii)$ is obvious. Conversely, suppose that F has isotropic mean Berwald curvature, $E = \frac{n+1}{2}c(x)F^{-1}h$. Then we have

$$\boldsymbol{S} = (n+1)[cF + \eta], \tag{3.3.18}$$

where $\eta = \eta_i(x)y^i$ is a 1-form on M. $L = \frac{\alpha^2}{\alpha-\beta} + \beta$, we have

$$Q = \frac{s^2+1}{s(s-2)}, \quad \Theta = -\frac{1}{2}\frac{s(s^3+3s-4)}{(s+s^2-1)(-s^3+2b^2)}, \quad \Psi = \frac{1}{-s^3+2b^2}. \tag{3.3.19}$$

By substituting (3.3.18) and (3.3.19) in (3.3.2), we have

$$\begin{aligned}
\boldsymbol{S} \;=\; & [-\frac{2(-3s^4+2sb^2+2s^3-2b^2+2s^2b^2-2s^2)}{s^2(s-2)^2(-s^3+2b^2)} \\
& \frac{2(-3s^6+4s^5+4s^2b^2-5s^4+8s^3+4sb^2-4b^2)}{s^2(-s^3+2b^2)^2(s-2)^2} \\
& \times(b^2-s^2) - \frac{(n+1)(s^2+1)(-4+s^3+3s)}{(s-2)(-1+s+s^2)(-s^3+2b^2)} + 2\lambda]s_0 + 2[\frac{1}{-s^3+2b^2} + \lambda]r_0 \\
& -[\frac{3s^2(b^2-s^2)}{\alpha(-s^3+2b^2)^2}]r_{00} - [\frac{(n+1)s(s^3+3s-4)}{2\alpha(-1+s+s^2)(-s^3+2b^2)}]r_{00}.
\end{aligned}$$

$$= (n+1)[c\alpha(1+s+\frac{1}{s}) + \eta]. \tag{3.3.20}$$

Multiplying (3.3.20) with $s(1+s+s^2)(s^3+2b^2)^2(s+2)\alpha^5$ implies that

$$\begin{aligned}
& M_1 + M_2\alpha^2 + M_3\alpha^4 + M_4\alpha^6 + M_5\alpha^8 + M_6\alpha^{10} + \alpha[M_7 + M_8\alpha^2 + M_9\alpha^4 \\
& \qquad + M_{10}\alpha^6 + M_{11}\alpha^8] + M_{12}\alpha^{10} = 0,
\end{aligned} \tag{3.3.21}$$

where

$$
\begin{aligned}
M_1 &= [-\beta^2 c(n+1) + 2\beta\lambda(s_0+r_0) - \beta\eta(n+1) + \frac{r_{00}}{2}(n+1)]\beta^9,\\
M_2 &= -\frac{1}{2}[10\beta^2 c(n+1) - 12\beta\lambda(s_0+r_0) + 12\beta s_0 + 6\beta\eta(n+1) + 3r_{00}(n+3)]\beta^7,\\
M_3 &= -[-5\beta^2 c(n+1) + 2\beta b^2 s_0(n+2) - 4\beta\eta b^2(n+1) + 8\beta\lambda b^2(s_0+r_0)\\
&\quad +2\beta(s_0(2n+3)+r_0) + r_{00}(2n-1)(b^2+2)]\beta^5,\\
M_4 &= -2[-2\beta^2 b^4 c(n+1) - 2\beta b^4\eta(n+1) + 4\beta b^4\lambda(s_0+r_0) - \beta((-ns_0+2r_0)+3s_0)\\
&\quad +4\beta b^4\lambda(s_0+r_0) + 4\beta\eta b^2(n+1) - 8\beta b^2\lambda(s_0+r_0) + 2\beta b^2((2n+3)s_0+r_0)\\
&\quad +r_{00}b^2(5n+8)]\beta^3,\\
M_5 &= -2b^2[-4\beta c(n+1) + 10\beta b^2 c(n+1) - 12b^2\lambda(s_0+r_0) + 3(ns_0-2r_0) + 6b^2\eta(n+1)]\beta^2,\\
M_6 &= 20b^4 c(n+1)\beta,\\
M_7 &= [2\beta\lambda(s_0+r_0) + \beta s_0(n+1) - \beta\eta(n+1) + r_{00}(n+4)]\beta^8,\\
M_8 &= [-4\beta^2 b^2 c(n+1) - 4\beta\lambda(s+0+r_0) - 4\beta\lambda(s_0+r_0) - 2\beta\eta(n-1) + 8\beta\lambda b^2(s_0+r_0)\\
&\quad +2\beta(r_0 - 2n(\eta b^2+s_0)) + r_{00}n((b^2+5) - 2r_{00}(b^2+2)]\beta^6,\\
M_9 &= [20\beta^2 b^2 c(n+1) - 2\beta^2 c(n+1) + 12\beta\eta b^2(n+1) - 24\beta\lambda b^2((s_0+r_0) + 3\beta(ns_{-}2r_0))\\
&\quad 3\beta s_0(2b^2-3) + 3r_{00}b^2(4+n)]\beta^4,\\
M_{10} &= 2b^2[-10\beta^2 c(n+1) - 2\beta b^2\eta(n+1) + 4\beta b^2\lambda(s_0+r_0) + \beta(4s_0 n + 2r_0 + 9s_0)\\
&\quad +4r_{00}(n+1)]\beta^2,\\
M_{11} &= 8b^2[b^2\eta(n+1) - 2b^2\lambda(s_0+r_0) - r_0 + ns_0]\beta,\\
M_{12} &= -8b^4 c(n+1).
\end{aligned}
$$

The term of (3.3.21) which is seemingly does not contain α^2 is M_1. Since β^9 is not divisible by α^2, then $c = 0$ which implies that

$$M_1 = M_7 = 0.$$

Therefore (3.3.21) reduces to following

$$M_2 + M_3\alpha^2 + M_4\alpha^4 + M_5\alpha^6 + M_6\alpha^8 = 0, \tag{3.3.22}$$

$$M_8 + M_9\alpha^2 + M_{10}\alpha^4 + M_{11}\alpha^6 + M_{12}\alpha^8 = 0. \tag{3.3.23}$$

By plugging $c = 0$ in M_2 and M_8, the only equations that don't contain α^2 are the following

$$-\beta[2\lambda(s_0 + r_0) - (n+1)\eta + 3r_{00}(n+3)] = \tau_1\alpha^2, \tag{3.3.24}$$

$$4\beta b^2[2\lambda(r_0 + s_0) - (n+1)\eta] + r_{00}(2n-1)(b^2+2) = \tau_2\alpha^2, \tag{3.3.25}$$

where $\tau_1 = \tau_1(x)$ and $\tau_2 = \tau_2(x)$ are scalar functions on M. By eliminating $[2\lambda(r_0 + s_0) - (n+1)\eta]$ from (3.3.24) and (3.3.25), we get

$$r_{00} = \tau\alpha^2, \tag{3.3.26}$$

where $\tau = \frac{\tau_2 - 4b^2\tau_1}{(b^2+2)(4b^2(2n-1)) - 3(n+3)}$.

By (3.3.24) or (3.3.25), it follows that

$$2\lambda(r_0 + s_0) - (n+1)\eta = 0. \tag{3.3.27}$$

By (3.3.26), we have $r_0 = \tau\beta$. Putting (3.3.26) and (3.3.27) in M_8 and M_9 yields

$$M_8 = [n(b^2+5) - 2(b^2+2)]\tau\alpha^2\beta^6, \tag{3.3.28}$$

$$M_9 = \left[[(6b^2 + 3n - 9)s_0 - 6r_0]\beta - 3b^2(n+4)r_{00}\tau\alpha^2\right]\beta^4. \tag{3.3.29}$$

By putting (3.3.28) and (3.3.29) into (3.3.23), we have

$$\begin{aligned} &[(6b^2 + 3n - 9)s_0 - 6r_0]\beta^5 - 3b^2(n+4)r_{00}\tau\alpha^2\beta^4 \\ &n(b^2+5) - 2(b^2+2)\tau\alpha^2\beta^6 - M_{10}\alpha^2 + M_{11}\alpha^4 + M_{12}\alpha^6 = 0. \end{aligned} \tag{3.3.30}$$

The only equations of (3.3.30) that do not contain α^2 is $[n(b^2+5)-2(b^2+2)\tau\beta+(6b^2+3n-9)s_0-6r_0]\beta^5$. Since β^6 is not divisible by α^2, then we have

$$[n(b^2+5)-2(b^2+2)\tau\beta^6+(6b^2+3n-9)s_0-6r_0]=0. \tag{3.3.31}$$

By lemma 3.3.1, we always have $s_j=0$. Then (3.3.31), reduces to following

$$[n(b^2+5)-2(b^2+2)]\tau\beta-6r_0=0. \tag{3.3.32}$$

Thus

$$[n(b^2+5)-2(b^2+2)]\tau b_i-6\tau b_i=0. \tag{3.3.33}$$

By multiplying (3.3.33) with b^i, we have

$$\tau=0.$$

Thus by (3.3.29), we get $\eta=0$ and then $\boldsymbol{S}=(n+1)cF$. By (3.3.26), we get $r_{ij}=0$. Therefore lemma 3.3.1, implies that $\boldsymbol{S}=0$. This completes the proof.

Theorem 3.3.3. *Let $F=\frac{\alpha^2}{\alpha-\beta}+\beta$ be a non-Riemannian metric on a manifold M of dimention n. Then F is of scalar flag curvature with isotropic S-curvature(3.1.2), if and only if it has isotropic Berwald curvature(3.1.3) with almost isotropic flag curvature(3.1.1). In this case, F must be locally Minkowskian.*

Proof: Let F be an isotropic Berwald metric (3.1.3) with almost isotropic flag curvature (3.1.1). It is proved that every isotropic Berwald metric (3.1.3) has isotropic S-curvature (3.1.2).

Conversely, suppose that F is of isotropic S-curvature (3.1.2) with scalar flag curvature $\boldsymbol{K}$. In [63], it is showed that every Finsler metric of isotropic S-curvature (3.1.2) has almost isotropic flag curvature (3.1.1). Now, we are going to prove that F is a isotropic Berwald metric. In [141], it is proved that F is an isotropic Berwald metric (3.1.3) if and only if it is a Douglas metric with isotropic mean Berwald curvature (3.2.1). On the other hand, every Finsler metric of isotropic S-curvature (3.1.2) has isotropic mean Berwald curvature (3.2.1). Thus for completing the proof, we must show that F is a Douglas metric. By proposition 3.2, we have $\boldsymbol{S}=0$. Therefore by theorem 1.1 in [64], F must be of isotropic flag curvature $\boldsymbol{K}=\sigma(x)$. By proposition 3.2, β is a Killing 1-form with constant length with respect to α, that is, $r_{ij}=s_j=0$. Then (3.3.10), (3.3.11) and (3.3.12) reduce to

$$G^i-\bar{G}^i=\alpha Q s^i_0,\quad J_i=-\tfrac{\Phi s_{i0}}{2\alpha\Delta},\quad \bar{J}=0.$$

By (3.3.9), we get

$$I_i b^i := -\tfrac{\Phi(\phi-s\phi^{'})}{2\Delta F}(b^2-s^2).$$

Now we consider two cases:

Case I: Let $dimM\geq 3$. In this case, by Schur lemma F has constant flag curvature and (3.3.17) holds, the equation (3.3.17) reduces to following

$$\frac{\Phi s_{i0}}{2\Delta\alpha}a^{ik}s_{k0}+\frac{\Phi s_{l0}}{2\Delta\alpha}(sQs^l_0+Q^{'}s^l_0(b^2-s^2))-\boldsymbol{K}F\frac{\Phi}{2\Delta}(\phi-s\phi^{'})(b^2-s^2)=0. \tag{3.3.34}$$

