A THESIS ENTITLED

THE STUDY OF GEODESIC ORBIT IN HOMOGENEOUS FINSLER SPACES

Submitted to the

Faculty of Science and Technology

KUVEMPU UNIVERSITY

For the Award of the Degree of

Doctor of Philosophy

in

MATHEMATICS

by

SUREKHA DESAI

Research Supervisor

Dr. S K NARASIMHAMURTHY

Professor

Department of P.G. Studies and Research in Mathematics,

Jnana Sahyadri, Shankaraghatta - 577 451,

Shivamogga, Karnataka, India.

February 2023

A THESIS ENTITLED

THE STUDY OF GEODESIC ORBIT IN HOMOGENEOUS FINSLER SPACES

Submitted to the

Faculty of Science and Technology

KUVEMPU UNIVERSITY

For the Award of the Degree of

Doctor of Philosophy

in

MATHEMATICS

by

SUREKHA DESAI

Research Supervisor

Dr. S K NARASIMHAMURTHY

Professor

Department of P.G. Studies and Research in Mathematics,

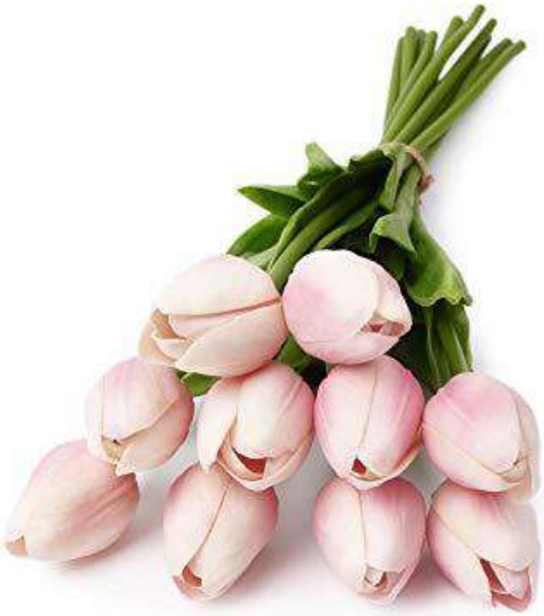
Jnana Sahyadri, Shankaraghatta - 577 451,

Shivamogga, Karnataka, India.

February 2023

Dedicated to

MY BELOVED PARENTS



DECLARATION

I hereby declare that the thesis entitled **The Study of Geodesic Orbit in Homogeneous Finsler Spaces**, submitted to the Faculty of Science and Technology, Kuvempu University for the award of the degree of Doctor of Philosophy in Mathematics is the result of research work carried out by me in the Department of Mathematics, Kuvempu University under the guidance of **Dr. S. K. Narasimhamurthy**, Professor, Department of P. G. Studies and Research in Mathematics, Kuvempu University, Jnanasahyadri, Shankaraghatta.

I further declare that this thesis or part thereof has not been previously formed the basis of the award of any degree, associateship etc., of any other University or Institution.

Place: Jnanasahyadri

Date: 28-02-2023

Surekha Desai

KUVEMPU
UNIVERSITY

Dr. S.K. NARASIMHAMURTHY *M.Sc., M.Phil., Ph.D.,*

Professor and Chairman
Dept. of, P.G Studies and Research in Mathematics,
Kuvempu University, **Jnana Sahyadri,**
SHANKARAGHATTA-577 451,
Shimoga, Karnataka, INDIA.

Phone: 094485 90043 (M), 08282-257310 (O),
e-mail: nmurthysk@gmail.com

CERTIFICATE

This is to certify that the thesis entitled **The Study of Geodesic Orbit in Homogeneous Finsler Spaces**, submitted to the Faculty of Science and Technology, Kuvempu University for the award of the degree of Doctor of Philosophy in Mathematics by **Surekha Desai** is the result of bonafide research work carried out by her under my guidance in the Department of P. G. Studies and Research in Mathematics, Kuvempu University, Jnanasahyadri, Shankaraghatta.

This thesis or part thereof has not been previously formed the basis of the award of Any degree, associateship etc., of any other University or Institution.

Place: Jnanasahyadri

Date: 28-02-2023

Prof. S K Narasimhamurthy

Research Supervisor

Dr. S.K. NARASIMHAMURTHY
M.Sc.,M.Phil.,Ph.D.,
Professor
Department of Studies in Mathematics
KUVEMPU UNIVERSITY, Jnana Sahyadri
SHANKARAGHATTA-577 451
Shivamogga, Karnataka, INDIA.

ACKNOWLEDGMENT

With great pleasure I take this opportunity to express my deep sense of gratitude and heart-felt thanks to several individuals from whom I recieved impetus, motivation and invaluable help during the course of my research work.

Firstly, I would like to thank the supreme god for everything in my life. The good and bad, some were blessings and some were lessons. I express my respectful thanks and deep sense of gratitude to my guide Dr. S. K. Narasimhamurthy, Professor and Chairman, Department of P. G. Studies and Research in Mathematics, Kuvempu University. Without his support, inspiration, suggestions, motivation and creative guidance, this work would not have been possible. I would like to thank him for his precious suggestions in completion of this work.

A special and deep sense of gratitude to Prof. C. S. Bagewadi, Emeritus Professor, Department of P. G. Studies and Research in Mathematics, Kuvempu University for his kind support, help, advice and precious suggestions.

I express my special gratitude to Prof. B.J. Gireesha, Prof. Venkatesha, Department of P. G. Studies and Research in Mathematics, Kuvempu University and Dr. Venkatesh P., Associate Professor, Sahyadri Science college, Shimoga, for their suggestions and kind co-operation.

I extend my sincere thanks to Prof. Pradeep Kumar, Presidency University and Dr. Ramesh M., Jain University, for valuable suggestions and discussions which are indeed helped throughtout my research work. Thanks to them for their concern on me.

I express my special grattitude to my colleague Mr. Raghavendra R. S., from the beginning of my research career to the present he has always supported me. I am very much thankful for his invaluable moral support, endless patience and support throughtout my research work.

My sincere thanks to all my senior and junior research colleagues and my friends Varnashree, Salma and my roomates Savithra and Supraja for their love, affection and kind support. I would like to offer my thanks to who directly and indirectly helped me in the completion of my research work. In addition, I would like to express my gratitude to non-teaching staff Mrs. Latha M. P., Mr. Shivakumar and Mr. Rakesh, and library of Kuvempu University for providing useful information for my research work.

Words fail to express the deep appreciation to my loving parents Shri. Hanamantray Desai, Smt. Aniladevi Desai, my brother Shivakumar, and my sisters Shobha, Renuka, Nirmala, Savita, my nephews Arush and Prabhas, brother-in-law and sister-in-law for supporting me spiritually throughout my Ph.D. study period and my life in general, without their inspiration, drive and support that they have given me, I might not be the person I am today.

Finally, I offer my regards to all of those who supported me in any respect during the completion of my Ph.D. Many Thanks!

Place: Jnana Sahyadri

Date: Surekha Desai

Nomenclature/Notations

$\partial_i \;=\; \dfrac{\partial}{\partial x^i},$

$\dot{\partial}_i \;=\; \dfrac{\partial}{\partial y^i},$

N^n – n-dimensional manifold,

L – Finsler metric,

g_{ij} – metric tensor,

F^n – Finsler space,

TN – Tangent bundle,

T_PN – Tangent space,

h_{ij} – Angular metric tensor,

l_i – Normalized element of support,

C_{ijk} – Cartan's tensor,

γ^i_{jk} – Christoffel symbol,

N^i_j – Non-linear connection,

$|$ – h-covariant derivative w.r.t Cartan's connection,

$\big|$ – v-covariant derivative w.r.t Cartan's connection,

$:$ – Covariant derivative w.r.t Berwald's connection,

R^i_{hjk} – Cartan's third curvature tensor,

G^i_{hjk} – hv-curvature tensor,

R^i_{jk} – h-torsion tensor field,

α – Riemannian metric,

β – Differential 1-form,

D^i_{jkl} – Douglas tensor,

Ric – Ricci curvature,

R_{ik} – Ricci curvature tensor,

${}^{\alpha}\mathrm{Ric}$ – Ricci curvature of α,

N^i – Normal vector,

P_{ijk} – Torsion tensor field,

$K(P, \eta)$ – Flag curvature,

B^i_{α} – Projection factor,

D^i_{hjk} – Douglas tensor.

Contents

Preface

Geometry is a part of Mathematics concerned with questions of size, shape and relative position of figures and with properties of space. Initially a body of practical knowledge concerning lengths, areas and volumes in the third century B.C., geometry was put into a axiomatic form by Euclid, whose treatment is known as Euclidean Geometry. Differential geometry has a long history as a field of Mathematics. The authors Schouten and Van Dantzing in 1930, first tried to transfer the results of differential geometry of spaces with Riemannian metric with affine connection to the case of spaces with complex structure.

The theory of spaces with a generalized metric was initiated by Finsler in 1918 under the influence of geometrization of variation calculus and was developed independently by Synge, Taylor and in particular, Berwald in the middle of 1920's as a generalization of Riemannian geometry. The study of Finsler spaces has important significance in physics. The concept of homogeneity is one of the fundamental notions in geometry although its means must be specified for the concrete situations. Homogeneous Finsler spaces emphasizes the relationship between Lie group and Finsler geometry. Let (N, F) be a connected Finsler space. The group of isometries of (N, F), denoted by $I(N, F)$ is a Lie transformation of N. We say that (N, F) is homogeneous Finsler space if the action of $I(N, F)$ on N is transitive. A homogeneous Finsler space emphasizes the relation between Lie groups and Finsler geometry. A geodesic vector is a non-zero vector that generates a geodesic curve. The non-zero vector of a geodesic orbit in homogeneous Finsler space was first described by Dariush Latifi.

This thesis comprises of six chapters commencing with introduction as **Chapter 1**, which consists of concise history of Finsler geometry, homogeneous Finsler spaces and its applications, definitions and description of significant terms involving formulae. It includes various types of curvatures such as Riemannian curvature, Flag curvature, S-curvature, Ricci curvature etc., and also geodesic orbit spaces, invariant Finsler metric, Projective change, Non-holonomic frames, and hypersurfaces.

Chapter 2 is devoted to study of the explicit formulae for the flag curvature of homogeneous Finsler spaces with some special (α, β)-metrics. First, we discusses a brief review of literature of Flag curvature. Further, by using Puttmann's formula we give the formula for flag curvature of naturally reductive homogeneous Finsler space with (α, β)-metric. Also, we have discussed the existence of homogeneous geodesics for the space (N, F) and obtained the following results:

- Let a compact Lie group G contains a closed subgroup H with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ of G and H respectively. Also an invariant Riemannian metric $\tilde{\alpha}$ on the homogeneous space G/H such that $\langle v, w\rangle = \langle\langle \psi(v), w\rangle\rangle$, where $\psi : \mathfrak{g} \to \mathfrak{g}, \forall\, v, w \in \mathfrak{g}$ is a positive definite endomorphism. Suppose that an invariant vector field $\tilde{u}$ on homogeneous space G/H is parallel with respect to Riemannian metric $\tilde{a}$ and $\tilde{u}_H = u$ and assume that $F = \alpha + \sqrt{\alpha^2 + \beta^2}$ be a special (α, β)-metric arising from $\tilde{\alpha}$ and $\tilde{u}$ such that its Chern connection of F and the Riemannian connection of $\tilde{a}$ are coincides, and a flag $\{P, \eta\}$ in $T_H(G/H)$ such that $\{\zeta, \eta\}$ is an orthonormal basis of P with respect to $\langle\cdot,\cdot\rangle$. Then the flag curvature of the flag $\{P, \eta\}$ is given by
$$K(P, \eta) = \frac{\big\langle \zeta, R(\zeta, \eta)\eta\big\rangle S_1 + \big\langle u, \zeta\big\rangle\big\langle u, R(\zeta, \eta)\eta\big\rangle S_2 + \big\langle \eta, R(\zeta, \eta)\eta\big\rangle S_3}{8 + 4\big\langle u, \eta\big\rangle^2 + 2\big\langle u, \zeta\big\rangle^2 + A_1},$$
where
$$\begin{aligned}
S_1 =& 2 + \frac{2 + \langle u, \eta\rangle^2}{\sqrt{1 + \langle u, \eta\rangle^2}}, \quad S_2 = \frac{(1 + \langle u, \eta\rangle^2)^{\frac{3}{2}} + 1}{(1 + \langle u, \eta\rangle^2)^{\frac{3}{2}}}, \quad S_3 = \frac{\langle u, \eta\rangle^3 \langle u, \zeta\rangle}{(1 + \langle u, \eta\rangle^2)^{\frac{3}{2}}},\\
S_4 =& 8 + 8\langle u, \eta\rangle^2 + 2\langle u, \zeta\rangle^2(1 + \langle u, \eta\rangle^2) + \langle u, \eta\rangle^4, \quad S_5 = 2\langle u, \zeta\rangle^2 - \langle u, \eta\rangle^2\langle u, \zeta\rangle^2,\\
A_1 =& \frac{S_4}{\sqrt{1 + \langle u, \eta\rangle^2}} + \frac{2\langle u, \zeta\rangle^2}{1 + \langle u, \eta\rangle^2} + \frac{S_5}{(1 + \langle u, \eta\rangle^2)^{\frac{3}{2}}} - \frac{\langle u, \eta\rangle^2\langle u, \zeta\rangle^2}{(1 + \langle u, \eta\rangle^2)^3}.
\end{aligned}$$

- Let a homogeneous Finsler space $(G/H, F)$ with (α, β)-metric $F = \alpha + \sqrt{\alpha^2 + \beta^2}$ be defined by an invariant Riemannian metric $\tilde{\alpha}$ and an invariant vector field u such

that the Chern connection of F coincides the Levi-Civita connection of $\tilde{\alpha}$. Then $(G/H, F)$ is naturally reductive if and only if the underlying Riemannian space $(G/H, \tilde{\alpha})$ is naturally reductive.