By assumption $\Phi\neq 0$. Thus by (3.3.34), we get

$$s_{i0}s^i_0+s_{l0}(\alpha Q s^l_0)_{.i}b^i-\boldsymbol{K}F\alpha(\phi-s\phi^{'})(b^2-s^2)=0. \tag{3.3.35}$$

The following holds

$$(\alpha Q s^l_0)_{.i}b^i=sQs^i_0+Q's^i_0(b^2-s^2)=0.$$

Then (3.3.35) can be rewritten as follows

$$s_{i0}s^i_0\Delta-\boldsymbol{K}\alpha^2\phi(\phi-s\phi^{'})(b^2-s^2)=0. \tag{3.3.36}$$

By (3.3.6), (3.3.19) and (3.3.36), we obtain

$$\left[1+\frac{s^2+1}{s-2}-\frac{2(b^2-s^2)(-1+s+s^2)}{s^2(s-2)^2}\right]s_{i0}s^i_0$$

$$-\boldsymbol{K}\alpha^2\Big(\frac{(-1+s+s^2)(s-2)}{s^2}(b^2-s^2)\Big)=0. \tag{3.3.37}$$

Multiflying (3.3.37) with $-s^2(s-2)^2\alpha^5$ yields

$$A+\alpha B=0,$$

where

$$\begin{aligned}A &= -\boldsymbol{K}20b^2\beta\alpha^6+(5\boldsymbol{K}\beta^3b^2+2b^2\beta s_{i0}s^i_0+20\boldsymbol{K}\beta^3)\alpha^4+(\beta^3s_{i0}s^i_0-5\boldsymbol{K}\beta^5+\boldsymbol{K}\beta^5b^2)\alpha^2\\ &\quad -\boldsymbol{K}\beta^7-s_{i0}s^i_0\beta^5\\ B &= 8\boldsymbol{K}b^2\alpha^6+(10\boldsymbol{K}b^2\beta^2-8\boldsymbol{K}\beta^2-2s_{i0}s^i_0b^2)\alpha^4+(-5\boldsymbol{K}b^2\beta^4+2s_{i0}s^i_0b^2\beta^2-10\boldsymbol{K}\beta^4)\alpha^2\\ &\quad +(5\boldsymbol{K}\beta^6-s_{i0}s^i_0\beta^4).\end{aligned} \tag{3.3.38}$$

Obviously, we have $A=0$ and $B=0$.

If $A=0$ and the fact that β^7 is not divisible by α^2, we get $\boldsymbol{K}=0$. Therefore (3.3.37) reduces to following

$$s_{i0}s^i_0=a_{ij}s^j_0s^i_0=0.$$

Because of positive-definiteness of the Riemannian metric α, we have $s^i_0=0$, i.e., β is closed. By $r_{00}=0$ and $s_0=0$, it follows that β is parallel with respect to α. Then $F=\frac{\alpha^2}{\alpha-\beta}+\beta$ is a Berwald metric. Hence F must be locally Minkowskian.

Case II: Let dim $M=2$. Suppose that F has isotropic Berwald curvature (3.1.3). In [144], it is proved that every isotropic Berwald metric [70] has isotropic S-curvature, $\boldsymbol{S}=(n+1)cF$. By proposition 3.2, $c=0$. Then by [144], F reduces to a Berwald metric. Since F is non-Riemannian, then by Szabo's rigidity theorem for Berwald surface (see [49] page 278), F must be locally Minkowskian.

3.4 Characterization of Weakly-Berwald (α, β)-metrics of Scalar flag curvature

In n-dimensional Finsler manifold ($n \geq 3$), we characterize the two important class of weakly-Berwald (α, β)-metrics of scalar flag curvature. So first we have to prove the following lemma.

Lemma 3.4.1. *Let (M, F) be an n-dimensional Finsler manifold ($n \geq 3$). Suppose that (α, β)-metrics $F = \frac{(\alpha+\beta)^2}{\alpha}$ and $F = \sqrt{c_1\alpha^2 + c_2\alpha\beta + c_3\beta^2}$ (where c_1, c_2 and c_3 are constants) are of non-Randers type. Then $\Phi \neq 0$.*

Proof: We just give the proof for $F = \frac{(\alpha+\beta)^2}{\alpha}$. Because the proof for $F = \sqrt{c_1\alpha^2 + c_2\alpha\beta + c_3\beta^2}$ (where c_1, c_2 and c_3 are constants) is also similar. So we omit it.

By a direct computation, we have

$$\Phi = -\frac{A\phi}{(1-ks^2)^4},$$

where

$$\phi = 1 + 2s + s^2,$$

$$A = -12ns^3 + 6(1+n)s^2 + 4n(1+2b^2) + 4(1-b^2) - 2(n+1)(1+2b^2).$$

Assume that $\Phi = 0$. Then A=0. Multiplying A=0 with α^3 yields

$[(4\beta n(2b^2+1) - 4\beta(b^2-1))\alpha^2 - 12n\beta^3] - \alpha[2(n+1)(2b^2+1) + 6\beta^2(n+1)] = 0.$

Hence we have,

$$(4\beta n(2b^2+1) - 4\beta(b^2-1))\alpha^2 - 12n\beta^3 = 0, \tag{3.4.1}$$

$$2(n+1)(2b^2+1) + 6\beta^2(n+1) = 0.$$

Clearly, observe that β^3 is not divisible by α^2. Since we have k=0 by (3.4.1), which is a contradiction with $k \neq 0$. So $\Phi \neq 0$.

By using this, now we prove the following:

Theorem 3.4.2. *Let (M, F) be an n-dimensional Finsler manifold ($n \geq 3$). Assume that (α, β)-metrics $F = \frac{(\alpha+\beta)^2}{\alpha}$ and $F = \sqrt{c_1\alpha^2 + c_2\alpha\beta + c_3\beta^2}$ (where c_1, c_2 and c_3 are constants) are of scalar flag curvature* $\boldsymbol{K}$=$\boldsymbol{K}$ *(x, y). Then F is weak Berwald metric if and only if F is Berwald metric and* $\boldsymbol{K}$*=0. In this case, F must be locally Minkowskian.*

Proof: By the lemma 3.4.1 and (3.3.7) we know that the metrics $F = \frac{(\alpha+\beta)^2}{\alpha}$ and $F = \sqrt{c_1\alpha^2 + c_2\alpha\beta + c_3\beta^2}$ (where c_1, c_2 and c_3 are constants) can not represents the Riemannian metrics, where $k \neq 0$ a constant and $\beta \neq 0$.

The sufficiency is obvious. We just prove the necessity.

First, we assume that the metric F is weak Berwald. By lemma 3.4.1, we know that $S = (n+1)c(x)F$ with *c(x)=0* and

$$r_{00} = 0, s_0 = 0. \tag{3.4.2}$$

Let (M, F) be an n-dimensional Finsler manifold of scalar flag curvature with flag curvature $\boldsymbol{K}$=$\boldsymbol{K}$(x, y). Suppose that the S-curvature is isotropic, $\boldsymbol{S}=(n+1)c(x)F(x,y)$, where *c=c(x)* is a scalar function on M. Then there is a scalar function $\sigma(x)$ on M such that

$$\boldsymbol{K} = \frac{3c_{x^m}(x)y^m}{F(x,y)} + \sigma(x). \tag{3.4.3}$$

F must be of isotropic flag curvature $\boldsymbol{K}=\sigma(x)$.

Further , by schur lemma[128], F must be of constant flag curvature.

From (3.4.2), we can simplify (3.3.10),(3.3.11) and (3.3.12) as follows

$G^i - \bar{G}^i = \alpha Q s^i_0, \quad J_i = -\frac{\Phi s_{i0}}{2\alpha\Delta}, \quad \bar{J} = 0.$

In addition, from (3.3.7), we obtain

$$I_i b^i := -\frac{\Phi(\phi - s\phi')}{2\Delta F}(b^2 - s^2).$$

Thus (3.3.17) can be expressed as follows

$$\tfrac{\Phi s_{i0}}{2\Delta\alpha}a^{ik}s_{k0}+\tfrac{\Phi s_{l0}}{2\Delta\alpha}(sQs^l_0+Q^{'}s^l_0(b^2-s^2))-\boldsymbol{K}F\tfrac{\Phi}{2\Delta}(\phi-s\phi^{'})(b^2-s^2)=0.$$

By lemma 3.4.1, we have

$$s_{i0}s^i_0+s_{l0}(\alpha Qs^l_0)_{.i}b^i-\boldsymbol{K}F\alpha(\phi-s\phi^{'})(b^2-s^2)=0.$$

Note that $F=\alpha\phi(s)$, $s=\beta/\alpha$. We have

$$s_{i0}s^i_0\Delta-\boldsymbol{K}\alpha^2\phi(\phi-s\phi^{'})(b^2-s^2)=0. \tag{3.4.4}$$

Case I: $F=\frac{(\alpha+\beta)^2}{\alpha}$. In this case,

$$\Delta=\tfrac{\phi(1+2b^2-3s^2)}{(s-1)^2}.$$

Then (3.4.4) becomes

$$(1+2b^2-3s^2)s_{i0}s^i_0-\boldsymbol{K}\alpha^2(b^2-s^2)(s+1)^3(s-1)^3=0.$$

Multiplying this equation with α^6 yields

$$\boldsymbol{K}b^2\alpha^8+\{(1+2kb^2)s_{i0}s^i_0-k\beta^2(1+3kb^2)\}\alpha^6+3\beta^2\{k\beta^2(1+kb^2)+ s_{i0}s^i_0\}\alpha^4-\boldsymbol{K}\beta^6(3+b^2)\alpha^2=-\boldsymbol{K}\beta^8. \tag{3.4.5}$$

Note that, the left of (3.4.5) is divisible by α^2. Hence we can obtain that the flag curvature $\boldsymbol{K}$=0, because $k\neq 0$ and β^8 is not divisible by α^2. Substituting $\boldsymbol{K}$=0 into (3.4.4), we have $s_{i0}s^i_0=a_{ij}(x)s^j_0s^i_0=0$. Because $(a_{ij}(x))$ is positive definite, we have $s^i_0=0$, i.e., β is closed. By (3.4.2), we know that β is parallel with respect to α. Then $F=\frac{(\alpha+\beta)^2}{\alpha}$ is a Berwald metric, where $k\neq 0$ a constant. Hence F must be locally Minkowskian.

Case II: $F=\sqrt{c_1\alpha^2+c_2\alpha\beta+c_3\beta^2}$ (where c_1, c_2 and c_3 are constants). In this case,

$$\Delta=\frac{4c_1^2+6c_1c_2s+3c_2^2s^2+2c_2c_3s^3-c_2^2b^2+4b^2c_3c_1}{(2c_1+c_2s)^3}. \tag{3.4.6}$$

Then (3.4.4) becomes

$$(4c_1^2+3c_2^2s^2-b^2c_2^2+2c_2c_3s^3+6c_1c_2s+4b^2c_1c_3)s_{i0}s^i_0-\boldsymbol{K}\alpha^2(b^2-s^2)(2c_1+c_2s)^3=0.$$

Implies $A+\alpha B=0$,

where

$$\begin{aligned}
A &= [-(12\boldsymbol{K}c_1^2c_2\beta b-8\boldsymbol{K}c_1^3\beta)b-8\boldsymbol{K}c_1^3\beta b]\alpha^4+[6s_{i0}s^i_0c_1c_2\beta-(\boldsymbol{K}c_2^3\beta^3b-6\boldsymbol{K}c_1c_2^2\beta^3)b \\
&\quad -(6\boldsymbol{K}c_1c_2^2\beta^2b-12\boldsymbol{K}c_1^2c_2\beta^2)\beta]\alpha^2+[2s_{i0}s^i_0c_2c_3\beta^3+\boldsymbol{K}c_2^3\beta^5], \\
B &= -8\boldsymbol{K}c_1^3b^2\alpha^4+[s_{i0}s^i_0(-b^2c_2^2+4c_1(c_1+c_3b^2))-6\boldsymbol{K}\beta^2c_1c_2(c_2b-2c_1)b-(12\boldsymbol{K}\beta c_1^2c_2b \\
&\quad -8\boldsymbol{K}\beta c_1^3)\beta]\alpha^2+[3s_{i0}s^i_0c_2^2\beta^4+\boldsymbol{K}\beta^4c_2^3b-(\boldsymbol{K}\beta^3c_2^3b-6\boldsymbol{K}\beta^3c_1c_2^2)\beta].
\end{aligned}$$

Obviously, we have A=0 and B=0.

By A=0 and clearly note that β^3 is not divisible by α^2. Then we obtain $s_{i0}s^i_0=0$. Hence β is closed. By (3.4.2), we know that β is parallel with respect to α. Then F is a Berwald metric. From (3.4.4), we find that $\boldsymbol{K}$=0. Hence F is locally Minkowskian.

3.5 Conclusion

In the past several years, Finsler geometry has carried out rapid and great progress. Various Riemannian curvatures and non-Riemannian curvatures in Finsler geometry have been studied deeply and their geometric meanings are better understood. Finsler geometry has been applied extensively in phisics, biology (ecology) and other fields in natural science. These are in part duo to the study of (α,β)-metrics.

In this chapter, we prove metric $F=\frac{\alpha^2}{\alpha-\beta}+\beta$ with some non-Riemannian curvature properties and be a non-Riemannian metric on a manifold M of dimention n. Then F is of scalar flag curvature with isotropic S-curvature (3.1.2), if and only if it has isotropic Berwald curvature (3.1.3) with almost isotropic flag curvature (3.1.1). In this case, F

must be locally Minkowskian and we characterize the two important class of weakly-Berwald (α, β)-metrics $F = \frac{(\alpha+\beta)^2}{\alpha}$ and $F = \sqrt{c_1\alpha^2 + c_2\alpha\beta + c_3\beta^2}$ (where c_1, c_2 and c_3 are constants) are of scalar flag curvature.

CHAPTER-4

NONHOLONOMIC FRAMES FOR FINSLER SPACE WITH SPECIAL (α, β)-METRICS

Content of this chapter

Publications based on this Chapter;

- **Thippeswamy K.R** and Narasimhamurthy S.K., *" Nonholonomic Frames for Finsler space with Special (α, β)-metrics", Communicated.*

- **Thippeswamy K.R** *and Narasimhamurthy S.K., " Finslerian Nonholonomic Frames for Finsler space with Special (α, β)-metrics", Communicated.*

Chapter 4

NONHOLONOMIC FRAMES FOR FINSLER SPACE WITH SPECIAL (α, β)-METRICS

4.1 Introduction

In 1982, P.R. Holland [[66] ,[67]] studies a unified formalism that makes use of a nonholonomic frame on space-time arising from consideration of a charged particle transferring in an external electromagnetic field. In fact, R.S. Ingarden [75] was first to factor out that the Lorentz force law can be written in this case as geodesic equation on a Finsler space known as Randers space. The creator Beil R.G.[[21][22]], have studied a gauge transformation viewed as a nonholonomic frame on the tangent bundle of a four dimensional base manifold. The geometry that follows from these considerations gives a unified strategy to gravitation and gauge symmetries. The above authors used the common Finsler thinking to learn about the existence of a nonholonomic frame on the vertical subbundle VTM of the tangent bundle of a base manifold M.

Consider $a_{ij}(x)$, the components of a Riemannian metric on the base manifold M, $a(x, y) > 0$ and $b(x, y) > 0$ two functions on TM and $B(x, y) = B_i(x, y)dx^i$ a vertical

1-form on TM. Then

$$g_{ij}(x,y) = a(x,y)a_{ij}(x) + b(x,y)B_i(x)B_j(x) \tag{4.1.1}$$

is a generalized Lagrange metric, called the Beil metric . We say additionally that the metric tensor g_{ij} is a *Beil deformation* of the Riemannian metric a_{ij}. It has been studied and applied by means of R.Miron and R.K. Tavakol in General Relativity for $a(x,y) = exp(2\sigma(x,y))$ and $b = 0$. The case $a(x,y) = 1$ with a number alternatives of b and B_i i used to be brought and studied with the aid of R.G. Beil for establishing a new unified field theory [22].

4.2 Preliminaries

4.2.1 Nonholonomic frame for Finsler metrics

In the existing section, we study an important class of Finsler spaces is the class of Finsler spaces with (α, β)-metrics [99]. The first Finsler spaces with (α, β)-metrics were introduced via the physicist G. Randers in 1941, are called Randers spaces[116]. Recently, R.G. Beil advised to think about a extra commonplace case, the classification of Lagrange spaces with (α, β)-metric, which used to be mentioned by means of R. Miron in [37]. Next we look for some different Finsler space with (α, β)-metrics.

Definition 2.2.1. *A Finsler space $F^n = (M, F(x,y))$ is stated to have an (α, β)-metric if there exists a 2-homogeneous function L of two variables such that the Finsler metric $F : TM \to R$ is given by,*

$$F^2(x,y) = L(\alpha(x,y), \beta(x,y)), \tag{4.2.1}$$

where $\alpha^2(x,y) = a_{ij}(x)y^iy^j$, α is a Riemannian metric on M and $\beta(x,y) = b_i(x)y^i$ is a 1-form on M.

Consider $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ the fundamental tensor of the Randers space (M, F). Taking into account the homogeneity of α and F we have the following formulae:

$$\begin{aligned} &p^i = \frac{1}{\alpha}y^i = a^{ij}\frac{\partial \alpha}{\partial y^j}; \qquad p_i = a_{ij}p^j = \frac{\partial \alpha}{\partial y^i}; \\ &l^i = \frac{1}{L}y^i = g_{ij}\frac{\partial L}{\partial y^j}; \qquad l_i = g^{ij}\frac{\partial L}{\partial y^j} = p_i + b_i; \\ &l^i = \frac{1}{L}p^i; \qquad l^il_i = p^ip_i = 1; \qquad l^ip_i = \frac{\alpha}{L}; \\ &p^il_i = \frac{L}{\alpha}; \quad b_ip^i = \frac{\beta}{\alpha}; \quad b_il^i = \frac{\beta}{L}. \end{aligned} \tag{4.2.2}$$

With recognize to these notations, the metric tensors a_{ij} and g_{ij} are related by [90],

$$g_{ij} = \frac{L}{\alpha}a_{ij} + b_ip_j + p_ib_j + b_ib_j - \frac{\beta}{\alpha}p_ip_j = \frac{L}{\alpha}(a_{ij} - p_ip_j) + l_il_j. \tag{4.2.3}$$

Theorem 4.2.1. *For a Finsler space (M, F), consider the matrix with the entries:*

$$Y^i_j = \sqrt{\frac{\alpha}{L}}\left(\delta^i_j - l_il_j + \sqrt{\frac{\alpha}{L}}p^ip_j\right), \tag{4.2.4}$$

defined on TM. Then $Y_j = Y^i_j(\frac{\partial}{\partial y^i})$, $j \in 1, 2, ..., n$ is an nonholonomic frame.

Theorem 4.2.2. *With recognize to frame the holonomic elements of the Finsler metric tensor $(a_{\alpha\beta})$ is the Randers metric (g_{ij}), i.e.,*

$$g_{ij} = Y^\alpha_i Y^\beta_j a_{\alpha\beta}. \tag{4.2.5}$$

Throughout this area we shall upward shove and lower indices only with the Riemannian metric $a_{ij}(x)$, i.e., $y_i = a_{ij}y^j$, $b^i = a^{ij}b_j$ and so on. For a Finsler space with (α, β)-metric $F^2(x,y) = L(\alpha(x,y), \beta(x,y))$ we have the Finsler invariants [90],

$$\rho_1 = \frac{1}{2\alpha}\frac{\partial L}{\partial \alpha}; \;\; \rho_0 = \frac{1}{2}\frac{\partial^2 L}{\partial \beta^2}; \;\; \rho_{-1} = \frac{1}{2\alpha}\frac{\partial^2 L}{\partial \alpha \partial \beta}; \;\; \rho_{-2} = \frac{1}{2\alpha^2}\left(\frac{\partial^2 L}{\partial \alpha^2} - \frac{1}{\alpha}\frac{\partial L}{\partial \alpha}\right), \tag{4.2.6}$$

where, subscripts $\{1, 0, -1, -2\}$ gives us the degree of homogeneity of these invariants. For a Finsler space with (α, β)-metric we have:

$$\rho_{-1}\beta + \rho_{-2}\alpha^2 = 0. \tag{4.2.7}$$

With respect to these notations, we have that the metric tensor g_{ij} of a Finsler space with (α, β)-metric is given by [90]:

$$g_{ij}(x, y) = \rho a_{ij}(x) + \rho_0 b_i(x) + \rho_{-1}(b_i(x)y_j + b_j(x)y_i) + \rho_{-2}y_i y_j. \tag{4.2.8}$$

From (4.2.8), we can see that g_{ij} is the result of two Finsler deformations:

$$\begin{aligned} &i) \quad a_{ij} \mapsto h_{ij} = \rho a_{ij} + \frac{1}{\rho_{-2}}(\rho_{-1}b_i + \rho_{-2}y_i)(\rho_{-1}b_j + \rho_{-2}y_j), \\ &ii) \quad h_{ij} \mapsto g_{ij} = h_{ij} + \frac{1}{\rho_{-2}}(\rho_0\rho_{-2} - \rho_{-1}^2)b_i b_j. \end{aligned} \tag{4.2.9}$$

The Finslerian nonholonomic frame that corresponds to the first deformation (4.2.9) is, according to the theorem 7.9.1 in [36], given by:

$$X_j^i = \sqrt{\rho_1}\delta_j^i - \frac{1}{B^2}\left(\sqrt{\rho_1} \pm \sqrt{\rho_1 + \frac{B^2}{\rho_{-2}}}\right)(\rho_{-1}b^i + \rho_{-2}y^i)(\rho_{-1}b_j + \rho_{-2}y_j), \tag{4.2.10}$$

where

$$B^2 = a_{ij}(\rho_{-1}b^i + \rho_{-2}y^i)(\rho_{-1}b^j + \rho_{-2}y^j) = \rho_{-1}^2 b^2 + \beta\rho_{-1}\rho_{-2}.$$

The metric tensors a_{ij} and h_{ij} are related by:

$$h_{ij} = X_i^k X_j^l a_{kl}. \tag{4.2.11}$$

Again the frame that corresponds to the second deformation (4.2.9) is given by:

$$Y_j^i = \delta_j^i - \frac{1}{C^2}\left(1 \pm \sqrt{1 + \frac{\rho_{-2}C^2}{\rho_0\rho_{-2} - \rho_{-1}^2}}\right) b^i b_j, \tag{4.2.12}$$

where

$$C^2 = h_{ij}b^i b^j = \rho_1 b^2 + \frac{1}{\rho_{-2}}(\rho_{-1}b^2 + \rho_{-2}\beta)^2.$$

The metric tensors h_{ij} and g_{ij} are related by the formula:

$$g_{mn} = Y^i_m Y^j_n h_{ij}. \tag{4.2.13}$$

Theorem 4.2.3. *[36] Let $F^2(x, y) = L(\alpha(x, y), \beta(x, y))$ be the metric function of a Finsler space with (α, β)-metric for which the condition (4.2.7) is true, then*

$$V^i_j = X^i_k Y^k_j, \tag{4.2.14}$$

is a Finslerian nonholonomic frame with X^i_k and Y^k_j are given by (4.2.10) and (4.2.12) respectively.