- Let a homogeneous Finsler space (N, F) with $F = \alpha + \sqrt{\alpha^2 + \beta^2}$ defined by the Riemannian metric $\alpha = a_{ij} dx^i \otimes dx^j$ and the vector field u corresponding to 1-form β. Then the homogeneous Finsler space (N, F) with the origin $p = \{H\}$ and with an Ad(H)-invariant decomposition $\mathfrak{g} = \mathfrak{l} + \mathfrak{h}$ is naturally reductive with respect to this decomposition if and only if for any vector $u \in \mathfrak{l} \backslash \{0\}$, the curve $\gamma(t)$ is geodesic of homogeneous Finsler manifold, here $\gamma(t)$ is $\exp tu(p)$.

Chapter 3 deals with the study of the existence of invariant vector fields of homogeneous Finsler spaces with (α, β)-metrics. The formula for S-curvature of homogeneous Finsler spaces with an (α, β)-metric is obtained. Further, using it, it is shown that these homogeneous Finsler spaces have isotropic S-curvature if and only if they have vanishing S-curvature. In the last section, the formulae for mean Berwald curvature of homogeneous Finsler spaces with (α, β)-metrics are obtained. We have proved the following results:

- Let G/H be reductive homogeneous Finsler space with a decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{h} + \mathfrak{l}$, and $F = \dfrac{\alpha^2}{(\alpha - \beta)} + \beta$ be a G-invariant Randers-Matsumoto metric on G/H. Then the S-curvature is given by

$$S(H, \eta) = \left[\frac{\begin{aligned} 6s^4 - (9n+15)s^3 + (6b^2n + 27n + 21)s^2 - (16b^2n + 10b^2 + 14n \\ + 14)s + 4b^2n + 10b^2 + 2n + 2 \end{aligned}}{2(s^2 - s - 1)(2b^2 - 3s + 1)^2} \right] \times \left(\frac{s^2 - 2s + 2}{1 - 2s} \langle [u, \eta]_{\mathfrak{l}}, u \rangle + \frac{1}{\alpha} \langle [u, \eta]_{\mathfrak{l}}, \eta \rangle \right),$$

where $u \in \mathfrak{l}$ corresponds to the 1-form β, $\mathfrak{l}$ is verified with the tangent space $T_H(G/H)$ of G/H at the origin H.

- Let G/H be reductive homogeneous Finsler space with a decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{h} + \mathfrak{l}$ and $F = \dfrac{\alpha^2}{(\alpha - \beta)} + \beta$ be a G-invariant Randers-Matsumoto metric on G/H. Then $(G/H, F)$ has isotropic S-curvature if and only if it has vanishing S-curvature.

- Let G/H be a reductive homogeneous Finsler space with a decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{h} + \mathfrak{l}$, and $F = \dfrac{\alpha^2}{(\alpha - \beta)} + \beta$ be a G-invariant special metric on G/H. Then the mean Berwald curvature E_{ij} of the homogeneous Finsler space with special (α, β)-metric is also derived.

In **Chapter 4**, we discuss the geodesic orbit of homogenous Finsler spaces and we have proved the necessary and sufficient conditions for a non-zero vector in these homogeneous spaces to be a geodesic vector with two different (α, β)-metrics. We have obtained the following results:

- Let N be a homogeneous Finsler space with Matsumoto metric $F = \dfrac{\alpha^2}{\alpha - \beta}$. Then a vector $\eta(\neq 0) \in \mathfrak{g}$ is a geodesic vector if and only if

$$\left\langle [\eta, \xi]_{\mathfrak{l}}, |\eta_{\mathfrak{l}}|\eta_{\mathfrak{l}} - 2\langle u, \eta_{\mathfrak{l}}\rangle\eta_{\mathfrak{l}} + |\eta_{\mathfrak{l}}|^2 u \right\rangle = 0,$$

holds for every $\xi \in \mathfrak{l}$.

- Let (N, F) be a homogeneous Finsler space with special metric $F = \alpha e^{\frac{\beta}{\alpha}} + \dfrac{\beta^2}{\alpha}$. Then a non-zero vector $\eta \in \mathfrak{g}$ is a geodesic vector if and only if

$$\begin{aligned}\Bigg\langle [\eta, \xi]_{\mathfrak{l}}, \Bigg(\exp\left(\frac{\langle u, \eta_{\mathfrak{l}}\rangle}{|\eta_{\mathfrak{l}}|}\right) - \frac{\langle u, \eta_{\mathfrak{l}}\rangle}{|\eta_{\mathfrak{l}}|}\exp\left(\frac{\langle u, \eta_{\mathfrak{l}}\rangle}{|\eta_{\mathfrak{l}}|}\right) - \frac{\langle u, \eta_{\mathfrak{l}}\rangle^2}{|\eta_{\mathfrak{l}}|^2}\Bigg)\eta_{\mathfrak{l}} \\ + \left(|\eta_{\mathfrak{l}}|\exp\left(\frac{\langle u, \eta_{\mathfrak{l}}\rangle}{|\eta_{\mathfrak{l}}|}\right) + 2\langle u, \eta_{\mathfrak{l}}\rangle\right)u\Bigg\rangle = 0,\end{aligned}$$

holds for every $\xi \in \mathfrak{l}$.

- For a homogeneous Finsler space (N, F), there exists at least one homogeneous geodesic, with Matsumoto metric $F = \dfrac{\alpha^2}{\alpha - \beta}$.

- For a homogeneous Finsler space (N, F), there exists at least one homogeneous geodesic, with metric $F = \alpha e^{\frac{\beta}{\alpha}} + \dfrac{\beta^2}{\alpha}$.

- Using above results, we discuss the geodesic vectors for a two-step nilpotent Lie group of dimension five with left-invariant (α, β)-metrics.

In **Chapter 5**, the concept of Ricci curvature in Finsler geometry is discussed. Curvature properties of homogeneous Finsler spaces with (α, β)-metrics are among the most significant topics in Finsler geometry. Here, we have obtained the formulae for Ricci curvature of homogeneous Finsler spaces with special (α, β)-metrics. Based on this formula, we have discussed the condition for vanishing S-curvature for the space $(G/H, F)$. We have obtained the following results:

- A compact homogeneous Finsler space G/H with G-invariant special metric $F = \alpha + \sqrt{\alpha^2 + \beta^2}$. Then the Ricci curvature is given by Eq. (5.3.1).

- Let $(N = G/H, F)$ be a compact connected homogeneous Finsler space with G-invariant special metric $F = \alpha + \sqrt{\alpha^2 + \beta^2}$ on G/H. Suppose that (N, F) has vanishing S-curvature. Then Ricci curvature is given by

$$\begin{aligned} Ric(Z) =& Ric^{\alpha}(Z) - \frac{c^2}{4}(C^n_{q0})^2 K_{19} + \frac{c}{4}\alpha(Z)\Big(2C^n_{m0}C^q_{qm} + C^n_{qm}C^0_{qm}\Big)K_{24} \\ &- \frac{c^2}{4}\alpha^2(Z)(C^n_{ik})^2 K_{26}, \end{aligned}$$

where, $Z(\neq 0) \in \mathfrak{l}$ and $K_{19} = \dfrac{-2(s^2\sqrt{s^2+1} - s^2 - \phi)}{\phi^2\sqrt{s^2+1}}, \quad K_{24} = \dfrac{2s}{\phi}, \quad K_{26} = \dfrac{-s^2}{\phi^2}.$

Chapter 6 focuses on some properties of Finsler space with (α, β)-metrics. In this chapter we discuss the nonholonomic Finsler frames, hypersurface and projective flatness of Finsler space with (α, β)-metrics. Nonholonomic frames have been studied by many geometers and the concept of nonholonomic Finsler frames was introduced by P. R. Holland in 1982, when he studied electromagnetism by considering the charged particles moving in an external electromagnetic field. Many researchers have worked on this concept with different (α, β)-metrics. In 1985, M. Matsumoto studied the theory of Finslerian hypersurfaces, a hyperplane of the first kind, a hyperplane of the second kind, and a hyperplane of the third kind are three different forms of Finslerian hypersurfaces that he investigated. Next we have discussed the projectively flat Finsler spaces with special (α, β)-metric. The condition for a Finsler space to be projectively flat was studies by L. Berwald and this work was completed by M. Matsumoto. We have discussed the following results:

- For the deformed Finsler metric $F = (\alpha+\beta)\left(\dfrac{\alpha^2}{\alpha-\beta}\right)$, we have obtained Finsler invariants $\rho, \rho_0, \rho_{-1}, \rho_{-2}$ which satisfies the condition, $\rho_{-1}\beta + \rho_{-2}\alpha^2 = 0$. Then we have constructed a nonholonomic Finsler frames, which are given by,

  $$V^i_j = X^i_k Y^k_j,$$

  where X^i_k and Y^k_j are given by Eqs. (6.2.13) and (6.2.14) respectively.

- For the deformed Finsler metric $F = (\alpha+\beta)\left(\alpha+\beta+\dfrac{\beta^2}{\alpha}\right)$, we have obtained Finsler invariants $\rho, \rho_0, \rho_{-1}, \rho_{-2}$ which satisfies the condition, $\rho_{-1}\beta + \rho_{-2}\alpha^2 = 0$. Then we have constructed a nonholonomic Finsler frames, which are given by,

  $$V^i_j = X^i_k Y^k_j,$$

  where X^i_k and Y^k_j are given by Eqs. (6.2.16) and (6.2.17) respectively.

- By considering the hypersurface of a Finsler space with generalized Matsumoto metric, we have obtained the following results:

  (a) The induced metric structure of the generalized Matsumoto metric on the hypersurface F^{n-1} and obtained the scalar function $b(x)$ given by $b_i(x(u)) = \sqrt{\dfrac{b^2}{1+(m^2+m)b^2}}N_i$ and $b^i = \sqrt{b^2\left((m^2+m)b^2+1\right)}N^i + \dfrac{mb^2}{\alpha}\eta^i$, where N_i is a unit normal vector.

  (b) For the generalized Matsumoto metric on the Finsler hypersurface F^{n-1} the second fundamental tensor is given by $M_{\alpha\beta} = \dfrac{m}{2\alpha}\sqrt{\dfrac{b^2}{1+(m^2+m)b^2}}h_{\alpha\beta}, \quad M_\alpha = 0$.

  (c) Further, using Matsumoto's results, we have discussed the properties of hypersurface $F^{n-1}(c)$ that it is a hyperplane of a first and second kind but not of third kind.

- A Finsler space F^n with an (α,β)-metric $F = \alpha + \sqrt{\alpha^2+\beta^2}$ provided $b^2 \neq 1$ is projectively flat if and only if the associated Riemannian space (N^n, α) is projectively flat and $b_{i;j} = 0$.

Chapter 1

Introduction

1.1 History and Development

Finsler geometry is a branch of differential geometry, has been started with Finsler's famous dissertation. In 1918, a German mathematician Paul Finsler studied Finsler geometry under the supervision of C. Caratheodory who intended to geometrize the calculus of variations. It is actually the geometry of a simple integral and is considered as old as the calculus of variations. The name 'Finsler Geometry' was first given by J. Taylor in 1927. Later, E. Cartan aquainted a system of axioms to give a Finsler connection from the fundamental function $F(x, \eta)$. In Riemannian geometry, the connection of choice was assembled by Levi-Civita, using the Christoffel symbols. It has two remarkably distinguished attributes, metric compatibility and torsion freeness. Furthermore, the Levi-Civita connection operates on the tangent bundle TN of our underlying manifold N. But we cannot say the same for its Finslerian counterpart. Finsler geometry is not considered as a generalization of Riemannian geometry, it is described as Riemannian geometry without quadratic restriction. The metric $ds^2 = F^2(x, dx) = g_{ij}(x)dx^i dx^j$, is called Riemannian metric. The geometry based on the this metric structure with n-form and F is positively homogeneous of degree 1 in dx_i is now called as Riemann-Finsler geometry or Finsler geometry in short.