4.3 Nonholonomic frame for Finsler spaces with (α, β)-metrics

In this part we consider Finlser metric with (α, β)-metrics, such as I^{st} Finsler frame product of Infinite series metric and Kropina metric and II^{nd} Finsler frame product of Cube root metric and Kropina metric and one of a kind (α, β)-metric then we construct Finslerian nonholonomic frame for these.

4.3.1 Finslerian Nonholonomic frame for (α,β)-metrics $L^2 = (\frac{\beta^4}{(\beta-\alpha)^2})(\frac{\alpha^2}{\beta})$:

In the first case, for a Finsler metric with the fundamental function $L^2 = (\frac{\beta^4}{(\beta-\alpha)^2})(\frac{\alpha^2}{\beta}) = L^2 = \frac{\alpha^2\beta^4}{\beta(\beta-\alpha)^2}$, the Finsler invariants (4.2.6) are given by:

$$\begin{aligned}\rho_1 &= \frac{-\beta^4}{(\alpha-\beta)^3}, \quad \rho_0 = \frac{3\alpha^4\beta}{(\alpha-\beta)^4},\\ \rho_{-1} &= -\frac{\beta^3(4\alpha-\beta)}{(\alpha-\beta)^4}, \quad \rho_{-2} = \frac{3\beta^4}{\alpha(\alpha-\beta)^4}.\\ B^2 &= \frac{\beta^6(4\alpha-\beta)(4\alpha^2b^2-\alpha\beta b^2-3\beta^2)}{\alpha(\alpha-\beta)^8}\end{aligned} \tag{4.3.1}$$

Using (4.3.1) in (4.2.10) we have,

$$\begin{aligned}X^i_j \;=\; & \sqrt{\frac{-\beta^4}{(\alpha-\beta)^3}}\delta^i_j - \frac{1}{\beta^6(4\alpha-\beta)(4\alpha^2b^2-\alpha\beta b^2-3\beta^2)}\times\\ & \left\{\alpha(\alpha-\beta)^8\left[\sqrt{\frac{-\beta}{(\alpha-\beta)^3}} \pm \frac{1}{3}\sqrt{\frac{-9\beta^4}{(\alpha-\beta)^3}+\frac{3\beta^2(4\alpha-\beta)(4\alpha^2b^2-\alpha\beta b^2-3\beta^2)}{(\alpha-\beta)^4}}\right]\right\}\\ & \left(-\frac{\beta^3(4\alpha-\beta)}{(\alpha-\beta)^4}b^i + \frac{3\beta y^i}{\alpha(\alpha-\beta)^4}\right)\cdot\left(-\frac{\beta^3(4\alpha-\beta)}{(\alpha-\beta)^4}b_j + \frac{3\beta^4 y_j}{\alpha(\alpha-\beta)^4}\right).\end{aligned} \tag{4.3.2}$$

Again using (4.3.1) in (4.2.12) we have,

$$Y^i_j = \delta^i_j - \frac{1}{C^2}\left(1 \pm \sqrt{1+\frac{C^2\alpha^2\beta}{(2\beta+\alpha)(\alpha^2+\alpha\beta+\beta^2)}}\right)b^ib_j; \tag{4.3.3}$$

where

$$C^2 = -\frac{\beta^4b^2}{(\alpha-\beta)^3} + \frac{(4\alpha^2b^2-\alpha\beta b^2-3\beta^2)^2\beta^2}{3\alpha(\alpha-\beta)^4}.$$

Theorem 4.3.1. *Consider a Finsler metric $L^2 = (\frac{\beta^4}{(\beta-\alpha)^2})(\frac{\alpha^2}{\beta})$, for which the condition (4.2.7) is true. Then*

$$V^i_j = X^i_k Y^k_j$$

is a Finslerian nonholonomic frame with X^i_k and Y^k_j are given by (4.3.2) and (4.3.3) respectively.

4.3.2 Finslerian Nonholonomic frame for (α, β)-metrics $L = (c_1\alpha^2\beta + c_2\beta^3)(\frac{\alpha^2}{\beta})$:

In the second case, for a Finsler metric with the fundamental function $L = (c_1\alpha^2\beta + c_2\beta^3)(\frac{\alpha^2}{\beta})$, the Finsler invariants (4.2.6) are given by:

$$\begin{aligned} &\rho_1 = 2c_1\alpha^2 + c_2\beta^2, \quad \rho_0 = c_2\alpha^2, \\ &\rho_{-1} = 2c_2\beta, \quad \rho_{-2} = 4c_1, \\ &B^2 = 4c_2\beta^2(c_2b^2 + 2c_1) \end{aligned} \tag{4.3.4}$$

Using (4.3.4) in (4.2.10) we have,

$$\begin{aligned} X^i_j &= \sqrt{\frac{2(c_1\alpha^2 + c_2\beta^2)}{\beta^2}}\delta^i_j \\ &- \frac{1}{4}\left[\left\{\frac{\left(\sqrt{2c_1\alpha^2 + c_2\beta^2} \pm \sqrt{\frac{2c_1^2\alpha^2 + c_1c_2\beta^2 + c_2\beta^2(c_2b^2+2c_1)}{c_1}}\right)}{c_2\beta^2(c_2b^2 + 2c_1)}\right\}\right. \\ &\left.(2c_2\beta b_i + 4c_1y_i)\,(2c_2\beta b_j + 4c_1y_j)\right] \end{aligned} \tag{4.3.5}$$

Again using (4.3.4) in (4.2.12) we have,

$$Y^i_j = \delta^i_j - \frac{1}{C^2}\left(1 \pm \sqrt{1 - \frac{C^2c_1}{c_2(c_1\alpha^2 - c_2^2\beta^2)}}\right)b^ib_j \tag{4.3.6}$$

where

$$C^2 = (2c_1\alpha^2c_2\beta^2)b^2 + \frac{b^2(c_2b^2 + 2c_1)^2}{c_1}.$$

Theorem 4.3.2. *Consider a Finsler space $L = (c_1\alpha^2\beta + c_2\beta^3)(\frac{\alpha^2}{\beta})$, for which the condition (4.2.7) is true. Then*

$$V^i_j = X^i_kY^k_j$$

is a Finslerian nonholonomic frame with X^i_k and Y^k_j are given by (4.3.5) and (4.3.6) respectively.

4.3.3 Nonholonomic frame for Infinite series (α, β)-metric:

In the third case, for a Finsler metric with the fundamental function $L = F^2 = \frac{\beta^4}{(\beta-\alpha)^2}$, the Finsler invariants (4.2.6) are given by:

$$\begin{aligned} \rho_1 &= \frac{\alpha^2 - \alpha\beta + \beta^2}{\alpha^4}, \quad \rho_0 = \frac{3\alpha^2 - 2\alpha\beta + 2\beta^2}{\alpha^2}, \\ \rho_{-1} &= \frac{\alpha^3 - 3\alpha\beta^2 + 4\beta^3}{\alpha^4}, \quad \rho_{-2} = \frac{\beta(\alpha^3 - 3\alpha\beta^2 + 4\beta^3}{\alpha^6}, \\ B^2 &= \frac{\alpha^3 - 3\alpha\beta^2 + 4\beta^3(b^2\alpha^2 - \beta^2)}{\alpha^{10}}. \end{aligned} \tag{4.3.7}$$

Using (4.3.7) in (4.2.10) we have,

$$\begin{aligned} X^i_j \;=\; & \sqrt{\frac{\alpha^2 - \alpha\beta + \beta^2)(\alpha^2 - \beta)^2}{\alpha^4}}\delta^i_j - \frac{\alpha^4}{(b^2\alpha^2 - \beta^2)} \\ & \left(\sqrt{(\alpha^2 - \alpha\beta + \beta^2)(\alpha^2 - \beta^2)} \pm \sqrt{(\alpha^2 - \alpha\beta + \beta^2)(\alpha^2 - \beta^2) - \frac{\alpha^3 - 3\alpha\beta^2 + 4\beta^3}{\beta}}\right) \\ & \left(b_i - \frac{\beta}{\alpha^2}y^i\right)\left(b_j - \frac{\beta}{\alpha^2}y^j\right) \end{aligned} \tag{4.3.8}$$

Again using (4.3.7) in (4.2.12) we have,

$$Y^i_j = \delta^i_j - \frac{1}{C^2}\left(1 \pm \sqrt{1 + \frac{\alpha^2\beta c^2}{\alpha^3 + 3\alpha\beta(-\alpha + \beta) - 2\beta^3}}\right) b^i b_j \tag{4.3.9}$$

where

$$C^2 \;=\; \frac{b^2(\alpha^6 + \alpha\beta + \beta^2)(\alpha^2 - \beta^2)}{\alpha^4} - \frac{(\alpha^3 - 3\alpha\beta^2 + 4\beta^3)(b^2\alpha^2 - \beta^2)^2}{\alpha^6\beta}.$$

Theorem 4.3.3. *Consider a Finsler meric $L = \left(\alpha - \beta + \frac{\beta^2}{\alpha}\right)^2$, for which the condition (4.2.7) is true, then*

$$V^i_j = X^i_k Y^k_j,$$

is a Finslerian nonholonomic frame with X^i_k and Y^k_j are given by (4.3.8) and (4.3.9) respectively.

4.4 Conclusion

The frequent Finsler concept used by the physicists Beil and Holland is the existence of a nonholonomic frame on the vertical subbundle VTM of the tangent bundle of a base manifold M. This nonholonomic frame relates a semi-Riemannian metric (the Minkowski or the Lorentz metric) with an induced Finsler metric. In 2001,Antonelli and Bucataru have decided such a nonholonomic frame for two important classes of Finsler spaces that are dual in the sense of Randers and Kropina spaces.

In this chapter, the fundamental tensor field might be taught as the result of two Finsler deformation. Then we can determine a corresponding frame for each of these two Finsler deformations. Consequently, a Finslerian nonholonomic frame for a Finlser spaces with (α, β)-metrics, such a I^{st} Finsler frame product of product of Infinite series metric and Kropina metric and II^{nd} Finsler frame product of Cube root metric and Kropina metric and special (α, β)-metric then we construct Finslerian nonholonomic frame for these. We study the different types of Finsler space with (α, β)-metrics which have nonholonomic frames.

CHAPTER-5

L-DUALLY OF RANDERS CHANGE OF MATSUMOTO METRIC

Content of this chapter

Publications based on this Chapter;

- **Thippeswamy K.R** and Narasimhamurthy S.K., *" L-Dual of Randers Change of Matsumato Metric", Acta Math.Univ.Comenianae Vol 1-8(2016).*

Chapter 5

L-DUALLY OF RANDERS CHANGE OF MATSUMOTO METRIC

5.1 Introduction

Matsumoto metric is an interesting (α, β)-metric introduced by using gradient of slope, speed and gravity used to be studied by [91]. This metric formulates the mannequin of a Finsler space. Many authors ([63],[91],[139]) have studied this metric via different perspectives.

The L-duality of Finsler and Lagrange spaces used to be introduced via R. Miron[101] and was intensively studied by using others. Concrete cases of Hemiltonians acquired by means of L-duality methods were also constructed. In special, the L-dual of some (α, β)-metrics like Randers and Kropina are quite interesting ([50], [51]). In 2007, Masca[72], has studied the L-dual of a Matsumoto space, very lately G.Shanker[90] have succeeded to compute the L-dual of a Generalized Matsumoto space. One of the remarkable results obtained are the concrete L-dual of Randers,Kropina and Matsumoto metrics([50], [51],[90]). However, the importance of L-duality is by far limited to computing the dual of some Finsler fundamental functions.

The cause of this chapter is to investigate the L-duality of the special (α, β)-metric $\frac{\alpha^2}{\alpha-\beta} + \beta$ which is regarded to be Randers change of Matsumoto metric.