The general development took a curious turn away from the basic aspects and later on methods of the theory were developed by P. Finsler. Finsler did not use tensor calculus. In 1925, the methods of tensor calculus were applied to the theory of Finsler spaces independently but almost at the same time by Synge [77], Taylor [79], who introduced a special parallelism and Berwald [11] introduced the concept of connection in the theory of Finsler spaces. It was found that the second derivative of $\frac{1}{2}F^2(x, dx)$ with respect to dx; functioned very well as components of a metric tensor, similar to Riemannian geometry, and that the connection coefficients could be derived from the differential equations of the geodesics, with the help of which Levi-Civita's generalized parallel displacement could be defined. Finsler geometry contains analogs for huge number of the common questions in Riemannian geometry. For instance, length, geodesics, curvature, connections, covariant derivative, and structure equations.

S. S. Chern (1911-2004), one of the greatest mathematicians of the 20th century, introduced a connection for Finsler metric in 1948, known as Chern connection which is a generalization of Levi-Civita connection in Riemannian geometry. Levi-Civita connection is metric compatible and torsion-free, but Chern connection is torsion-free and almost metric compatible.

L. Berwald introduced the notion of flag curvature, which is the natural generalization of sectional curvature in Riemannian geometry. In 1972 [45], generalizing the Randers type and Kropina type metrics, Matsumoto introduced the concept of (α, β)-metrics, α is a Riemannian metric and β is a 1-form. In 1997, [73] Zhongmin Shen, while studying the volume comparison in Riemann-Finsler geometry, introduced the notion of S-curvature. S-curvature is a non-Riemannian quantity, plays an important role in Finsler geometry and is related subtly with flag curvature of Finsler metrics.

In later years, Finsler geometry got more attention by physicists. A Norwegian physicist, G. Randers [65] used the concept of Finsler geometry to study the theory of electromagnetism and gravitation and introduced a metric called "Randers metric". Another physicist, R. S. Ingarden used Finsler structure to study electron microscope in 1948. Further, important contribution came from the German mathematician, H. Rund [67] who studied parallelism based on Minkowski geometry while Cartan studied the parallelism from the viewpoint of Euclidean geometry. One of the famous geometer R. Miron, a fellow researcher of Matsumoto, build a field of orthonormal frames associated to an n-dimensional Finsler space, in 1974. It was named Miron frame by Matsumoto in his monograph "Foundations of Finsler geometry and special Finsler spaces [47]". In his lecture at the University of Brasov, Miron introduced the Finsler connection as linear connections in the tangent bundle TN which is tangent to a manifold N. In 1989, M. Matsumoto [48] proved that a slope of a mountain with respect to time measure is a Finsler structure.

The importance of S-curvature in Riemann-Finsler geometry can be seen in several papers [75, 74]. S-curvature is used to measure the rate of change of the volume form of a Finsler space along geodesics.

Finsler geometry has abundance of applications in physics, mechanics and information geometry. During 1980s, R. S. Ingarden, P. L. Antonelli and among others developed new applications in biology, optics, quantum physics, psychology, geosciences and geodesy [19].

This theory was applied to the electron microscope by R. S. Ingarden and P. L. Antonelli used the Finsler geometry in biology. Finsler geometry has its roots in various problems from Differential equations, Calculus of variations, Mechanics, Theoretical physics.

Finsler geometry was first connected in gravitational hypothesis, and this application prompt redresses to observational outcomes anticipated by general relativity. The main application of Finsler geometry is the geometrization of electromagnetism and gravitation. Some (α, β)-metrics are essential for Cosmology, in application perspective. These days Finsler geometry has discovered a plentitude of applications in both physics and practical applications.

1.2 Basic concepts of Finsler space

Definition 1.2.1. Let N be an n-dimensional smooth manifold, $F : TN \to [0, +\infty)$ be a non-negative function on the tangent bundle TN. F is called a Finsler metric on N if it satisfies the following conditions:

1. $F(x, \lambda\eta) = \lambda F(x, \eta), \forall \lambda > 0$ (Positively homogeneous);
2. $F(x, \eta)$ is a C^∞ function on the slit tangent bundle $N_0 = TN \setminus \{0\}$;
3. For any non-zero vector $\eta \neq 0$, the Hessian matrix $g_{ij}(x, \eta) = \dfrac{1}{2}\dfrac{\partial^2 F^2}{\partial \eta^i \partial \eta^j}$ is positively definite.

A differentiable manifold N equipped with a Finsler metric F is called a Finsler manifold or Finsler space denoted by (N, F).

Lemma 1.2.1. *[76] Let $F = \alpha\phi(s), s = \dfrac{\beta}{\alpha}$, where ϕ is a smooth function on an open interval $(-b_0, b_0)$, α is a Riemannian metric and β is a 1-form with $\|\beta\|_\alpha < b_0$. Then F*

is a Finsler metric if and only if ϕ satisfies the conditions:

$$\phi(s) > 0, \quad \phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad \forall |s| \leq b < b_0.$$

Definition 1.2.2. A Finsler metric F on a differentiable manifold N is called an (α, β)-metric, where α is a Riemannian metric, $\alpha = \sqrt{a_{ij}(x)\eta^i\eta^j}$ and β is a 1-form, $\beta = b_i(x)\eta^i$, if F is a positively homogeneous function of degree one in α and β.

M. Matsumoto has introduced the concept of (α, β)-metric in 1972 [45]. In physics and biology, (α, β)-metrics have several applications.

Here are some notations are related to (α, β)-metrics

$$r_{ij} = \frac{b_{i;j} + b_{j;i}}{2}, \qquad s_{ij} = \frac{b_{i;j} - b_{j;i}}{2}, \qquad r^i_j = a^{iq} r_{qj},$$

$$r_i = b^j r_{ji} = b_j r^j_i, \qquad s_i = b^j s_{ji} = b_j s^j_i, \qquad r = r_{ij} b^i b^j = b^i r_i,$$

$$r_{00} = r_{ij}\eta^i\eta^j, \qquad r_{i0} = r_{ij}\eta^j, \qquad s_{i0} = s_{ij}\eta^j,$$

$$r_0 = r_i\eta^i, \qquad s_0 = s_i\eta^i, \qquad s^i_j = a^{iq} s_{qj},$$

where ';' denotes the contravariant derivative, with respect to the Levi-Civita connection on Riemannian metric α and $a^{ij} = (a_{ij})^{-1}, \ b^i = a^{ij} b_j$.

Ricci curvature:

The Ricci curvature is an important geometrical entity. The notion of Riemannian curvature can be extended to Finsler metrics for Riemannian spaces. For a non-zero vector $z \in T_x N$, a linear map $R_z : T_x N \to T_x N$ is the Riemannian curvature [17], which can be defined as follows:

$$R_z(u) = R^i_k(z) u^k \frac{\partial}{\partial x^i}, \qquad u = u^i \frac{\partial}{\partial x^i},$$

where
$$R^i_k(z) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial z^k} z^j + 2G^j \frac{\partial^2 G^i}{\partial z^j \partial z^k} - \frac{\partial G^i}{\partial z^j}\frac{\partial G^j}{\partial z^k},$$

and G^i are geodesic coefficients given by

$$G^i = \frac{1}{4} g^{im} \left\{ (F^2)_{x^k z^m} z^k - (F^2)_{x^m} \right\}, \quad i = 1, 2, \ldots, n.$$

Definition 1.2.3. [17] Let (N, F) be a Finsler space. A scalar function Ric : $TN \to R$ such that $\mathrm{Ric}(z) = tr(R_z)$ is called Ricci curvature of Finsler space (N, F) and $tr(R_z)$ is the trace of its Riemannian curvature, where $z \in TN$.

Definition 1.2.4. The Berwald curvature of a Finsler metric F is defined, in local coordinates, as follows:

$$B := B^i_{jkl}\, dx^j \otimes dx^k \otimes dx^l \otimes \frac{\partial}{\partial x^i},$$

where $B^i_{jkl} = \dfrac{\partial^3 G^i}{\partial \eta^j \partial \eta^k \partial \eta^l}$ and G^i are the geodesic spray coefficients. A Finsler metric F is called a Berwald metric if its Berwald curvature is zero.

S-curvature:

Let V be a n-dimensional real vector space with basis α_i and F be a Minkowski norm on V. Let Vol(B) be the volume of a subset B of R^n, and B^n be the open unit ball. The function $\tau = \tau(\eta)$ is defined as

$$\tau(\eta) = \ln \left(\frac{\sqrt{\det(g_{ij}(\eta))}}{\sigma_F} \right), \quad \eta \in V - \{0\},$$

where $\sigma_F = \dfrac{\text{Vol}(B^n)}{\text{Vol}\{(\eta^i) \in R^n : F(\eta^i, \alpha_i) < 1\}}$ is called the distortion of (V, F).

For a Finsler space (N, F), distortion of Minkowski norm F_x on T_xN is $\tau = \tau(x, \eta)$, $x \in N$. Let γ be a geodesic with $\gamma(0) = x, \dot{\gamma}(0) = \eta$, where $\eta \in T_xN$, then S-curvature denoted as $S(x, \eta)$ is the rate of change of distortion along the geodesic γ, i.e.,

$$S(x, \eta) = \frac{d}{dt}\{\tau(\gamma(t), \dot{\gamma}(t))\}\Big|_{(t=0)}.$$

Here, $S(x, \eta)$ is illustrated as positively homogeneous of degree one, i.e., for $\lambda > 0$, we have $S(x, \lambda\eta) = \lambda S(x, \eta)$.

A Finsler space's S-curvature is interconnected to a volume form. The Busemann-Hausdorff ($dV_{BH} = \sigma_{BH}(x)dx$) and the Holmes-Thompson ($dV_{HT} = \sigma_{HT}(x)dx$) volume forms are significant volume forms in Finsler geometry:

$$\sigma_{BH}(x) = \frac{\text{Vol}(B^n)}{\text{Vol}(A)}, \quad \text{and} \quad \sigma_{HT}(x) = \frac{1}{\text{Vol}(B^n)} \int_A \det(g_{ij}) d\eta, \text{ respectively,}$$

where $A = \left\{\eta^i \in R^n : F\left(x, \eta^i \dfrac{\partial}{\partial x^i}\right) < 1\right\}$. If we consider a Riemannian metric instead of Finsler metric F, then dV_{HT} and dV_{BH} are reduced to single Riemannian volume form $dV_{HT} = dV_{BH} = \sqrt{\det(g_{ij}(x))}dx$. Subsequently, the function $T(s) = \phi(\phi - s\phi')^{(n-2)}\{(\phi - s\phi') + (b^2 - s^2)\phi''\}$, $dV = dV_{BH}$ (or dV_{HT}) is given by $dV = f(b)dV_\alpha$, where

$$f(b) = \begin{cases} \dfrac{\int_0^\pi sin^{n-2}tdt}{\int_0^\pi \frac{sin^{n-2}t}{\phi(bcost)^n}dt}, & if \quad dV = dV_{BH}, \\ \dfrac{\int_0^\pi (sin^{n-2}t)T(bcost)dt}{\int_0^\pi sin^{n-2}tdt}, & if \quad dV = dV_{HT} \end{cases}$$

and $dV_\alpha = \sqrt{\det(a_{ij})}dx$ is the Riemannian volume form of α.

In a local coordinate system, the formula for the S-curvature is given by Cheng and

Shen [16] and is written as:

$$S = \left(2\Psi - \frac{f'(b)}{bf(b)}\right)(r_0 + s_0) - \frac{\Phi}{2\alpha\Delta^2}(r_{00} - 2\alpha Q s_0), \tag{1.2.1}$$

where,

$$Q = \frac{\phi'}{\phi - s\phi'}, \quad \Delta = 1 + sQ + (b^2 - s^2)Q', \quad \Psi = \frac{Q'}{2\Delta},$$
$$\Phi = (sQ' - Q)(n\Delta + 1 + sQ) - (b^2 - s^2)(1 + sQ)Q''.$$

It is commonly known that $r_0 + s_0 = 0$ if b, the Riemannian length, is constant. Hence,

$$S = -\frac{\Phi}{2\alpha\Delta^2}(r_{00} - 2\alpha Q s_0).$$

Flag curvature:

Flag curvature in Finsler geometry, is a generalization of notion of sectional curvature of Riemannian geometry. We consider a flag on a Finsler manifold (N, F). Installing a flag at a point $x \in N$, gives a non-zero tangent vector $\eta \in P \subset T_xN$, which we call the flagpole. The flag (P, η) is described by one edge along the flagpole and another transverse edge, say $\zeta = \zeta^i \frac{\partial}{\partial x^i} \in P$ such that $P = span\{\eta, \zeta\}$. Then the flag curvature of the flag (P, η) is given by [7]

$$K(P, \eta) = \frac{g_\eta(\zeta, R(\zeta, \eta)\eta)}{g_\eta(\eta, \eta)g_\eta(\zeta, \zeta) - g_\eta^2(\eta, \zeta)}. \tag{1.2.2}$$

1.3 Nonholonomic Frames

Definition 1.3.1. Let U be an open set of TN and $V_i : u \in U \to V_i(u) \in V_uTN$, $i \in \{1, 2, \ldots, n\}$ be a vertical frame over U. If $V_i(u) = V_j^i(u) \frac{\partial}{\partial \eta^j}\Big|_u$, then $V_i^j(u)$ are the entries of the invertible matrix for all $u \in U$. Denote by $V_k^j(u)$ the inverse of this matrix. This means that: $V_j^i V_k^j = \delta_k^j$. We call V_j^i a nonholonomic Finsler Frame.

1.4 Hypersurface of Finsler spaces

Let F^{n-1} be a hypersurface which contains an equation $x^i = x^i(u^\alpha), \alpha = 1, 2, \ldots, (n-1)$, where u_α be a Gaussian coordinates on the hypersurface F^{n-1}. Let us consider that the matrix of the projection factor $B_\alpha^i = \dfrac{\partial x^i}{\partial u^\alpha}$ is of rank $(n-1)$, then

$$\eta^i = B_\alpha^i(u)v^\alpha. \tag{1.4.1}$$

Here η^i is the supporting element of F^n is tangential to F^{n-1} and thus $v = v^\alpha$ is the element of support of F^{n-1} at the point u^α. Denote η^i of (1.4.1) by $\eta^i(u, v)$. Then

$$F_*(u, v) = F(x(u), \eta(u, v)),$$

gives rise to the fundamental function of M^{n-1}, induced from one of the ambient space. Thus we obtain the $(n-1)$-dimensional Finsler space $F^{n-1} = (M^{n-1}, F_*(u, v))$, called the Finslerian hypersurface of F^n.

The metric tensor $g_{\alpha\beta}$ and Cartan tensor $C_{\alpha\beta\gamma}$ of F^{n-1} are given by

$$g_{\alpha\beta} = g_{ij}B_\alpha^i B_\beta^j, \qquad C_{\alpha\beta\gamma} = C_{ijk}B_\alpha^i B_\beta^j B_\gamma^k, \tag{1.4.2}$$

where $B_\alpha^i = \dfrac{\partial x^i}{\partial u^\alpha}$ is of rank $(n-1)$. A unit normal vector $N^i(u, v)$ is at each point u_α of F^{n-1} is expressed by,

$$\left.\begin{aligned} g_{ij}(x(u, v), \eta(u, v))B_\alpha^i N^i &= 0, \\ g_{ij}(x(u, v), \eta(u, v))N^i N^j &= 1. \end{aligned}\right\} \tag{1.4.3}$$

In terms of angular metric tensor h_{ij}, we have

$$h_{\alpha\beta} = h_{ij}B^i_\alpha B^j_\beta, \qquad h_{ij}B^i_\alpha N^j = 0, \qquad h_{ij}N^iN^j = 1. \tag{1.4.4}$$

If B^α_i is the inverse of B^i_α , then

$$\left.\begin{aligned} &B^\alpha_i = g^{\alpha\beta}g_{ij}B^j_\beta, \qquad B^i_\alpha B^\beta_i = \delta^\beta_\alpha, \qquad B^i_\alpha N^i = 0,\\ &N_i = g_{ij}N^j, \qquad\quad B^i_\alpha B^\alpha_j + N^iN_j = \delta^i_j. \end{aligned}\right\} \tag{1.4.5}$$

where $g^{\alpha\beta}$ is the inverse of $g_{\alpha\beta}$ of F^{n-1}.

From Eqs. (1.4.3) and (1.4.5), we have

$$B^i_\alpha B^\beta_i = \delta^\beta_\alpha, \qquad B^i_\alpha N_i = 0, \qquad N^iB^\alpha_i = 0, \qquad N^iN_i = 1. \tag{1.4.6}$$

And also we have,

$$B^i_\alpha B^\alpha_j + N^iN_j = \delta^i_j. \tag{1.4.7}$$

The induced Cartan's connection $IC\Gamma = (\Gamma^{*\alpha}_{\beta\ \gamma}, G^\alpha_\beta, C^\alpha_{\beta\ \gamma})$ of F^{n-1} generated from the Cartan's connection $C\Gamma = (\Gamma^{*i}_{j\ k}, \Gamma^{*i}_{0\ k}, C^i_{jk})$ is given by [46],

$$\begin{aligned} \Gamma^{*\alpha}_{\beta\ \gamma} &= B^\alpha_i(B^i_{\beta\gamma} + \Gamma^{*i}_{j\ k}B^j_\beta B^k_\gamma) + M^\alpha_\beta H_\gamma, \qquad G^\alpha_\beta = B^\alpha_i(B^i_{0\beta} + \Gamma^{*i}_{0\ j}B^j_\beta),\\ C^\alpha_{\beta\ \gamma} &= B^\alpha_i C^i_{jk}B^j_\beta B^k_\gamma, \end{aligned}$$

where, $$M_{\beta\gamma} = N_iC^i_{jk}B^j_\beta B^k_\gamma, \qquad M^\alpha_\beta = g^{\alpha\gamma}M_{\beta\gamma}, \qquad H_\beta = \dot{N}_i(B^i_{0\beta} + \Gamma^{*i}_{0j}B^j_\beta)$$

and $$B^i_{\beta\gamma} = \frac{\partial B^i_\beta}{\partial u^\gamma}, \qquad B^i_{0\beta} = B^i_{\alpha\beta}v^\alpha.$$

Here the second fundamental v-tensor and normal curvature vector are $M_{\beta\gamma}$ and H_β respectively, the second fundamental h-tensor $H_{\beta\gamma}$ is stated as [46]

$$H_{\beta\gamma} = N_i(B^i_{\beta\gamma} + \Gamma^{*i}_{jk}B^j_\beta B^k_\gamma) + M_\beta H_\gamma, \tag{1.4.8}$$

where, $$M_\beta = N_iC^i_{jk}B^j_\beta N^k. \tag{1.4.9}$$

The second fundamental h-tensor and the normal curvature vector are $H_{\alpha\beta}$ and H_α of F^{n-1} respectively, are given by

$$H_{\alpha\beta} = N_i(B^i_{\alpha\beta} + F^i_{jk}B^j_\alpha B^k_\beta) + M_\alpha H_\beta \tag{1.4.10}$$

and
$$H_\alpha = N_i(B^i_{0\alpha} + G^i_j B^j_\alpha), \tag{1.4.11}$$

where $M_\alpha = C_{ijk}B^i_\alpha N^j N^k$, $\quad B^i_{\alpha\beta} = \dfrac{\partial x^i}{\partial u^\alpha \partial u^\beta}$ and $B^i_{0\alpha} = B^i_{\beta\alpha}v^\beta$. It clearly expresses that $H_{\alpha\beta}$ is not symmetric and

$$H_{\alpha\beta} - H_{\beta\alpha} = M_\alpha H_\beta - M_\beta H_\alpha. \tag{1.4.12}$$

The Eqs. (1.4.10) and (1.4.11) gives

$$H_{0\alpha} = H_{\beta\alpha}v^\beta = H_\alpha, \qquad H_{\alpha 0} = H_{\alpha\beta}v^\beta = H_\alpha + M_\alpha H_0, \tag{1.4.13}$$

Here,
$$M_{\alpha\beta} = C_{ijk}B^i_\alpha B^j_\beta N^k, \tag{1.4.14}$$

where $M_{\alpha\beta}$ is the second fundamental v-tensor. The projection factors B^α_i and N^i contain the h and v-covariant derivatives evolve with respect to induced Cartan connection $IC\Gamma$ respectively are given by

$$B^i_{\alpha|\beta} = H_{\alpha\beta}N^i, \quad B^i_{\alpha\big|\beta} = M_{\alpha\beta}N^i, \quad N^i_{|\beta} = -H_{\alpha\beta}B^\alpha_j g^{ij} = -M_{\alpha\beta}B^\alpha_j g^{ij} \tag{1.4.15}$$

and in terms of h and v-covariant derivatives of vector field X^i are as follows:

$$X_{i|\beta} = X_{i|j}B^j_\beta + X_{i\big|j}N^j H_\beta, \qquad X_{i|\beta} = X_{i|j}.B^j_\beta. \tag{1.4.16}$$

The following lemmas state the different kinds of hypersurfaces and their characteristic conditions which are defined by M. Matsumoto [46].

Lemma 1.4.1. *A hypersurface F^{n-1} of a Finsler space F^n is a hyperplane of the 1^{st} kind if and only if $H_\alpha = 0$.*

Lemma 1.4.2. *A hypersurface F^{n-1} of a Finsler space F^n is a hyperplane of the 2^{nd} kind if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$.*

Lemma 1.4.3. *A hypersurface F^{n-1} of a Finsler space F^n is a hyperplane of the 3^{rd} kind if and only if $H_\alpha = 0$ and $M_{\alpha\beta} = H_{\alpha\beta} = 0$.*

1.5 Projectively flat Finsler Space

A Finsler space $F^n = (N^n, F)$ is called a locally Minkowski space [47] if N^n is covered by coordinate neighborhood system x^i in each of which F is a function of η^i only. A Finsler space $F^n = (N^n, F)$ is called projectively flat if F^n is projective to a locally Minkowski space.

1.6 Homogeneous Finsler space

S. Lie, W. Killing, and E. Cartan developed the Lie group theory in the late ninteenth century. Later on, H. Weyl, E. Cartan, and O. Schreier developed global Lie group theory in the year 1920. Cartan applied Lie theory to the theory of Riemannian geometry for classification of globally symmetric Riemannian spaces and in 1930, these groups were named "Lie groups" by E. Cartan. Lie stated that a Lie group, which is a non-linear object is determined by a linear object called "Lie algebra". If the action of a Lie group on a manifold is transitive, then such a manifold is called as homogeneous manifold. Here, transitive means that as far as properties preserved by a Lie group are concerned, any two points of the manifold are alike. The Lie group G is an analytic manifold whose group operations are analytic, i.e., the maps $G \times G \to G$, defined by $(c, d) \to cd$, and $G \to G$, defined by $c \to c^{-1}$ are analytic [66].

Homogeneous Finsler spaces emphasizes the relationship between Lie group and Finsler geometry. Let (N, F) be a connected Finsler space. The group of isometries of (N, F), denoted by $I(N, F)$ is a Lie transformation of N. We say that (N, F) is homogeneous Finsler space if the action of $I(N, F)$ on N is transitive. A geodesic vector is a non-zero vector that generates a geodesic curve. The non-zero vector of a geodesic orbit in homogeneous Finsler space was first described by Dariush Latifi [41]. In 2002, S. Deng and Z. Hou [20] have generalized Myers-Steenrod theorem (1939) to Finslerian case, extended the application of Lie theory to the scope of all homogeneous Riemannian manifolds. Finsler geometry can be studied with Lie theory using this result. In homogeneous Riemannian manifolds, the study of homogeneous geodesics is carried out by O. Kowalski, S. Nikcevic and Z. Vlasek [37]. Let (N, F) be a Finsler space and let G be the group of isometries $I(N, F)$. If each of its geodesics is an orbit of a one-parameter subgroup of G, the space (N, F) is a Finsler geodesic orbit space.

Definition 1.6.1. [19] Let (N, F) be a Finser space and G a Lie group. A diffeomorphism $\phi : N \to N$ is called an isometry if $F(\phi(p),\ d\phi_p(u)) = F(p, u),\ p \in N,\ u \in T_pN.$

We denote the group of isometries of (N, F) by $I(N, F)$.

Definition 1.6.2. Let G be a Lie group and N be a smooth manifold. If G has a smooth action on N, then G is called a Lie transformation group of N.

Definition 1.6.3. [19] If the action of the group of isometries $I(N, F)$ of a Finsler space (N, F) is transitive on N, then N is said to be a homogeneous Finsler space.

Definition 1.6.4. A Riemannian homogeneous space $(G/H, g)$ is said to be naturally reductive if there exists a connected Lie group G of isometries acting transitively on G/H is a reductive decomposition $\mathfrak{g} = \mathfrak{l} + \mathfrak{h}$ of $\mathfrak{g}$ satisfies the following condition:

$$\langle [\eta, \omega]_{\mathfrak{l}}, \xi \rangle + \langle \eta, [\omega, \xi]_{\mathfrak{l}} \rangle = 0, \forall\, \eta, \xi, \omega \in \mathfrak{l}. \tag{1.6.1}$$

Here, the subscript $\mathfrak{l}$ indicates the projection of an element of $\mathfrak{g}$ into $\mathfrak{l}$. And $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathfrak{l}$ induced by the metric g [38].

Definition 1.6.5. Let G be a Lie group with the identity element e and $\mathfrak{g}$ be its Lie algebra. The exponential map $\exp : \mathfrak{g} \to G$ is characterized by

$$\exp(tY) = \Psi(t), \forall\, t \in R,$$

where $\Psi : R \to G$ is defined as a unique one-parameter subgroup of G with $\dot{\Psi}(0) = Y_e$.

In a reductive homogeneous manifold, at the origin $eH = H$, we can recognize the tangent space $T_H(G/H)$ of G/H with $\mathfrak{l}$ through the map,

$$Y \to \frac{d}{dt} \exp(tY)H\Big|_{(t=0)}, Y \in \mathfrak{l},$$

since G/H is recognized as N and for any Lie group G that contains Lie algebra described as T_eG.