5.2 The Legendre Transformation

Let $F^n = (M, F)$ be an n-dimensional Finsler space. The fundamental function $F(x, y)$ is called an (α, β)-metric if F is homogeneous of α and β of degree one, where $\alpha^2 = a(y, y) = a_{ij}y^i y^j, y = y^i \frac{\partial}{\partial x^i}|_x \in T_x M$ is Riemannian metric, and $\beta = b_i(x)y^i$ is a 1-form on $\tilde{TM} = TM - 0$.

A Finsler space with fundamental function:

$$F(x, y) = \alpha(x, y) + \beta(x, y) \tag{5.2.1}$$

is called a Randers space, where as the space having the fundamental function:

$$F(x, y) = \frac{\alpha^2(x, y)}{\beta(x, y)} \tag{5.2.2}$$

is called a Kropina space.

A Finsler space with fundamental function:

$$F(x, y) = \frac{\alpha^2(x, y)}{\alpha(x, y) - \beta(x, y)} \tag{5.2.3}$$

is called a Matsumoto space.

The generalized metrics:

$$F(x, y) = \frac{\alpha^{m+1}(x, y)}{\beta^m(x, y)}, (m \neq 0, -1) \tag{5.2.4}$$

and

$$F(x, y) = \frac{\alpha^{m+1}(x, y)}{(\alpha(x, y) - \beta(x, y))^m}, (m \neq 0, -1) \tag{5.2.5}$$

are called generalized Kropina and Matsumoto metrics respectively and the spaces equipped with the corresponding metrics are called generalized m-Kropina and generalized Matsumoto space respectively.

Definition 1. *A Cartan space C^n is a pair (M, H) which consists of a real n-dimensional C^∞-manifold M and a Hamiltonian function $H : T^x M \setminus \{0\} \longrightarrow \Re$, where $(T^m M, \pi^x, M)$ is the cotangent bundle of M such that $H(x, p)$ has the following properties:*

1. It is two homogeneous with respect to $p_i (i, j, k, = 1, 2, ..., n)$.

2. The tensor field $g^{ij}(x, p) = \frac{1}{2}\frac{\partial^2 H}{\partial p_i \partial p_j}$ is nondegenerate.

Let $C^n = (M, K)$ be an n-dimensional Cartan space having the fundamental function $K(x, p)$. We also consider Cartan spaces having the metric function of the following forms ([50]):

$$K(x,p) = \sqrt{a^{ij}(x)p_i p_j} + b^i(x)p_i. \tag{5.2.6}$$

or

$$K(x,p) = \frac{a^{ij}p_i p_j}{b^i p_i}. \tag{5.2.7}$$

or

$$K(x,p) = \frac{a^{ij}p_i p_j}{\sqrt{a^{ij}(x)p_i p_j} - b^i(x)p_i}. \tag{5.2.8}$$

with $a_{ij}a^{jk} = \delta_i^k$ and we will again call these spaces Randers, Kropina and Matsumoto spaces respectively on the cotangent bundle T^*M.

Definition 2. *A regular Lagrangin (Hamiltonian) on a domain $D \subset TM(D^* \subset T^*M)$ is a real smooth function $L : D \longrightarrow \Re(H : D^* \longrightarrow \Re)$ such that the matrix with entries*

$$g_{ab}(x,y) = \dot{\partial}_a \dot{\partial}_b L(x,y)(g^{*ab}(x,y) = \dot{\partial}^a \dot{\partial}^b H(x,y))$$

is everywhere nondegenerate on $D(D^*)$.

A Lagrange (Hamilton) manifold is a pair $(M, L(H))$, where M is a smooth manifold and $L(H)$ is regular Lagrangian (Hamiltonian) on $D(D^*)$.

Example 1:

(a) Every Finsler space $F^n = (M, F(x, y))$ is a Lagrange manifold with $L = \frac{1}{2}F^2$.

(b) Every Cartan space $C^n = (M, \bar{F}(x, p))$ is a Hamilton manifold with $H = \frac{1}{2}\bar{F}^2$. (Here $\bar{F}$ is positively 1-homogeneous in p_i and the tensor $\bar{g}^{ab} = \frac{1}{2}\dot{\partial}_a\dot{\partial}_b\bar{F}^2$ is nondegenerate).

(c) (M, L) and (M, H) with

$$L(x, y) = \frac{1}{2}a_{ij}(x)y^i y^j + b_i(x)y^i + c(x)$$

and

$$H(x, y) = \frac{1}{2}\bar{a}^{ij}(x)p_i p_j + \bar{b}^i(x)p_i + \bar{c}(x).$$

are Lagrange and Hamilton manifolds respectively. (Here $a_{ij}, \bar{a}^{ij}$ are the fundamental tensors of Riemannian manifold, b_i are components of covector field,$\bar{b}^i$ are the components of a vector fields, C and $\bar{C}$ are the smooth functions on M).

Let $L(x, y)$ be a regular Lagrangian on a domain $D \subset TM$ and let $H(x, p)$ be a regular Hamiltonian on a domain $D^* \subset T^*M$. If L is a differential map, we can consider the fiber derivative of L, locally given by the diffeomorphism between the open set $U \subset D$ and $U^* \subset D^*$ ([101],[102]):

$$\varphi(x, y) = (x^i, \dot{\partial}_a L(x, y)). \tag{5.2.9}$$

which is called the Legendre transformation. We can define, in this case, the function $H : U^* \mapsto R :$

$$H(x, y) = p_a y^a - L(x, y). \tag{5.2.10}$$

where $y = y^a$ is the solution of the equation:

$$p_a = \dot{\partial}_a L(x, y). \tag{5.2.11}$$

In the same manner, the fiber derivative of H is locally given by:

$$\varphi(x,p) = (x^i, \dot{\partial}^a H(x,p)), \tag{5.2.12}$$

where φ is a diffeomorphism between the same open sets $U \subset D$ and $U^* \subset D^*$ and we can consider the function $L : U \mapsto R$, such that

$$L(x,y) = p_a y^a - H(x,p), \tag{5.2.13}$$

where $p = (p_a)$ is the solution of the equations:

$$y^a = \dot{\partial}^a H(x,p). \tag{5.2.14}$$

The Hamiltonian given by (5.2.10) is the Legendre transformation of the Lagrangian L and the Lagrangian given by (5.2.13) is called the Legendre transformation of the Hamiltonian H.

If (M,K) is a Cartan space, then (M,H) is a Hamiltonian manifold ([101],[102]), where $H(x,p) = \frac{1}{2}K^2(x,p)$ is 2-homogenous on a domain of T^*M. So we get the following transformation of H on U :

$$L(x,y) = p_a y^a - H(x,p) = H(x,p). \tag{5.2.15}$$

Theorem 5.2.1. *The scalar field $L(x,y)$ defined by (5.2.16) is a positively 2-homogeneous regular Lagrangian on U.*

Therefore, we get Finsler metric F of U, so that

$$L = \frac{1}{2}F^2 \tag{5.2.16}$$

Thus for the Cartan space (M,K) we always can locally associate a Finsler space (M,F) which will be called the L-dual of a Cartan space $(M, C_{|U^*})$ vice versa, we can associate, locally, a Cartan space to each and every Finsler space which will be called the L-dual of a Finsler space $(M, F_{|U})$.

5.3 The L-Dual of Randers change of Matsumoto metric

In this section, we consider the Finsler (α,β)-metric $F = \frac{\alpha^2}{\alpha-\beta} + \beta$ we put, $\alpha^2 = y_i y^i, \beta = b_i y^i, \beta^* = b^i p_i, p^i = a^{ij} p_j, \alpha^{*2} = p_i p^i = a^{ij} p_i p_j$.

We have :

$$p_i = \frac{1}{2}\dot{\partial}^i F^2 = F\left[\frac{2Fy^i}{\alpha^2} - \frac{F}{\alpha(\alpha-\beta)}(y^i - b_i\alpha)\right] \tag{5.3.1}$$

Contracting (5.3.1) by p^i and b^i respectively, we get

$$\alpha^{*2} = F\left[\frac{2F^3}{\alpha^2} - \frac{F}{\alpha(\alpha-\beta)}(F^2 - \alpha\beta^*) + \beta^*\right], \tag{5.3.2}$$

$$\beta^* = F\left[\frac{2F\beta}{\alpha^2} - \frac{F}{\alpha(\alpha-\beta)}(\beta - b^2\alpha) + b^2\right], \tag{5.3.3}$$

In [50], for a Finsler (α,β)-metric F on a manifold M, there is a positive function $\phi = \phi(s)$ on $(-b_0; b_0)$ with $\phi(0) = 1$ and $F = \alpha\phi(s), s = \frac{\beta}{\alpha}$, where $\alpha = \sqrt{a_{ij}y^i y^j}$ and $\beta = b_i y^i$ with $||\beta||_x < b_0, \forall x \in M$. ϕ satisfies $\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, (|s| \leq b_0)$.

A Randers change of Matsumoto metric is a special (α,β)-metric with $\phi = [1 + s - \frac{1}{s}]$. Using Shens notation[127] $s = \frac{\beta}{\alpha}$, (5.3.1) and (5.3.2) become

$$\alpha^{*2} = F\left[\frac{2F}{(1-s)^2} - \frac{1}{(1-s)^3} + \frac{\beta^*}{(1-s)^2} + \beta^*\right], \tag{5.3.4}$$

and

$$\beta^* = F\left[\frac{2s}{(1-s)} - \frac{1}{(1-s)^2}(s - b^2) + b^2\right]. \tag{5.3.5}$$

Putting $(1-s) = t$, so that $s = (1-t)$ in (5.3.4) and (5.3.5), we get

$$\alpha^{*2} = F\left[\frac{2F^2}{t^2} - \frac{F^2}{t^3} + F(1 + \frac{1}{t^2})\beta^*\right], \tag{5.3.6}$$

and

$$\beta^* = F\left[\frac{2(1-t)}{t} - \frac{1}{t^2}(1 - t - b^2) + b^2\right]. \tag{5.3.7}$$

Now, we have following two cases:

***Case* I.** For $b^2 = 1$, from (5.2.7), we get

$$F = \left[\frac{\beta^* t}{(3-t)}\right]. \tag{5.3.8}$$

From (5.3.6) and (5.3.8), we get

$$(K-1)s^3 + 3Ks^2 + 4s - (4K+5) = 0, \tag{5.3.9}$$

where $K = \frac{\alpha^{*2}}{\beta^{*2}}$.

Solving (5.3.9) for s, using maple, we get

$$s = \left[\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right]^{\frac{1}{3}} + \left[\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right]^{\frac{1}{3}} - \frac{A}{3}, \tag{5.3.10}$$

where

$$p = \left[\frac{A^2 - 2A^3 + B}{3}\right], q = \left[\frac{3C - AB}{3}\right], \tag{5.3.11}$$

and

$$A = \frac{3K}{(K-1)}, B = \frac{4}{(K-1)}, C = \frac{4K+5}{1-K}. \tag{5.3.12}$$

From (5.3.8), we get

$$F = \frac{\beta^*\left[1 - \left[\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right]^{\frac{1}{3}} - \left[\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right]^{\frac{1}{3}} + \frac{A}{3}\right]}{2 + \left[\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right]^{\frac{1}{3}} + \left[\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right]^{\frac{1}{3}} - \frac{A}{3}}. \tag{5.3.13}$$

From (5.2.15) and (5.2.16), we get

$$H(x,p) = \frac{\frac{\beta^{*2}}{2}\left[1 - \left[\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right]^{\frac{1}{3}} - \left[\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right]^{\frac{1}{3}} + \frac{A}{3}\right]^2}{\left[2 + \left[\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right]^{\frac{1}{3}} + \left[\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right]^{\frac{1}{3}} - \frac{A}{3}\right]^2}. \tag{5.3.14}$$

Hence we have the following theorem :

Theorem 5.3.1. *Let (M, F) be a Randers change of Matsumoto space and $b = (a_{ij}b^i b^j)^{\frac{1}{2}}$ the Riemannian length of b_i. Then if $b^2 = 1$, the L-dual of (M, F) is the space having the fundamental function (5.3.14).*

***Case* II.** For $b^2 \neq 1$, from (5.3.7), we get

$$F = \left[\frac{\beta^* t^2}{(b^2-2)t^2 + 3t + (b^2-1)}\right], \tag{5.3.15}$$

from (5.3.6) and (5.3.15), we get

$$s^4 + A_1 s^3 + A_2 s^2 + A_3 s + A_4 = 0, \tag{5.3.16}$$

where

$$A_1 = (4Kb^4 - 10Kb^2 - 4b^2 + 4K + 6)/A, A_2 = (-8Kb^4 + 12Kb^2 + 8b^2 - K - 7)/A$$

,

$$A_3 = (8Kb^4 - 4Kb^2 - 8b^2 + 1)/A, A_4 = (-4Kb^4 + 4b^2 + 1)/A, A = (-Kb^4 + 4Kb^2 + b^2 - 4K - 2)$$

.