One parameter subgroups are the mappings $t \to \exp tz$, where z is an element of Lie algebra [30].

Definition 1.6.6. [38] A homogeneous space G/H of a connected Lie group G is called reductive if the following conditions are satisfied:

- In the Lie algebra $\mathfrak{g}$ of G, there exists a subspace $\mathfrak{l}$ such that $\mathfrak{g} = \mathfrak{l} + \mathfrak{h}$ (direct sum of vector subspaces).

- $\mathrm{Ad}(h)\mathfrak{l} \subset \mathfrak{l}$, for all $h \in H$, where $\mathfrak{h}$ is the subalgebra of $\mathfrak{g}$ corresponding to the identity component H_0 of H and $\mathrm{Ad}(h)$ denotes the adjoint representation of H in $\mathfrak{g}$.

The following geometrical property and the above-mentioned condition (1.6.1) both are equivalent:

For any vector $z \in \mathfrak{l} \setminus \{0\}$, the curve $\gamma(t) = \tau(\exp tz)(p)$ is a geodesic with respect to the Riemannian connection. Here, exp and $\tau(h)$ denote the Lie exponential map of G and the left transformation of G/H induced by $h \in G$, respectively. Thus, for a naturally reductive homogeneous space, every geodesic on $(G/H, g)$ is an orbit of a one-parameter subgroup of the group of isometries [44]. Let $(G/H, g)$ be a homogeneous Riemannian manifold with a fixed origin p, and

$$\mathfrak{g} = \mathfrak{l} + \mathfrak{h},$$

a reductive decomposition. A homogeneous geodesic through the origin $p \in G/H$ is a geodesic $\gamma(t)$ which is an orbit of a one-parameter subgroup of G, that is

$$\gamma(t) = \exp(tz)(p), \ t \in R,$$

where z is a nonzero vector of $\mathfrak{g}$. Kowalski and Venhecke [40] proved the following characterization of geodesic vectors.

Lemma 1.6.1. *[41] A vector $\eta \in \mathfrak{g}$ is a geodesic vector if and only if*

$$g_{\eta_{\mathfrak{l}}}([\eta, \xi]_{\mathfrak{l}}, \eta_{\mathfrak{l}}) = 0, \forall \xi \in \mathfrak{l}.$$

Geodesic orbit space

Definition 1.6.7. [41] Let (N, F) be a homogeneous Finsler space. If every geodesic in N is an orbit of a one-parameter group of isometries, i.e., there exists a transitive group G of isometries such that every geodesic in N is of the form $\exp(tz)p$ with $z \in \mathfrak{g},\ p \in N$. Then N is said to be geodesic orbit space.

Chapter 2

Flag curvature and geodesic orbit of homogeneous Finsler space

In this chapter, we have deduced the formula for flag curvature of homogeneous Finsler space (N, F) with metric $F = \alpha + \sqrt{\alpha^2 + \beta^2}$ and studied the condition for naturally reductive of homogeneous space (N, F). Further, we have discussed the existence of homogeneous geodesics for space (N, F). In the last part, we have obtained the formula for flag curvature of naturally reductive homogeneous Finsler space (N, F).

2.1 Introduction

Flag curvature was first introduced by Berwald (1926). It has an important role in characterizing Finsler spaces. The notion of naturally reductive Riemannian metrics was first introduced by Kobayashi and Nomizu [38]. It is well-known that the geodesics of a naturally reductive homogeneous space are the orbits of one-parameter subgroups of isometries [5]. In the field of mechanics homogeneous geodesics has important applications.

In recent years, many authors have given the formula for flag curvature of a naturally reductive homogeneous Finsler space with different (α, β)-metrics [18, 44, 41, 57] and also discussed the naturally reductive of homogeneous Finsler space. We have studied the existence of homogeneous geodesics for the homogeneous Finsler space with metric $F = \alpha + \sqrt{\alpha^2 + \beta^2}$.

Using the lemma 1.6.1, as a result of Puttmann's work, the curvature tensor of invariant metrics $\langle \cdot, \cdot \rangle$ in compact homogeneous spaces G/H has the following formula:

$$\begin{aligned} \langle R(u,v)w, z\rangle =& \frac{1}{2}\Big\{\langle\langle B_-(u,v), [w,z]\rangle\rangle + \langle\langle [u,v], B_-(w,z)\rangle\rangle\Big\} \\ & \frac{1}{4}\Big\{\langle [u,z], [v,w]_{\mathfrak{l}}\rangle - \langle [u,w], [v,z]_{\mathfrak{l}}\rangle - 2\langle [u,v], [w,z]_{\mathfrak{l}}\Big\} \\ & \langle\langle B_+(u,z), \psi^{-1}B_+(v,w)\rangle\rangle - \langle\langle B_+(u,w), \psi^{-1}B_+(v,z)\rangle\rangle, \end{aligned} \tag{2.1.1}$$

where the bilinear maps B_+ (symmetric) and B_- (skew-symmetric) are defined by

$$\begin{aligned} B_+(u,v) =& \frac{1}{2}\Big([u, \psi v] + [v, \psi u]\Big), \\ B_-(u,v) =& \frac{1}{2}\Big([\psi u, v] + [u, \psi v]\Big), \end{aligned}$$

and $[\,,\,]_{\mathfrak{l}}$ is the projection of $[\,,\,]$ to $\mathfrak{l}$.

2.2 Flag curvature of homogeneous Finsler space

The present section deals with the flag curvature of a homogeneous Finsler space with metric $F = \alpha + \sqrt{\alpha^2 + \beta^2}$.

Let $(G/H, \alpha)$ be a homogeneous Riemannian manifold. Then the Lie algebra has a decomposition $\mathfrak{g} = \mathfrak{l} + \mathfrak{h}$, where $\mathfrak{h}$ and $\mathfrak{g}$ are the Lie algebras of H and G respectively. We consider $\mathfrak{l}$ along the $T_0(G/H)$, the tangent space at $H = o$ (origin) (means isomorphism between the $\mathfrak{l}$ and $T_0(G/H)$). Here a left-invariant metric denoted by $\bar{\alpha}$ on G was generalized by a G-invariant Riemannian metric on G/H.

The concept of a naturally reductive homogeneous Riemannian space is a generalization of the concept of bi-invariant Riemannian metric denoted by $\bar{\alpha_0}$ on G. The values of $\bar{\alpha_0}$ and $\bar{\alpha}$ are the inner products on G and denote them as $\langle\langle \cdot, \cdot \rangle\rangle$ and $\langle \cdot, \cdot \rangle$ respectively. The inner product $\langle \cdot, \cdot \rangle$ induces an endomorphism ψ of $\mathfrak{g}$ such that $\langle v, w \rangle = \langle\langle \psi(v), w \rangle\rangle, \forall\, v, w \in \mathfrak{g}$. In fact, if $H = e$ then $\mathfrak{l} = \mathfrak{g}$, Eq. (1.6.1) is just the condition for a bi-invariant Riemannian metric on G.

Theorem 2.2.1. *Let a compact Lie group G contains a closed subgroup H with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ of G and H respectively. Also an invariant Riemannian metric $\tilde{\alpha}$ on the homogeneous space G/H such that $\langle v, w \rangle = \langle\langle \psi(v), w \rangle\rangle$, where $\psi : \mathfrak{g} \to \mathfrak{g}, \forall\, v, w \in \mathfrak{g}$ is a positive definite endomorphism. Suppose that an invariant vector field $\tilde{u}$ on homogeneous space G/H is parallel with respect to Riemannian metric $\tilde{\alpha}$ and $\tilde{u}_H = u$ and assume that $F = \alpha + \sqrt{\alpha^2 + \beta^2}$ be a special (α, β)-metric arising from $\tilde{\alpha}$ and $\tilde{u}$ such that its Chern connection of F and the Riemannian connection of $\tilde{\alpha}$ are coincides, and a flag $\{P, \eta\}$ in $T_H(G/H)$ such that $\{\zeta, \eta\}$ is an orthonormal basis of P with respect to $\langle \cdot, \cdot \rangle$. Then the flag curvature of the flag $\{P, \eta\}$ is given by*

$$K(P, \eta) = \frac{\langle \zeta, R(\zeta, \eta)\eta \rangle S_1 + \langle u, \zeta \rangle \langle u, R(\zeta, \eta)\eta \rangle S_2 + \langle \eta, R(\zeta, \eta)\eta \rangle\, S_3}{8 + 4\langle u, \eta \rangle^2 + 2\langle u, \zeta \rangle^2 + A_1}, \tag{2.2.1}$$

where

$$S_1 = 2 + \frac{2 + \langle u, \eta \rangle^2}{\sqrt{1 + \langle u, \eta \rangle^2}}, \quad S_2 = \frac{(1 + \langle u, \eta \rangle^2)^{\frac{3}{2}} + 1}{(1 + \langle u, \eta \rangle^2)^{\frac{3}{2}}}, \quad S_3 = \frac{\langle u, \eta \rangle^3 \langle u, \zeta \rangle}{(1 + \langle u, \eta \rangle^2)^{\frac{3}{2}}},$$

$$S_4 = 8 + 8\langle u, \eta \rangle^2 + 2\langle u, \zeta \rangle^2 (1 + \langle u, \eta \rangle^2) + \langle u, \eta \rangle^4, \quad S_5 = 2\langle u, \zeta \rangle^2 - \langle u, \eta \rangle^2 \langle u, \zeta \rangle^2,$$

$$A_1 = \frac{S_4}{\sqrt{1 + \langle u, \eta \rangle^2}} + \frac{2\langle u, \zeta \rangle^2}{1 + \langle u, \eta \rangle^2} + \frac{S_5}{(1 + \langle u, \eta \rangle^2)^{\frac{3}{2}}} - \frac{\langle u, \eta \rangle^2 \langle u, \zeta \rangle^2}{(1 + \langle u, \eta \rangle^2)^3}.$$

Proof. We can write $F(\eta) = \sqrt{\langle \eta, \eta \rangle} + \sqrt{\langle \eta, \eta \rangle + \langle u, \eta \rangle^2}, \forall u \in \mathfrak{l}$. By the formula,

$$g_\eta(\zeta, \omega) = \frac{1}{2} \frac{\partial^2}{\partial s \partial t} F^2(\eta + s\zeta + t\omega)\Big|_{t=s=0}, \tag{2.2.2}$$

$$\begin{aligned} g_\eta(\zeta, \omega) =& 2\langle \zeta, \omega \rangle + \langle u, \zeta \rangle \langle u, \omega \rangle + \frac{1}{\langle \eta, \eta \rangle^2 + \langle u, \eta \rangle^2 \langle \eta, \eta \rangle} \\ & \times \Bigg[\sqrt{\langle \eta, \eta \rangle^2 + \langle u, \eta \rangle^2 \langle \eta, \eta \rangle} \Big\{ 4\langle \eta, \zeta \rangle \langle \eta, \omega \rangle + 2\langle \eta, \eta \rangle \langle \zeta, \omega \rangle + 2\langle u, \eta \rangle \langle u, \omega \rangle \langle \eta, \zeta \rangle \\ & + \langle u, \zeta \rangle \langle u, \omega \rangle \langle \eta, \eta \rangle + 2\langle u, \eta \rangle \langle u, \zeta \rangle \langle \eta, \omega \rangle + \langle u, \eta \rangle^2 \langle \zeta, \omega \rangle \Big\} \\ & - \Big\{ 2\langle \eta, \eta \rangle \langle \eta, \omega \rangle + \langle u, \eta \rangle \langle u, \omega \rangle \langle \eta, \eta \rangle + \langle u, \eta \rangle^2 \langle \eta, \omega \rangle \Big\} \\ & \times \Big\{ \frac{2\langle \eta, \eta \rangle \langle \eta, \zeta \rangle + \langle u, \eta \rangle \langle u, \zeta \rangle \langle \eta, \eta \rangle + \langle u, \eta \rangle^2 \langle \eta, \zeta \rangle}{\sqrt{\langle \eta, \eta \rangle^2 + \langle u, \eta \rangle^2 \langle \eta, \eta \rangle}} \Big\} \Bigg], \end{aligned}$$

on simplification of the above term, we get

$$g_\eta(\zeta, \omega) = 2\langle \zeta, \omega \rangle + \langle u, \zeta \rangle \langle u, \omega \rangle + \frac{L_1}{\sqrt{\langle \eta, \eta \rangle^2 + \langle u, \eta \rangle^2 \langle \eta, \eta \rangle}} - \frac{L_2}{(\langle \eta, \eta \rangle^2 + \langle u, \eta \rangle^2 \langle \eta, \eta \rangle)^{\frac{3}{2}}}, \tag{2.2.3}$$

where,

$$\begin{aligned} L_1 =& 4\langle \eta, \zeta \rangle \langle \eta, \omega \rangle + 2\langle \eta, \eta \rangle \langle \zeta, \omega \rangle + 2\langle u, \eta \rangle \langle u, \omega \rangle \langle \eta, \zeta \rangle + \langle u, \zeta \rangle \langle u, \omega \rangle \langle \eta, \eta \rangle \\ & + 2\langle u, \eta \rangle \langle u, \zeta \rangle \langle \eta, \omega \rangle + \langle u, \eta \rangle^2 \langle \zeta, \omega \rangle, \\ L_2 =& 4\langle \eta, \eta \rangle^2 \langle \eta, \zeta \rangle \langle \eta, \omega \rangle + 2\langle u, \eta \rangle \langle u, \zeta \rangle \langle \eta, \omega \rangle \langle \eta, \eta \rangle^2 + 2\langle u, \eta \rangle^2 \langle \eta, \eta \rangle \langle \eta, \zeta \rangle \langle \eta, \omega \rangle \\ & + 2\langle u, \eta \rangle \langle \eta, \eta \rangle^2 \langle u, \omega \rangle \langle \eta, \zeta \rangle + \langle u, \eta \rangle^2 \langle \eta, \eta \rangle^2 \langle u, \zeta \rangle \langle u, \omega \rangle + \langle u, \eta \rangle^3 \langle \eta, \eta \rangle \langle u, \omega \rangle \langle \eta, \zeta \rangle \\ & + 2\langle u, \eta \rangle^2 \langle \eta, \eta \rangle \langle \eta, \zeta \rangle \langle \eta, \omega \rangle + \langle u, \eta \rangle^3 \langle u, \zeta \rangle \langle \eta, \eta \rangle \langle \eta, \omega \rangle + \langle u, \eta \rangle^4 \langle \eta, \zeta \rangle \langle \eta, \omega \rangle. \end{aligned}$$

For an orthonormal basis $\{\zeta, \eta\}$ with respect to $\langle\cdot,\cdot\rangle$, Eq. (2.2.3) can be reduces to

$$g_\eta(\zeta,\omega) = 2\langle\zeta,\omega\rangle + \langle u,\zeta\rangle\langle u,\omega\rangle + \frac{L_3}{\sqrt{1+\langle u,\eta\rangle^2}} - \frac{L_4}{(1+\langle u,\eta\rangle^2)^{\frac{3}{2}}}, \tag{2.2.4}$$

where,

$$L_3 = 2\langle\zeta,\omega\rangle + \langle u,\zeta\rangle\langle u,\omega\rangle + 2\langle u,\eta\rangle\langle u,\zeta\rangle\langle\eta,\omega\rangle + \langle u,\eta\rangle^2\langle\zeta,\omega\rangle,$$

$$L_4 = 2\langle u,\eta\rangle\langle u,\zeta\rangle\langle\eta,\omega\rangle + \langle u,\eta\rangle^2\langle u,\zeta\rangle\langle u,\omega\rangle + \langle u,\eta\rangle^3\langle u,\zeta\rangle\langle\eta,\omega\rangle.$$

From Eq. (2.2.3), we get the following:

$$g_\eta(\eta,\eta) = 2 + \langle u,\eta\rangle^2 + 2\sqrt{1+\langle u,\eta\rangle^2},$$

$$g_\eta(\zeta,\zeta) = 2 + \langle u,\zeta\rangle^2 + \frac{2+\langle u,\zeta\rangle^2 + 3\langle u,\eta\rangle^2 + \langle u,\eta\rangle^4}{(1+\langle u,\eta\rangle^2)^{\frac{3}{2}}},$$

$$g_\eta(\eta,\zeta) = \langle u,\eta\rangle\langle u,\zeta\rangle + \frac{\langle u,\eta\rangle\langle u,\zeta\rangle}{(1+\langle u,\eta\rangle^2)^{\frac{3}{2}}}.$$

Therefore,

$$\begin{aligned} g_\eta(\eta,\eta)g_\eta(\zeta,\zeta) - g_\eta^2(\eta,\zeta) =& \left\{2+\langle u,\eta\rangle^2 + 2\sqrt{1+\langle u,\eta\rangle^2}\right\}\Bigg\{2+\langle u,\zeta\rangle^2 \\ &+ \frac{2+\langle u,\zeta\rangle^2+3\langle u,\eta\rangle^2+\langle u,\eta\rangle^4}{(1+\langle u,\eta\rangle^2)^{\frac{3}{2}}}\Bigg\} - \Bigg\{\langle u,\eta\rangle\langle u,\zeta\rangle \\ &+ \frac{\langle u,\eta\rangle\langle u,\zeta\rangle}{(1+\langle u,\eta\rangle^2)^{\frac{3}{2}}}\Bigg\}^2, \\ =& 8 + 4\langle u,\eta\rangle^2 + 2\langle u,\zeta\rangle^2 + \frac{8+8\langle u,\eta\rangle^2 + 2\langle u,\zeta\rangle^2(1+\langle u,\eta\rangle^2)}{\sqrt{1+\langle u,\eta\rangle^2}} \\ &+ \frac{2\langle u,\zeta\rangle^2}{1+\langle u,\eta\rangle^2} + \frac{2\langle u,\zeta\rangle^2}{(1+\langle u,\eta\rangle^2)^{\frac{3}{2}}} - \frac{\langle u,\eta\rangle^2\langle u,\zeta\rangle^2}{(1+\langle u,\eta\rangle^2)^{\frac{3}{2}}} \\ &- \frac{\langle u,\eta\rangle^2\langle u,\zeta\rangle^2}{(1+\langle u,\eta\rangle^2)^3} + \frac{\langle u,\eta\rangle^4}{\sqrt{1+\langle u,\eta\rangle^2}}, \end{aligned} \tag{2.2.5}$$

and

$$\begin{aligned} g_\eta(\zeta,R(\zeta,\eta)\eta) =& \langle\zeta,R(\zeta,\eta)\eta\rangle\left[2+\frac{2+\langle u,\eta\rangle^2}{\sqrt{1+\langle u,\eta\rangle^2}}\right] + \langle u,\zeta\rangle\langle u,R(\zeta,\eta)\eta\rangle \\ &\times\left[\frac{(1+\langle u,\eta\rangle^2)^{\frac{3}{2}}+1}{(1+\langle u,\eta\rangle^2)^{\frac{3}{2}}}\right] + \frac{\langle u,\eta\rangle^3\langle u,\zeta\rangle\langle\eta,R(\zeta,\eta)\eta\rangle}{(1+\langle u,\eta\rangle^2)^{\frac{3}{2}}}. \end{aligned} \tag{2.2.6}$$

Here, by using Puttmann's formula [63], we obtain the following:

$$\langle \zeta, R(\zeta,\eta)\eta\rangle = \frac{1}{2}\left(\langle\langle[\psi\zeta,\eta]+[\zeta,\psi\eta],[\eta,\zeta]\rangle\rangle\right)+\frac{3}{4}\langle[\eta,\zeta],[\eta,\zeta]_{\mathfrak{l}}\rangle+\langle\langle[\zeta,\psi\zeta],\psi^{-1}([\eta,\psi\eta])\rangle\rangle$$
$$-\frac{1}{4}\langle\langle[\zeta,\psi\eta]+[\eta,\psi\zeta],\psi^{-1}([\eta,\psi\zeta]+[\zeta,\psi\eta])\rangle\rangle. \tag{2.2.7}$$

By substituting Eqs. (2.2.5) to Eq. (2.2.7) in (1.2.2), we obtain the required Eq. (2.2.1).

□

2.3 Naturally reductive homogeneous Finsler space

Let G be a connected Lie group. Then there exists a bi-invariant Finsler metric on G if and only if there exists a Minkowski norm F on $\mathfrak{g}$ such that the below condition (2.3.1) is considered as a natural generalization of the condition (1.6.1), that is

$$g_\eta([\xi,\zeta]_{\mathfrak{l}},\omega)+g_\eta(\zeta,[\xi,\omega]_{\mathfrak{l}})+2C_\eta([\xi,\eta]_{\mathfrak{l}},\zeta,\omega)=0, \quad \forall\eta\neq 0,\ \xi,\zeta,\omega\in\mathfrak{l}. \tag{2.3.1}$$

The naturally reductive homogeneous Finsler space was first proposed by D. Latifi [41].

Definition 2.3.1. A homogeneous manifold $N=G/H$ with an invariant Finsler metric F is called naturally reductive if there exists an $\mathrm{Ad}(h)$-invariant decomposition $\mathfrak{g}=\mathfrak{l}\oplus\mathfrak{h}$ such that Eq. (2.3.1) holds.

Definition 2.3.2. [21] A homogeneous space $(N=G/H,F)$ with an invariant Finsler metric is called naturally reductive if there exists an invariant Riemannian metric $\tilde{\alpha}$ such that $(N,\tilde{\alpha})$ is naturally reductive and the Chern connection of F coincides with the Levi-Civita connection of $\tilde{\alpha}$.

In this definition, the authors have assumed that the metric must be of Berwald type.

Remark: If $(G/H, F)$ is naturally reductive following the definition 2.3.1, then it implies that $(G/H, F)$ must be naturally reductive in the sense of definition 2.3.2 [22].

Theorem 2.3.1. *Let a homogeneous Finsler space* $(G/H, F)$ *with* (α, β)*-metric* $F = \alpha + \sqrt{\alpha^2 + \beta^2}$ *be defined by an invariant Riemannian metric* $\tilde{\alpha}$ *and an invariant vector field* u *such that the Chern connection of* F *coincides the Levi-Civita connection of* $\tilde{\alpha}$*. Then* $(G/H, F)$ *is naturally reductive if and only if the underlying Riemannian space* $(G/H, \tilde{\alpha})$ *is naturally reductive.*

Proof. Let $\eta \neq 0, \omega \in \mathfrak{l}$. From Eq. (2.2.3), we get

$$\begin{aligned} g_\eta(\eta, [\eta, \omega]_\mathfrak{l}) =& \langle \eta, [\eta, \omega]_\mathfrak{l} \rangle \left\{ 2 + \frac{L_5}{\sqrt{\langle \eta, \eta \rangle^2 + \langle u, \eta \rangle^2 \langle \eta, \eta \rangle}} - \frac{L_6}{(\langle \eta, \eta \rangle^2 + \langle u, \eta \rangle^2 \langle \eta, \eta \rangle)^{\frac{3}{2}}} \right\} \\ & + \langle u, [\eta, w]_\mathfrak{l} \rangle \left\{ \langle u, \eta \rangle + \frac{L_7}{\sqrt{\langle \eta, \eta \rangle^2 + \langle u, \eta \rangle^2 \langle \eta, \eta \rangle}} \right. \\ & \left. - \frac{L_8}{(\langle \eta, \eta \rangle^2 + \langle u, \eta \rangle^2 \langle \eta, \eta \rangle)^{\frac{3}{2}}} \right\}, \end{aligned} \tag{2.3.2}$$

where,

$$\begin{aligned} L_5 =& 6\langle \eta, \eta \rangle + 3\langle u, \eta \rangle^2, \quad L_6 = 4\langle \eta, \eta \rangle^3 + 6\langle u, \eta \rangle^2 \langle \eta, \eta \rangle^2 + 2\langle u, \eta \rangle^4 \langle \eta, \eta \rangle, \\ L_7 =& 3\langle u, \eta \rangle \langle \eta, \eta \rangle, \qquad L_8 = 2\langle u, \eta \rangle \langle \eta, \eta \rangle^3 + 2\langle u, \eta \rangle^3 \langle \eta, \eta \rangle^2. \end{aligned}$$

Since F is of Berwald type, $(G/H, F)$ and $(G/H, \tilde{\alpha})$ have the connection that coincides. Thus Eq. (2.3.2) implies that

$$\langle u, [\eta, \omega]_\mathfrak{l} \rangle = 0, \forall \omega \in \mathfrak{l}. \tag{2.3.3}$$

Now, let $(G/H, F)$ be naturally reductive, as per the reference [41], we can write Eq. (2.3.1) as follows:

$$g_\eta([\eta, \zeta]_\mathfrak{l}, \omega) + g_\eta(\zeta, [\eta, \omega]_\mathfrak{l}) + 2C_\eta([\eta, \eta]_\mathfrak{l}, \zeta, \omega) = 0, \eta \neq 0.$$

which implies

$$g_\eta([\eta,\zeta]_{\mathfrak{l}},\omega)+g_\eta(\zeta,[\eta,\omega]_{\mathfrak{l}})=0, \tag{2.3.4}$$

from Eq. (2.3.4), we have

$$g_\eta([\eta,\omega]_{\mathfrak{l}},\eta)=0. \tag{2.3.5}$$

From Eqs. (2.3.2), (2.3.3), and (2.3.5), we obtain

$$\langle[\eta,\omega]_{\mathfrak{l}},\eta\rangle=0. \tag{2.3.6}$$

From Eq. (2.2.3), also using Eqs. (2.3.3) and (2.3.6) we have

$$g_\eta([\eta,\zeta]_{\mathfrak{l}},\omega)=\langle[\eta,\zeta]_{\mathfrak{l}},\omega\rangle\left\{2+\frac{2\langle\eta,\eta\rangle+\langle u,\eta\rangle^2}{\sqrt{\langle\eta,\eta\rangle^2+\langle u,\eta\rangle^2\langle\eta,\eta\rangle}}\right\},$$

similarly,

$$g_\eta(\zeta,[\eta,\omega]_{\mathfrak{l}})=\langle\zeta,[\eta,\omega]_{\mathfrak{l}}\rangle\left\{2+\frac{2\langle\eta,\eta\rangle+\langle u,\eta\rangle^2}{\sqrt{\langle\eta,\eta\rangle^2+\langle u,\eta\rangle^2\langle\eta,\eta\rangle}}\right\}.$$