Using maple, after long computations solving (5.3.16) for s,we get

$$s = \left[\frac{-H_2 + \sqrt{H_2^2 - 4H_1H_3}}{2H_1} - \frac{A_1}{4}\right], \tag{5.3.17}$$

where

$$H_1 = B_1 + 2C, H_2 = -B_2, H_3 = B_1^2 - B_3 + 2B_1C + C^2,$$

$$B_1 = \frac{-3A_1^2}{8} + A_2, B_2 = \frac{A_1^3}{8} - \frac{A_1A_2}{8} + A_3, B_3 = \frac{-3A_1^4}{256} - \frac{A_3A_1}{4} + \frac{A_1^2A_2}{16} + A_4,$$

$$C = \left(\frac{-P_2}{2} + \sqrt{\frac{P_2^2}{4} + \frac{P_1^3}{27}}\right)^{1/3} + \left(\frac{-P_2}{2} - \sqrt{\frac{P_2^2}{4} + \frac{P_1^3}{27}}\right)^{1/3} - \frac{D_1}{3},$$

$$P_1 = \left[\frac{D_1^2}{3} - \frac{2D_1^3}{3} + D_2\right], P_2 = \left[D_3 - \frac{D_1D_2}{3}\right],$$

and

$$D_1 = \frac{5}{2}B_1, D_2 = 2B_1^2 - B_3, D_3 = \frac{4B_1^3 - B_2^2 - 4B_1B_3}{8}.$$

From (5.3.15), we get

$$F = \frac{\beta^* \left[1 - \frac{-H_2+\sqrt{H_2^2-4H_1H_3}}{2H_1} + \frac{A_1}{4}\right]^2}{(b^2-2)\left[\frac{-H_2+\sqrt{H_2^2-4H_1H_3}}{2H_1} + \frac{A_1}{4}\right]^2 + 3\left[\frac{-H_2+\sqrt{H_2^2-4H_1H_3}}{2H_1} + \frac{A_1}{4}\right] + (b^2-1)}. \quad (5.3.18)$$

From (5.2.15) and (5.2.16), we get

$$H(x,p) = \frac{\frac{\beta^{*2}}{2}\left[1 - \frac{-H_2+\sqrt{H_2^2-4H_1H_3}}{2H_1} + \frac{A_1}{4}\right]^4}{\left[(b^2-2)\left[\frac{-H_2+\sqrt{H_2^2-4H_1H_3}}{2H_1} + \frac{A_1}{4}\right]^2 + 3\left[\frac{-H_2+\sqrt{H_2^2-4H_1H_3}}{2H_1} + \frac{A_1}{4}\right] + (b^2-1)\right]^2}. \quad (5.3.19)$$

Hence we have the following theorem :

Theorem 5.3.2. *Let (M, F) be a Randers change of Matsumoto space and $b = (a_{ij}b^ib^j)^{\frac{1}{2}}$ the Riemannian length of b_i. Then if $b^2 \neq 1$, the L-dual of (M, F) is the space having the fundamental function (5.3.19).*

5.4 Conclusion

As we know, Finsler geometry is just Riemannian geometry without the quadratic restriction. Therefore, it is natural to extending the construction of locally dually flat metrics for Finsler geometry. In Finsler geometry, Z.Shen extends the idea of locally dually flatness metric in Finsler information geometry, which play a very important role in studying many applications in Finsler information structure.

In this chapter, we proved the following two consequences first one is Let (M, F) be a Randers change of Matsumoto space and $b = (a_{ij}b^ib^j)^{\frac{1}{2}}$ the Riemannian length of b_i.

Then if $b^2 = 1$, the L-dual of (M, F) is the space having the fundamental function(5.3.14) and second one is Let (M, F) be a Randers change of Matsumoto space and $b = (a_{ij}b^i b^j)^{\frac{1}{2}}$ the Riemannian length of b_i. Then if $b^2 \neq 1$, the L-dual of (M, F) is the space having the fundamental function(5.3.19).

CHAPTER-6

CONFORMAL CHANGE OF FINSLER SUBSPACES

Content of this chapter

Publications based on this Chapter;

- **Thippeswamy K.R** and Narasimhamurthy S.K., *"Killing Vector Field in Conformal Change of Finsler Spaces", Communicated.*
- **Thippeswamy K.R** *and Narasimhamurthy S.K., " Finslerian Subspaces subjected to Conformal β-Change", Communicated.*

Chapter 6

CONFORMAL CHANGE OF FINSLER SUBSPACES

6.1 Introduction

Let (M, L) be a Finsler space, where M is an n-dimentional differentiable manifold equipped with a fundamental function L. Given a function σ, the change

$$\bar{L}(x,y) \longrightarrow e^{\sigma(x)}L(x,y), \tag{6.1.1}$$

is called a conformal change. The conformal theory of Finsler spaces has been initiated by many authors. For a differential 1-form $\beta(x,y) = b_i(x)y^i$ on M, Randers, introduced a special Finsler space defined by β-change $\bar{L} = L + \beta$, where L is Riemannian. In 2008, S. Abed ([61], [62]) introduced the transformation $\bar{L} = e^{\sigma(x)}L + \beta$, thus generalizing the conformal, Randers and generalized Randers changes.

General change of Finsler metrics defined by:

$$L(x,y) \rightarrow \bar{L}(x,y) = f(e^{\sigma(x)}L(x,y), \beta(x,y))$$

where f is a positively homogeneous function of degree one in $\bar{L} := e^{\sigma}L$ and β. This change will be referred to as a generalized β-conformal change. It is clear that this change is a generalization of the above mentioned changes and deals simultaneously with β-change and conformal change. It combines also the special case of Shibata ($\bar{L} = f(L, \beta)$) and that of Abed ($\bar{L} = e^{\sigma} L, \beta$) .

In 1984, C. Shibata [132] studied β-change of Finsler metrics and discussed certain invariant tensors under such a change. In 1979, Singh, et. al. [140] studied a Randers space $F^n(M, L(x, y) = (g_{ij}(x)y^i y^j)^{\frac{1}{2}} + b_i(x)y^i), n \geq 2$ which undergoes a change $L(x, y) \mapsto L^*(x, y) = L^2(x, y) + (\alpha_i(x)y^i)^2$.

6.2 Preliminaries

Let $F^n = (M, L), n \geq 2$ be an n-dimensional C^∞ Finsler manifold with fundamental function $L = L(x, y)$. Consider the following change of Finsler structures which will be referred to as a generalized β-conformal change:

$$L(x, y) \longrightarrow \bar{L}(x, y) = f(e^{\sigma(x)} L(x, y), \beta(x, y)), \tag{6.2.1}$$

where f is a positively homogeneous function of degree one in $e^{\sigma} L$ and 1-form β where, $\beta = b_i(x) dx^i$.

We define

$$f_1 := \frac{\partial f}{\partial L}, f_2 := \frac{\partial f}{\partial \beta}, f_{12} := \frac{\partial^2 f}{\partial L \partial \beta}, ...,$$

where $\tilde{L} = e^{\sigma} L$.

The angular metric tensor $\bar{h}_{ij}$ of the space $\bar{F}^n$ is given by [146]

$$\bar{h}_{ij} = e^{\sigma} p h_{ij} + q_0 m_i m_j \tag{6.2.2}$$

where

$$\begin{aligned} p &= f f_1/L, \quad q = f f_2, \quad q_0 = f f_{22}, \quad p_0 = f_2^2 + q_0, \quad q_{-1} = f f_{12}/L, \\ p_{-1} &= q_{-1} + p f_2/f, \quad q_{-2} = f(e^{\sigma} f_{11} - f_1/L)/L^2, \quad p_{-2} = q_{-2} + e^{\sigma} p^2/f^2, \\ m_i &= b_i - \beta y^i/L^2 \neq 0, \qquad \sigma_i = \partial_i \sigma. \end{aligned} \tag{6.2.3}$$

h_{ij} being the angular metric tensor of F^n. The fundamental metric tensor $\bar{g}_{ij}$ and its inverse $\bar{g}^{ij}$ of $\bar{F}^n$ are expressed as

$$\bar{g}_{ij} = e^\sigma p g_{ij} + p_0 b_i b_j + e^\sigma p_{-1}(b_i y_j + b_j y_i) + e^\sigma p_{-2} y_i y_j, \tag{6.2.4}$$

$$\bar{g}^{ij} = (e^{-\sigma}/p) g^{ij} - s_0 b^i b^j - s_{-1}(b^i y^j + b^j y^i) - s_{-2} y^i y^j, \tag{6.2.5}$$

where

$$\begin{aligned} s_0 &= e^{-\sigma} f^2 q_0/(\varepsilon p L^2), \quad s_{-1} = p_{-1} f^2/(\varepsilon p L^2), \\ s_{-2} &= p_{-1}(e^\sigma m^2 p L^2 - b^2 f^2)/(\varepsilon p \beta L^2), \\ \varepsilon &= f^2(e^\sigma p + m^2 q_0)/L^2 \neq 0, \quad m^2 = g^{ij} m_i m_j. \end{aligned} \tag{6.2.6}$$

g_{ij} and g^{ij} respectively being the metric tensor and inverse metric tensor of F^n. The Cartan tensor $\bar{C}_{ijk}$ and the associate Cartan tensor $\bar{C}^l_{ij}$ of $\bar{F}^n$ are given by the following expressions:

$$\bar{C}_{ijk} = e^\sigma p C_{ijk} + \frac{1}{2} e^\sigma p_{-1}(h_{ij} m_k + h_{jk} m_i + h_{ki} m_j) + \frac{1}{2} p_{02} m_i m_j m_k, \tag{6.2.7}$$

The $(h)hv$-torsion tensor $\bar{C}^l_{ij}$ is expressed in terms of C^l_{ij} as

$$\bar{C}^l_{ij} = C^l_{ij} + M^l_{ij}, \tag{6.2.8}$$

where

$$\begin{aligned} M^l_{ij} &= \frac{1}{2p}[e^{-\sigma} m^l - p m^2(s_0 b^l + s_{-1} y^l)](e^\sigma p_{-1} h_{ij} + p_{02} m_i m_j) \\ &- e^\sigma (s_0 b^l + s_{-1} y^l)(p C_{isj} b^s + p_{-1} m_i m_j) + \frac{p_{-1}}{2p}(h^l_i m_j + h^l_j m_i), \end{aligned} \tag{6.2.9}$$

$$h^i_j = g^{il} h_{lj}, \qquad p_{02} = \frac{\partial p_0}{\partial \beta}$$

C_{ijk} and C^l_{ij} respectively being the Cartan tensor and associate Cartan tensor of F^n. The spray coefficients $\bar{G}^i$ of $\bar{F}^n$ in terms of the spray coefficients G^i of F^n are expressed as

$$\bar{G}^i = G^i + D^i, \tag{6.2.10}$$

where

$$\begin{aligned}
D^i &= \frac{\sigma_0}{2p}\{[2p-\beta p_{-1}-e^{\sigma}p^2L^2s_{-2}-ps_{-1}(2e^{\sigma}p\beta+e^{\sigma}p_{-1}L^2m^2)]y^i-2e^{\sigma}p^2\beta s_0b^i\}\\
&+ \frac{q}{p}e^{-\sigma}F^i_0-\frac{1}{2}L^2\sigma^i+\frac{1}{2}(e^{\sigma}pE_{00}-2qF_{\beta 0}+e^{\sigma}pL^2\sigma_\beta)(s_0b^i+s_{-1}y^i), \qquad (6.2.11)\\
E_{jk} &= (1/2)(b_{j|k}+b_{k|j}), \qquad F_{jk}=(1/2)(b_{j|k}-b_{k|j}), \qquad F^i_j=g^{ik}F_{kj},
\end{aligned}$$

the symbol '|' denote the h-covariant derivative with respect to the Cartan connection $C\Gamma$ and the lower index '$_0$' (except in s_0) denote the contraction by y^i.