Adding the terms $g_\eta([\eta,\zeta]_{\mathfrak{l}},\omega)$ and $g_\eta(\zeta,[\eta,\omega]_{\mathfrak{l}})$, we obtain

$$g_\eta(\zeta,[\eta,\omega]_{\mathfrak{l}})+g_\eta([\eta,\zeta]_{\mathfrak{l}},\omega)=\{\langle\zeta,[\eta,\omega]_{\mathfrak{l}}\rangle+\langle[\eta,\zeta]_{\mathfrak{l}},\omega\rangle\}\left\{2+\frac{2\langle\eta,\eta\rangle+\langle u,\eta\rangle^2}{\sqrt{\langle\eta,\eta\rangle^2+\langle u,\eta\rangle^2\langle\eta,\eta\rangle}}\right\}. \tag{2.3.7}$$

Since, $\left\{2+\frac{2\langle\eta,\eta\rangle+\langle u,\eta\rangle^2}{\sqrt{\langle\eta,\eta\rangle^2+\langle u,\eta\rangle^2\langle\eta,\eta\rangle}}\right\}\neq 0$, we have

$$\langle\zeta,[\eta,\omega]_{\mathfrak{l}}\rangle+\langle[\eta,\zeta]_{\mathfrak{l}},\omega\rangle=0.$$

Hence, the Riemannian metric $(G/H,\tilde{\alpha})$ is naturally reductive.

On the other hand, let the Riemannian space $(G/H, \tilde{\alpha})$ be naturally reductive. Thus, by using Eq. (2.2.3), we can write

$$\begin{aligned} g_\eta([\xi,\zeta]_{\mathfrak{l}},\omega) =& \langle[\xi,\zeta]_{\mathfrak{l}},\omega\rangle \left\{2 + \frac{2\langle\eta,\eta\rangle + \langle u,\eta\rangle^2}{\sqrt{\langle\eta,\eta\rangle^2 + \langle u,\eta\rangle^2\langle\eta,\eta\rangle}}\right\} \\ &+ 2\langle[\xi,\zeta]_{\mathfrak{l}},\eta\rangle \left\{\frac{2\langle\eta,\omega\rangle + \langle u,\eta\rangle\langle u,\omega\rangle}{\sqrt{\langle\eta,\eta\rangle^2 + \langle u,\eta\rangle^2\langle\eta,\eta\rangle}}\right\} \\ &- \frac{\langle[\xi,\zeta]_{\mathfrak{l}},\eta\rangle}{(\langle\eta,\eta\rangle^2 + \langle u,\eta\rangle^2\langle\eta,\eta\rangle)^{\frac{3}{2}}}[4\langle\eta,\eta\rangle^2\langle\eta,\omega\rangle + 4\langle u,\eta\rangle^2\langle\eta,\eta\rangle\langle\eta,\omega\rangle \\ &+ 2\langle u,\eta\rangle\langle\eta,\eta\rangle^2\langle u,\omega\rangle + \langle u,\eta\rangle^3\langle\eta,\eta\rangle\langle u,\omega\rangle + \langle u,\eta\rangle^4\langle\eta,\omega\rangle], \end{aligned} \tag{2.3.8}$$

and similarly,

$$\begin{aligned} g_\eta(\zeta,[\xi,\omega]_{\mathfrak{l}}) =& \langle[\xi,\omega]_{\mathfrak{l}},\zeta\rangle \left\{2 + \frac{2\langle\eta,\eta\rangle + \langle u,\eta\rangle^2}{\sqrt{\langle\eta,\eta\rangle^2 + \langle u,\eta\rangle^2\langle\eta,\eta\rangle}}\right\} \\ &+ 2\langle[\xi,\omega]_{\mathfrak{l}},\eta\rangle \left\{\frac{2\langle\eta,\zeta\rangle + \langle u,\eta\rangle\langle u,\zeta\rangle}{\sqrt{\langle\eta,\eta\rangle^2 + \langle u,\eta\rangle^2\langle\eta,\eta\rangle}}\right\} \\ &- \frac{\langle[\xi,\omega]_{\mathfrak{l}},\eta\rangle}{(\langle\eta,\eta\rangle^2 + \langle u,\eta\rangle^2\langle\eta,\eta\rangle)^{\frac{3}{2}}}[4\langle\eta,\eta\rangle^2\langle\eta,\zeta\rangle + 4\langle u,\eta\rangle^2\langle\eta,\eta\rangle\langle\eta,\zeta\rangle \\ &+ 2\langle u,\eta\rangle\langle\eta,\eta\rangle^2\langle u,\zeta\rangle + \langle u,\eta\rangle^3\langle\eta,\eta\rangle\langle u,\zeta\rangle + \langle u,\eta\rangle^4\langle\eta,\zeta\rangle]. \end{aligned} \tag{2.3.9}$$

By the fundamental Cartan tensor

$$C_\eta(\xi,\zeta,\omega) = \frac{1}{2}\frac{d}{dt}[g_{\eta+t\omega}(\xi,\zeta)]\Big|_{t=0},$$

we have,

$$\begin{aligned} 2C_\eta([\xi,\eta]_{\mathfrak{l}},\zeta,\omega) =& 2\langle[\xi,\eta]_{\mathfrak{l}},\zeta\rangle \left\{\frac{2\langle\eta,\omega\rangle + \langle u,\eta\rangle\langle u,\omega\rangle}{\sqrt{\langle\eta,\eta\rangle^2 + \langle u,\eta\rangle^2\langle\eta,\eta\rangle}}\right\} \\ &+ 2\langle[\xi,\eta]_{\mathfrak{l}},\omega\rangle \left\{\frac{2\langle\eta,\zeta\rangle + \langle u,\eta\rangle\langle u,\zeta\rangle}{\sqrt{\langle\eta,\eta\rangle^2 + \langle u,\eta\rangle^2\langle\eta,\eta\rangle}}\right\} \\ &- \frac{\langle[\xi,\eta]_{\mathfrak{l}},\zeta\rangle}{(\langle\eta,\eta\rangle^2 + \langle u,\eta\rangle^2\langle\eta,\eta\rangle)^{\frac{3}{2}}}[4\langle\eta,\eta\rangle^2\langle\eta,\omega\rangle + 4\langle u,\eta\rangle^2\langle\eta,\eta\rangle\langle\eta,\omega\rangle \\ &+ 2\langle u,\eta\rangle\langle\eta,\eta\rangle^2\langle u,\omega\rangle + \langle u,\eta\rangle^3\langle\eta,\eta\rangle\langle u,\omega\rangle + \langle u,\eta\rangle^4\langle\eta,\omega\rangle] \end{aligned}$$

$$- \frac{\langle [\xi,\eta]_{\mathfrak{l}},\omega\rangle}{(\langle u,\eta\rangle^2+\langle u,\eta\rangle^2\langle \eta,\eta\rangle)^{\frac{3}{2}}}[4\langle \eta,\eta\rangle^2\langle \eta,\zeta\rangle+4\langle u,\eta\rangle^2\langle \eta,\eta\rangle\langle \eta,\zeta\rangle$$
$$+2\langle u,\eta\rangle\langle \eta,\eta\rangle^2\langle u,\zeta\rangle+\langle u,\eta\rangle^3\langle \eta,\eta\rangle\langle u,\zeta\rangle+\langle u,\eta\rangle^4\langle \eta,\zeta\rangle]. \tag{2.3.10}$$

Since, $(G/H,\tilde{\alpha})$ is naturally reductive, adding Eqs. (2.3.8), (2.3.9) and (2.3.10), we obtain

$$g_\eta([\xi,\zeta]_{\mathfrak{l}},\omega)+g_\eta([\xi,\omega]_{\mathfrak{l}},\zeta)+2C_\eta([\xi,\eta]_{\mathfrak{l}},\zeta,\omega)=0.$$

Thus, the homogeneous Finsler space $(G/H,F)$ is also naturally reductive. Hence the proof. □

The next section deals with geodesic orbit of homogeneous Finsler space.

2.4 Existence of geodesic orbit on homogeneous Finsler space

D. Latifi, Toomanian [44], and some other authors discussed the results of homogeneous geodesics in a homogeneous Finsler manifold. In Finsler space, the basic formula characterizing the geodesic vector was given in [41]. Let $(G/H,g)$ be a homogeneous Riemannian manifold with a fixed origin p, and $\mathfrak{g}=\mathfrak{l}+\mathfrak{h}$ a reductive decomposition.

The following theorem shows the existence of homogeneous geodesics.

Theorem 2.4.1. *Let a homogeneous Finsler space (N,F) with $F=\alpha+\sqrt{\alpha^2+\beta^2}$ defined by the Riemannian metric $\alpha=a_{ij}dx^i\otimes dx^j$ and the vector field u corresponding to 1-form β. Then the homogeneous Finsler space (N,F) with the origin $p=\{H\}$ and with an $\mathrm{Ad}(h)$-invariant decomposition $\mathfrak{g}=\mathfrak{l}+\mathfrak{h}$ is naturally reductive with respect to this* *decomposition if and only if for any vector* $u \in \mathfrak{l} \setminus \{0\}$*, the curve* $\gamma(t) = \exp tu(p)$ *is geodesic of the homogeneous Finsler space* (N, F)*.*

Proof. Suppose that a homogeneous Finsler space (N, F) is naturally reductive, then the Riemannian metric α is naturally reductive, and the connection of Finsler metric F and Riemannian metric α are coinciding. This means Finsler metric is of Berwald metric. Let the decomposition $\mathfrak{g} = \mathfrak{l} + \mathfrak{h}$ be naturally reductive, thus

$$g_\eta([u,\zeta]_\mathfrak{l},\omega) + g_\eta([u,\omega]_\mathfrak{l},\zeta) + 2C_\eta([u,\eta]_\mathfrak{l},\zeta,\omega) = 0,$$

where $\eta \neq 0, u, \zeta, \omega \in \mathfrak{l}$. Then for $\xi \in \mathfrak{l}$,

$$g_\xi([\xi,\eta]_\mathfrak{l},\xi) = a(\xi,[\xi,\eta]_\mathfrak{l}) = 0, \forall\, \eta \in \mathfrak{l}.$$

Thus, according to D. Latifi [41], each geodesic of $(G/H, F)$ obtained from the fixed origin $p = \{H\}$ is nothing but $\exp(tu)p, u \in \mathfrak{l}$.

On the other hand, assume that for any vector $u \in \mathfrak{l} \setminus \{0\}$, the curve $\gamma(t)$ is a geodesic of homogeneous Finsler space $(G/H, F)$, where $\gamma(t) = \exp tu(p)$. Here the first thing is to prove the metric F is a Berwald metric. The smooth mapping $\pi : G \to G/H$ is a canonical projection that instigates an isomorphism between the tangent space $T_\zeta N$ and the subspace $\mathfrak{l}$. In fact, the tangent space can be identified with $\mathfrak{l}$ by the correspondence

$$u \in \mathfrak{l} \to \frac{d}{dt}\exp(tu)\cdot\zeta\Big|_{t=0}.$$

And then, due to the geodesics of (N, F), the exponential mapping $Exp|_\zeta$ of (N, F) at ζ is $Exp(u) = \pi(\exp(u)),\ u \in \mathfrak{l}$, which is a smooth mapping everywhere. Therefore F must be a Berwald metric [6].

Since F is of Berwald type, $(G/H, F)$ and $(G/H, \alpha)$ have the same connection and also have the same geodesic. Therefore, for any $\omega \in \mathfrak{l}$, and the curve $\exp(tu)p$ is a homogeneous geodesic of $(G/H, \alpha)$, from the lemma 1.6.1, we conclude that, the vector $\omega \in \mathfrak{l}$ is a geodesic vector, hence $\langle [\omega, \eta]_{\mathfrak{l}}, \omega \rangle = 0, \forall\ \eta \in \mathfrak{l}$.

Consider, any $\eta, \omega, \xi \in \mathfrak{l}$, set $\xi' = \xi + \eta,\ \eta' = \eta + \omega$.