The relation between the coefficients $\bar{N}^i_j$ of Cartan nonlinear connection in $\bar{F}^n$ and the coefficients N^i_j of the corresponding Cartan nonlinear connection in F^n is given by

$$\bar{N}^i_j = N^i_j + D^i_j, \qquad (6.2.12)$$

where

$$D^i_j = \frac{e^{-\sigma}}{p}A^i_j-(s_0b^i+s_{-1}y^i)A_{tj}b^t-(qb_{0|j}+e^{\sigma}pL^2\sigma_j)(s_{-1}b^i+s_{-2}y^i), \quad (6.2.13)$$

$$\begin{aligned}
A_{ij} &= E_{00}B_{ij}+F_{i0}Q_j+qF_{ij}+E_{j0}Q_i-2(e^{\sigma}pC_{sij}+V_{sij})D^s\\
&+ \frac{1}{2}\sigma_0[2e^{\sigma}pg_{ij}+2e^{\sigma}p_{-1}m_jy_i-2\beta B_{ij}+e^{\sigma}p_{-1}(b_iy_j-b_jy_i)] \qquad (6.2.14)\\
&- \frac{1}{2}\sigma_i(e^{\sigma}L^2p_{-1}m_j+2e^{\sigma}py_j)+\frac{1}{2}\sigma_j(2e^{\sigma}py_i+e^{\sigma}L^2p_{-1}m_i),
\end{aligned}$$

$$A^i_j = g^{li}A_{lj}, \qquad 2B_{ij}=e^{\sigma}p_{-1}h_{ij}+p_{02}m_im_j, \qquad Q_i=e^{\sigma}p_{-1}y_i+p_0b_i$$

. The coefficients $\bar{F}^i_{jk}$ of Cartan connection $C\bar{\Gamma}$ in $\bar{F}^n$ and the coefficients F^i_{jk} of the corresponding Cartan connection $C\Gamma$ in F^n are related as

$$\bar{F}^i_{jk} = F^i_{jk} + D^i_{jk}, \qquad (6.2.15)$$

where

$$\begin{aligned} D^i_{jk} &= \{(e^{-\sigma}/p)g^{it} - (s_0 b^i + s_{-1}y^i)b^t - (s_{-1}b^i + s_{-2}y^i)y^t\}\{F_{tk}Q_j + F_{tj}Q_k + E_{jk}Q_t \\ &+ \frac{1}{2}\Theta_{(j,k,t)}(2e^{\sigma}pC_{jkm}D^m_t + 2V_{jkm}D^m_t - K_{jk}\sigma_t - 2B_{jk}b_{0|t})\} \qquad (6.2.16) \\ V_{ijk} &= \frac{1}{2}e^{\sigma}p_{-1}(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + \frac{1}{2}p_{02}m_i m_j m_k \\ K_{ij} &= A_1 g_{ij} + A_2 b_i b_j + A_3(b_i y_j + b_j y_i) + A_4 y_i y_j, \\ A_1 &= e^{\sigma}(2p - \beta p_{-1}), \quad A_2 = -\beta p_{02}, \quad A_3 = e^{\sigma}p_{-1} + (\beta^2/L^2)p_{02}, \\ A_4 &= e^{\sigma}p_{-2} - (\beta^3/L^4)p_{02}, \Theta_{(j,k,t)}\{A_{jkt}\} = A_{jkt} - A_{ktj} - A_{tjk}, \end{aligned}$$

The tensor D^i_{jk} has the properties:

$$D^i_{j0} = B^i_{j0} = D^i_j; \quad D^i_{00} = 2D^i, \quad where \quad B^i_{jk} = \partial_k D^i_j.$$

6.3 Conformal transformation of Finsler space with Killing vector fields

Let us consider an infinitesimal transformation

$$'x^i = x^i + \epsilon v^i(x), \qquad (6.3.1)$$

where ϵ is an infinitesimal constant and $v^i(x)$ is a contravariant vector field.

The vector field $v^i(x)$ is said to be a Killing vector field in F^n if the metric tensor of the Finsler space with respect to the infinitesimal transformation (6.3.1) is Lie invariant, that is,

$$\pounds_v g_{ij} = 0, \qquad (6.3.2)$$

$\pounds_v$ being the operator of Lie differentiation. Equivalently, the vector field $v^i(x)$ is Killing in F^n if

$$v_{i|j} + v_{j|i} + 2C^l_{ij} v_{l|0} = 0. \tag{6.3.3}$$

where $v_i = g_{il} v^l$.

Now, we prove the following result which gives a necessary and sufficient condition for a Killing vector field in F^n to be Killing in $\bar{F}^n$:

Theorem 6.3.1. *A Killing vector field $v^i(x)$ in F^n is Killing in $\bar{F}^n$ if and only if*

$$M^l_{ij} v_{l|0} + C_{rjt} v^t D^r_i + C_{rit} v^t D^r_j + v_r(e^{-\sigma} F^{-r}_{ij} - F^r_{ij}) + e^{\sigma} \bar{C}^l_{ij}(2C_{rlt} v^t D^r + v_r D^r_l) = 0, \tag{6.3.4}$$

where $\bar{C}^l_{ij}$ is the associate Cartan tensor of $\bar{F}^n$.

Proof: Assume that $v^i(x)$ is Killing in F^n. Then (6.3.3) is satisfied. By definition, the h-covariant derivatives of v_i with respect to $C\bar{\Gamma}$ and $C\Gamma$ are respectively given as

$$(a) \quad v_{i||j} = \partial_j v_i - e^{\sigma}(\dot{\partial}_r v_i)\bar{G}^r_j - e^{\sigma} v_r \bar{F}^r_{ij}, \qquad (b) \quad v_{i|j} = \partial_j v_i - (\dot{\partial}_r v_i) G^r_j - v_r F^r_{ij}, \tag{6.3.5}$$

where $\partial_j = \partial/\partial x^j$ and $'\|'$ denote the h-covariant differentiation with respect to $C\bar{\Gamma}$. Equation (6.3.5)(a), by virtue of (6.2.10), (6.2.15) and (6.3.5)(b), takes the form

$$v_{i||j} = v_{i|j} - 2C_{rit} v^t D^r_j - v_r(e^{-\sigma} F^{-r}_{ij} - F^r_{ij}). \tag{6.3.6}$$

Now, from (6.3.6), we have

$$\begin{aligned} v_{i||j} + v_{j||i} + 2e^{\sigma} \bar{C}^l_{ij} v_{l||0} &= v_{i|j} + v_{j|i} + 2e^{\sigma} C^l_{ij} v_{l|0} - 2C_{rit} v^t D^r_j - 2C_{rjt} v^t D^r_i \\ &- 2v_r(e^{-\sigma} F^{-r}_{ij} - F^r_{ij}) - 2e^{\sigma} \bar{C}^l_{ij}(2C_{rlt} v^t D^r + v_r D^r_l). \end{aligned} \tag{6.3.7}$$

Using (6.2.10) in (6.3.7) and applying (6.3.3), we get

$$\begin{aligned} v_{i||j} + v_{j||i} + 2e^{\sigma} \bar{C}^l_{ij} v_{l||0} &= 2M^l_{ij} v_{l|0} - 2C_{rit} v^t D^r_j - 2C_{rjt} v^t D^r_i \\ &- 2v_r(e^{-\sigma} F^{-r}_{ij} - F^r_{ij}) - 2e^{\sigma} \bar{C}^l_{ij}(2C_{rlt} v^t D^r + v_r D^r_l). \end{aligned} \tag{6.3.8}$$

Proof completes with the observation that $v^i(x)$ is Killing in $\bar{F}^n$ if and only if $v_{i||j}+v_{j||i}+2e^{\sigma}\bar{C}^l_{ij}v_{l||0}=0$, that is, if and only if (6.3.4) holds.

If a vector field $v^i(x)$ is Killing in F^n and $\bar{F}^n$, then from Theorem 3.1, (6.3.4) holds, which on transvection by y^i yields

$$2C_{rlt}v^tD^r+v_r(e^{-\sigma}F^{-r}_{ij}-F^r_{ij})=0. \tag{6.3.9}$$

Equation (6.3.4), in view of (6.3.9), enables us to state the following:

Corollary 6.3.2. *If a vector field $v^i(x)$ is Killing in F^n and $\bar{F}^n$, then*

$$C_{rit}v^tD^r_j+C_{rjt}v^tD^r_i+v_r(e^{-\sigma}F^{-r}_{ij}-F^r_{ij})-M^l_{ij}v_{l|0}=0. \tag{6.3.10}$$

As another important consequence of Theorem 6.3.1, we have the following:

Corollary 6.3.3. *If a vector field $v^i(x)$ is Killing in F^n and $\bar{F}^n$, then the vector $v_i(x,y)$ is orthogonal to the vector $D^i(x,y)$.*

Proof: As $v^i(x)$ is Killing in F^n and $\bar{F}^n$, (6.3.4) holds, which on transvection by y^i gives (6.3.9). Again transvecting (6.3.9) by y^j , it follows that $v_rD^r=0$. This proves the result.

6.4 Finslerian Subspaces given by conformal β-change

Let M^n be an n-dimensional smooth manifold and $F^n=(M^n,L)$ be an n-dimensional Finsler space equipped with a fundamental function $L(x,y)$ on M^n. Then the metric tensor $g_{ij}(x,y)$ and Cartans C-tensor $C_{ijk}(x,y)$ are given by

$$g_{ij}=(\partial^2L^2/\partial y^i\partial y^j)/2, \quad C_{ijk}=(\partial g_{ij}/\partial y^k)/2,$$

and we can introduce in F^n the Cartan connection $C\Gamma = (F^i_{jk}, G^i_j, C^i_{jk})$. An m-dimensional subspace M^m of the underlying smooth manifold M^n may be parametrically represented by the equation $x^i = x^i(u^\alpha)(i = 1, 2, ..., n)$, where u^α are Gaussian coordinates on M^m and Greek indices run from 1 to m. Here, we shall assume that the matrix consisting of the projection factors $B^i_\alpha = \partial x^i/\partial u^\alpha$ is of rank m. The following notations are also employed : $B^i_{\alpha\beta} := \partial^2 x^i/\partial u^\alpha \partial u^\beta, B^i_{0\beta} := v^\alpha B^i_{\alpha\beta}, B^{ij...}_{\alpha\beta....} := B^i_\alpha B^j_\beta$....... If the supporting element y^i at a point (u^α) of M^m is assumed to be tangential to M^m, we may then write $y^i = B^i_\alpha(u)v^\alpha$, so that v^α is thought of as the supporting element of M^m at the point (u^α). Since the function $\underline{L}(u, v) := L(x(u), y(u, v))$ gives rise to a Finsler metric of M^m, we get an m-dimensional Finsler space $F^m = (M^m, \underline{L}(u, v))$.

At each point (u^α) of F^m, the unit normal vectors $N^i_\alpha(u, v)$ are defined by

$$g_{ij} B^i_\alpha N^j_a = 0, \quad g_{ij} N^i_a N^j_b = \delta_{ab} \quad (a, b, ... = m + 1, ..., n). \tag{6.4.1}$$

If (B^α_i, N^a_i) is the inverse matrix of (B^i_α, N^i_a), we have

$$B^i_\alpha B^\beta_i = \delta^\beta_\alpha, \quad B^i_\alpha N^a_i = 0, \quad N^i_a B^\alpha_i = 0, \quad N^i_a N^b_i = \delta^b_a, \tag{6.4.2}$$

and further

$$B^i_\alpha B^\alpha_j + N^i_a N^a_j = \delta^i_j. \tag{6.4.3}$$

Making use of the inverse matrix $(g^{\alpha\beta})$ of $(g_{\alpha\beta})$, we get $B^\alpha_i = g^{\alpha\beta} g_{ij} B^j_\beta$. By (6.4.1) and (6.4.3), we also have $\delta_{ab} N^b_i = g_{ij} N^j_a$.

For the induced Cartan connection $C\Gamma = (F^\alpha_{\beta\gamma}, G^\alpha_\beta, C^\alpha_{\beta\gamma})$ on F^m, the second fundamental h-tensor $H^a_{\alpha\beta}$ and the normal curvature vector H^a_α in a normal direction N^i_a are given by

$$H^a_{\alpha\beta} = N^a_i(B^i_{\alpha\beta} + F^i_{jk}B^{jk}_{\alpha\beta}) + M^a_{\alpha b}H^b_\beta, \tag{6.4.4}$$
$$H^a_\alpha = N^a_i(B^i_{0\alpha} + G^i_jB^j_\alpha),$$

where $M^a_{\alpha b} := C^j_{ik}B^i_\alpha N^a_j N^k_b$ and $B^i_{0\alpha} = B^i_{\beta\alpha}v^\beta$. Contracting $H^a_{\beta\alpha}$ by v^β, we immediately get

$$H^a_{0\alpha} := H^a_{\beta\alpha}v^\beta = H^a_\alpha. \tag{6.4.5}$$

Lets introduce in $F^n = (M^n, \bar{L})$ the Cartan connection $C\bar{\Gamma} = (\bar{F}^i_{jk}, \bar{G}^i_j, \bar{C}^i_{jk})$ from a generalized conformal β-change of the metric.

We now consider a Finslerian subspace $F^m = (M^m, \bar{L}(u,v))$ of F^n and another Finslerian subspace $\bar{F}^m = (M^m, \bar{L}(u,v))$ of the $\bar{F}^n$ given by the generalized conformal β-change. Let N^i_a be unit normal vectors at each point of F^m, and (B^α_i, N^a_i) be the inverse matrix of (B^i_α, N^i_a). The functions $B^i_\alpha(u)$ may be considered as components of m linearly independent vectors tangent to F^m and they are invariant under the generalized conformal β-change. The unit normal vectors $\bar{N}^i_a(u,v)$ of $\bar{F}^m$ are uniquely determined by

$$\bar{g}_{ij}B^i_\alpha\bar{N}^j_a = 0, \quad \bar{g}_{ij}\bar{N}^i_a\bar{N}^j_b = \delta_{ab}. \tag{6.4.6}$$

The fundamental tensor $\bar{g}_{ij} = (\partial^2\bar{L}^2/\partial y^i\partial y^j)/2$ of the Finsler space $\bar{F}^n$ given by (6.2.4), (6.2.5).

Now contracting (6.4.1) by v^α, we immediately get

$$y_iN^i_a = 0 \tag{6.4.7}$$

Further contracting (6.2.5) by $N^i_aN^j_b$ and paying attention to (6.4.1), (6.4.6) and (6.4.7), we have

$$\bar{g}_{ij}N^i_aN^j_b = e^\sigma p\delta_{ab} + p_0(b_iN^i_a)(b_jN^j_b). \tag{6.4.8}$$

Putting a = b, then we obtain

$$\bar{g}_{ij}(\pm N^i_a/\sqrt{e^\sigma p + p_0(b_iN^i_a)^2})(\pm N^j_a/\sqrt{e^\sigma p + p_0(b_iN^i_a)^2}) = 1, \tag{6.4.9}$$

provided $e^\sigma p + p_0(b_iN^i_a)^2 > 0$. Therefore we can put

$$\bar{N}^i_a = N^i_a/\sqrt{e^\sigma p + p_0(b_iN^i_a)^2}, \tag{6.4.10}$$

where we have chosen the sign " + " in order to fix an orientation. On using (6.4.1) and (6.4.7), the first condition of (6.4.6) gives us

$$(b_iN^i_a)(p_0b_jB^j_\alpha + e^\sigma y_jB^j_\alpha) = 0. \tag{6.4.11}$$

Now, assuming that $p_0b_jB^j_\alpha + e^\sigma p_{-1}y_jB^j_\alpha = 0$ and contracting this by v^α, we find $p_0\beta + e^\sigma p_{-1}L^2 = 0$. By (6.2.4) this equation lead us to $ff_\beta = 0$, where we have used $Lf_{L\beta} + \beta f_{\beta\beta} = 0$ and $Lf_L + \beta f_\beta = f$ owing to the homogeneity of f. Thus we have $f_\beta = 0$ because of $f \neq 0$. This fact means $\bar{L} = f(L)$ and contradicts the definition of a generalized conformal β-change of metric. Consequently (6.4.11) gives us

$$b_iN^i_a = 0. \tag{6.4.12}$$

Therefore (6.4.10) is rewritten as

$$\bar{N}^i_a = N^i_a/\sqrt{e^\sigma p} \quad (p > 0). \tag{6.4.13}$$

and then it is clear $\bar{N}^i_a$ satisfies (6.4.6). Summarizing the above, we obtain

Theorem 6.4.1. *For a field of linear frame $(B^i_1, ..., B^i_m, N^i_{m+1}, ..., N^i_n)$ of F^n, there exists a field of linear frame $(B^i_1, ..., B^i_m, \bar{N}^i_{m+1}, ..., \bar{N}^i_n)$ of $\bar{F}^n$ given by the generalized conformal β-change such that (6.4.6) is satisfied along $\bar{F}^m$, and then we get (6.4.12).*

The quantities $\bar{B}^\alpha_i$ are uniquely defined along $\bar{F}^m$ by

$$\bar{B}^\alpha_i = \bar{g}^{\alpha\beta}\bar{g}_{ij}B^j_\beta, \tag{6.4.14}$$

where $\bar{g}^{\alpha\beta}$ is the inverse matrix of $\bar{g}_{\alpha\beta}$. Let $(\bar{B}^\alpha_i, \bar{N}^a_i)$ be the inverse matrix of $(B^i_\alpha, \bar{N}^i_a)$, we have

$$B^i_\alpha \bar{B}^\beta_i = \delta^\beta_\alpha, \quad B^i_\alpha \bar{N}^a_i = 0, \quad \bar{N}^i_a \bar{B}^\alpha_i = 0, \quad \bar{N}^i_a \bar{N}^b_i = \delta^b_a, \tag{6.4.15}$$

and further

$$B^i_\alpha \bar{B}^\alpha_j + \bar{N}^i_a \bar{N}^a_j = \delta^i_j. \tag{6.4.16}$$

we also get $\delta_{ab}\bar{N}^b_i = \bar{g}_{ij}\bar{N}^j_a$, that is,

$$\bar{N}^a_i = \sqrt{e^\sigma p} N^a_i. \tag{6.4.17}$$

Now assuming that the covector field $b_i(x)$ is gradient, we have from (6.2.13)

$$N^a_i D^i = 0. \tag{6.4.18}$$

Differentiating (6.4.18) by y^j and contracting it by B^j_α, we get

$$N^a_i D^i_j B^j_\alpha = 0. \tag{6.4.19}$$

If each geodesic of F^m with respect to the induced metric is also a geodesic of F^n, then F^m is called totally geodesic. A totally geodesic subspace F^m is characterized by each $H^a_\alpha = 0$. From (6.4.4) and (6.4.17) we have

$$\bar{H}^a_\alpha = \sqrt{e^\sigma p}(H^a_\alpha + N^a_i D^i_j B^j_\alpha). \tag{6.4.20}$$

Thus from (6.4.19) we obtain $\bar{H}^a_\alpha = \sqrt{e^\sigma p} H^a_\alpha$. Hence we have

Theorem 6.4.2. *Assume that the covector field $b_i(x)$ is gradient. Then the subspace F^m is totally geodesic, if and only if the subspace $\bar{F^m}$ is totally geodesic.*

From (6.4.4), (6.4.17) and Lemma 2.1, we have $\bar{H}^a_\alpha = \sqrt{e^\sigma p} H^a_\alpha$. Thus we obtain

Theorem 6.4.3. *Let $b_i(x)$ be parallel with respect to $C\Gamma$ on F^n. Then the subspace F^m is totally geodesic, if and only if the subspace $\bar{F^m}$ is totally geodesic.*

If each h-path of F^m with respect to the induced connection is also an h-path of F^n, then F^m is called totally h-autoparallel. A totally h-autoparallel subspace F^m is characterized by each $H^a_{\alpha\beta} = 0$. From (6.4.4), (6.4.5), (6.4.17) and Lemma 2.1, we obtain

Theorem 6.4.4. *Let $b_i(x)$ be parallel with respect to $C\Gamma$ on F^n. Then the subspace F^m is totally h-autoparallel, if and only if the subspace $\bar{F^m}$ is totally h-autoparallel.*

6.5 Conclusion

The infinitesimal symmetries of space-time are expressed by so-called Killing vector fields in general relativity. Therefore, it is an important problem to determine the Killing vector fields of different classes of generalized metrics. In a Euclidean space, translations are distinguished from other types of isometries by the property that their orbits are straight lines. This property is used to generalize the notion of translations to more general classes of metrics, translations are Killing vector fields whose integral curves are at the same time geodesics.

In this chapter, we consider a general Finsler space $F^n(M, L)$ which undergoes conformal and β-change, that is $L(x, y) \to \bar{L}(x, y) = f(e^{\sigma(x)}L(x, y), \beta(x, y))$ where $\beta(x, y) = b_i(x)y^i$ is a 1-form. We study Finslerian subspace $F^m = (M^m, \bar{L}(u, v))$ of F^n and another Finslerian subspace $\bar{F}^m = (M^m, \bar{L}(u, v))$ of the $\bar{F}^n$ subjected to the generalized conformal β-change. Further, we consider a Finsler subspace is totally geodesic and totally h-autoparallel and we also examine the classical approach to the problem of existence of Killing vector fields and study how they vary from point to point and how they are related to Killing vector fields defined on the whole manifold and as its consequences we obtained Corollaries 6.3.2 and 6.3.3. Since the Killing equation (6.3.2) is a necessary and sufficient condition for the transformation (6.3.1) to be a motion in F^n, condition (6.3.4) obtained in Theorem 5.3.1 may be taken as the necessary and sufficient condition for the vectorfield $V^i(x)$, generating a motion in F^n, to generate a motion in $\bar{F}^n$ as well. It is clear that vector field $v^i(x)$, generating an affine motion in F^n , generates an affine motion in $\bar{F}^n$ if condition (6.3.4) holds. Our study has applications to link various transformations in F^n with the corresponding transformations in $\bar{F}^n$.

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