Then, we have

$$0 = \langle [\xi', \eta']_{\mathfrak{l}}, \xi' \rangle = \langle \eta, [\xi, \omega]_{\mathfrak{l}} \rangle + \langle [\eta, \omega]_{\mathfrak{l}}, \xi \rangle.$$

This implies that $(G/H, \alpha)$ is naturally reductive. So $(G/H, F)$ is naturally reductive. □

Now, we can discuss the Flag curvature of naturally reductive homogeneous Finsler space:

2.5 Flag curvature of naturally reductive homogeneous Finsler space

As per the study of Deng and Hou in [21] authors found the formula for flag curvature of naturally reductive homogeneous (α, β)-metric spaces. In the sense of Deng and Hou, for special (α, β)-metric $F = \alpha + \sqrt{\alpha^2 + \beta^2}$, we derive the formulae for flag curvature of naturally reductive homogeneous Finsler space.

Theorem 2.5.1. *A naturally reductive homogeneous Finsler space G/H with (α, β)-metric $F = \alpha + \sqrt{\alpha^2 + \beta^2}$, defined by an invariant Riemannian metric $\tilde{\alpha}$ and an invariant vector field $\tilde{u}$ on G/H, suppose that $\tilde{u}_H = u$. And (P, η) be a flag in $T_H(G/H)$ such that $\{\zeta, \eta\}$ is an orthonormal basis of P with respect to $\langle \cdot, \cdot \rangle$. Then the flag curvature of the* *flag* (P,η) *is given by*

$$K(P,\eta)=\left[\frac{\langle\zeta,\frac{1}{4}[\eta,[\zeta,\eta]_{\mathfrak{l}}]_{\mathfrak{l}}+[\eta,[\zeta,\eta]_{\mathfrak{h}}]\rangle S_1+\langle u,\frac{1}{4}[\eta,[\zeta,\eta]_{\mathfrak{l}}]_{\mathfrak{l}}+[\eta,[\zeta,\eta]_{\mathfrak{h}}]\rangle S_2+\langle\eta,\frac{1}{4}[\eta,[\zeta,\eta]_{\mathfrak{l}}]_{\mathfrak{l}}+[\eta,[\zeta,\eta]_{\mathfrak{h}}]\rangle S_3}{8+4\langle u,\eta\rangle^2+2\langle u,\zeta\rangle^2+A_1}\right], \tag{2.5.1}$$

where $A_1,\ S_1,\ S_2,\ S_3,\ S_4$ *and* S_5 *are as defined in Eq.* (2.2.1).

Proof. Since (N,F) is naturally reductive, and using proposition 3.4, from [38], we have

$$R(\zeta,\eta)\eta=\frac{1}{4}[\eta,[\zeta,\eta]_{\mathfrak{l}}]_{\mathfrak{l}}+[\eta,[\zeta,\eta]_{\mathfrak{h}}],\forall\zeta,\eta\in\mathfrak{l}.$$

Substitute the above equation in Eq. (2.2.1) and after simplification, we get Eq. (2.5.1).

□

If $H=\{e\}$, then we have the following corollary:

Corollary 2.5.2. *Let* $F=\alpha+\sqrt{\alpha^2+\beta^2}$ *be defined by a bi-invariant Riemannian metric* $\tilde{\alpha}$ *on a Lie group* G *and a left-invariant vector field* u *on* G *such that the Chern connection of* F *and Riemannian connection of* $\tilde{\alpha}$ *coincides. Suppose that* (P,η) *be a flag in* $T_e(G)$ *such that* $\{\zeta,\eta\}$ *is an orthonormal basis of* P *with respect to* $\langle\cdot,\cdot\rangle$*. Then the flag curvature of the flag* (P,η) *is given by*

$$K(P,\eta)=\left[\frac{\langle\zeta,\frac{1}{4}[\eta,[\zeta,\eta]]\rangle S_1+\langle u,\frac{1}{4}[\eta,[\zeta,\eta]]\rangle S_2+\langle\eta,\frac{1}{4}[\eta,[\zeta,\eta]]\rangle S_3}{8+4\langle u,\eta\rangle^2+2\langle u,\zeta\rangle^2+A_1}\right], \tag{2.5.2}$$

where $A_1,\ S_1,\ S_2,\ S_3,\ S_4$ *and* S_5 *are as mentioned in Eq.* (2.2.1).

Proof. Since $\langle\cdot,\cdot\rangle$ is bi-invariant, we have

$$R(\zeta,\eta)\eta=\frac{1}{4}[\eta,[\zeta,\eta]].$$

Substituting the above term in Eq. (2.2.1) and after simplifying this, we get Eq. (2.5.2).

□

2.6 Conclusion

We have the following results:

- Let a compact Lie group G contains a closed subgroup H with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ of G and H respectively. Also an invariant Riemannian metric $\tilde{\alpha}$ on the homogeneous space G/H such that $\langle v, w\rangle = \langle\langle \psi(v), w\rangle\rangle$, where $\psi : \mathfrak{g} \to \mathfrak{g}, \forall\, v, w \in \mathfrak{g}$ is a positive definite endomorphism. Suppose that an invariant vector field $\tilde{u}$ on homogeneous space G/H is parallel with respect to Riemannian metric $\tilde{a}$ and $\tilde{u}_H = u$ and assume that $F = \alpha + \sqrt{\alpha^2 + \beta^2}$ be a special (α, β)-metric arising from $\tilde{\alpha}$ and $\tilde{u}$ such that its Chern connection of F and the Riemannian connection of $\tilde{a}$ are coincides, and a flag $\{P, \eta\}$ in $T_H(G/H)$ such that $\{\zeta, \eta\}$ is an orthonormal basis of P with respect to $\langle\cdot,\cdot\rangle$. Then the flag curvature of the flag $\{P, \eta\}$ is given by

$$K(P,\eta) = \frac{\big\langle\zeta, R(\zeta,\eta)\eta\big\rangle S_1 + \big\langle u,\zeta\big\rangle\big\langle u, R(\zeta,\eta)\eta\big\rangle S_2 + \big\langle\eta, R(\zeta,\eta)\eta\big\rangle S_3}{8 + 4\big\langle u,\eta\big\rangle^2 + 2\big\langle u,\zeta\big\rangle^2 + A_1},$$

where

$$\begin{aligned} S_1 =& 2 + \frac{2+\langle u,\eta\rangle^2}{\sqrt{1+\langle u,\eta\rangle^2}}, \quad S_2 = \frac{(1+\langle u,\eta\rangle^2)^{\frac{3}{2}}+1}{(1+\langle u,\eta\rangle^2)^{\frac{3}{2}}}, \quad S_3 = \frac{\langle u,\eta\rangle^3\langle u,\zeta\rangle}{(1+\langle u,\eta\rangle^2)^{\frac{3}{2}}}, \\ S_4 =& 8 + 8\langle u,\eta\rangle^2 + 2\langle u,\zeta\rangle^2(1+\langle u,\eta\rangle^2) + \langle u,\eta\rangle^4, \quad S_5 = 2\langle u,\zeta\rangle^2 - \langle u,\eta\rangle^2\langle u,\zeta\rangle^2, \\ A_1 =& \frac{S_4}{\sqrt{1+\langle u,\eta\rangle^2}} + \frac{2\langle u,\zeta\rangle^2}{1+\langle u,\eta\rangle^2} + \frac{S_5}{(1+\langle u,\eta\rangle^2)^{\frac{3}{2}}} - \frac{\langle u,\eta\rangle^2\langle u,\zeta\rangle^2}{(1+\langle u,\eta\rangle^2)^3}. \end{aligned}$$

- Let a homogeneous Finsler space $(G/H, F)$ with (α, β)-metric $F = \alpha + \sqrt{\alpha^2 + \beta^2}$ be defined by an invariant Riemannian metric $\tilde{\alpha}$ and an invariant vector field u such

that the Chern connection of F coincides the Levi-Civita connection of $\tilde{\alpha}$. Then $(G/H, F)$ is naturally reductive if and only if the underlying Riemannian space $(G/H, \tilde{\alpha})$ is naturally reductive.

- Let a homogeneous Finsler space (N, F) with $F = \alpha + \sqrt{\alpha^2 + \beta^2}$ defined by the Riemannian metric $\alpha = a_{ij} dx^i \otimes dx^j$ and the vector field u corresponding to 1-form β. Then the homogeneous Finsler space (N, F) with the origin $p = \{H\}$ and with an Ad(H)-invariant decomposition $\mathfrak{g} = \mathfrak{l} + \mathfrak{h}$ is naturally reductive with respect to this decomposition if and only if for any vector $u \in \mathfrak{l} \setminus \{0\}$, the curve $\gamma(t)$ is geodesic of homogeneous Finsler manifold, here $\gamma(t)$ is $\exp tu(p)$.

- A naturally reductive homogeneous Finsler space G/H with (α, β)-metric $F = \alpha + \sqrt{\alpha^2 + \beta^2}$, defined by an invariant Riemannian metric $\tilde{\alpha}$ and an invariant vector field $\tilde{u}$ on G/H, suppose that $\tilde{u}_H = u$. And (P, η) be a flag in $T_H(G/H)$ such that $\{\zeta, \eta\}$ is an orthonormal basis of P with respect to $\langle \cdot, \cdot \rangle$. Then the flag curvature of the flag (P, η) is given by

$$K(P,\eta) = \left[\frac{\begin{array}{r} \left\langle \zeta, \frac{1}{4}[\eta,[\zeta,\eta]_{\mathfrak{l}}]_{\mathfrak{l}} + [\eta,[\zeta,\eta]_{\mathfrak{h}}] \right\rangle S_1 + \left\langle u, \frac{1}{4}[\eta,[\zeta,\eta]_{\mathfrak{l}}]_{\mathfrak{l}} + [\eta,[\zeta,\eta]_{\mathfrak{h}}] \right\rangle S_2 \\ + \left\langle \eta, \frac{1}{4}[\eta,[\zeta,\eta]_{\mathfrak{l}}]_{\mathfrak{l}} + [\eta,[\zeta,\eta]_{\mathfrak{h}}] \right\rangle S_3 \end{array}}{8 + 4\langle u,\eta\rangle^2 + 2\langle u,\zeta\rangle^2 + A_1} \right],$$

where $A_1,\ S_1,\ S_2,\ S_3,\ S_4$ and S_5 are as defined in Eq. (2.2.1).

Chapter 3

Geodesic orbit of a homogeneous Finsler space with (α, β)-metric

In this chapter, we have studied the condition for a non-zero vector to be a geodesic vector for Matsumoto and special metrics. We have discussed the existence of reductive decomposition in homogeneous Finsler space. Later on, in the last sections, we have proved the necessary and sufficient condition for a non-zero vector in a homogeneous Finsler space to be a geodesic vector, and also we have studied the geodesic orbit of a homogeneous Finsler spaces for invariant (α, β)-metrics on a simply connected nilpotent Lie groups of dimension five.

3.1 Introduction

In Finsler spaces, the theory of Lie groups plays an important role. A geodesic vector is a non-zero vector of a geodesic curve. The non-zero vector of a geodesic orbit in homogeneous Finsler space was first described by Dariush Latifi [41]. In homogeneous Riemannian manifolds, the study of homogeneous geodesics is carried out by O. Kowalski, S. Nikcevic and Z. Vlasek [37]. Furthermore, they have proved that for any homogeneous

Riemannian manifolds, invariant metrics over Lie groups, there exists at least one homogeneous geodesic over the identity, through the origin, which is a generalized work of J. Kajzer [35]. Later on, using an invariant Randers metric and a 3-dimensional connected Lie group, Razavi [42, 43] studied homogeneous geodesics in collaboration with Latifi. In 2014 [82], Zaili Yan and S. Deng gave the conditions for Randers space to be a geodesic orbit space. In 2019 [34], Hosseini and Moghaddam studied homogeneous geodesics in homogeneous Kropina spaces.

When a Lie group N has a 2-step nilpotent Lie algebra $\mathfrak{g}$, it is said to be 2-step nilpotent. If $[x, [y, z]] = 0$, for any $x, y, z \in \mathfrak{g}$, the Lie algebra $\mathfrak{g}$ is called a 2-step nilpotent Lie algebra. Two-step homogeneous nilmanifolds with left-invariant metrics have been studied in the last few years [26, 31, 58, 52]. All 2-step nilpotent Riemannian nilmanifolds with five dimensions and their isometry groups have been classified by S. Homolya and O. Kowalski.

The connected homogeneous Finsler space $N = G/H$, where G and H represent the Lie group of isometries of N and the isotropy subgroup of a point in N, respectively. There exists $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{h}$ an $\mathrm{Ad}(h)$-invariant decomposition is a reductive decomposition of homogeneous Finsler space [39]. Lie algebras of H and G can be represented by $\mathfrak{h}$ and $\mathfrak{g}$ respectively, also $\mathfrak{l}$ be the subspace of $\mathfrak{g}$.

Definition 3.1.1. Let $\mathfrak{g}$ be a Lie algebra and N is a simply connected Lie group with Lie algebra $\mathfrak{g}$. A Finsler metric $F : TN \to [0, \infty)$ is called left-invariant if

$$F((L_a)_{*e}z) = F(z), \quad \forall\ a \in N, z \in \mathfrak{g},$$

where L_a is the left translation and e is the unit element of the Lie group.

Theorem 3.1.1. *[41] If (N, F) is a Finsler geodesic orbit space, then it has vanishing S-curvature.*

Now, in the next section we discuss the geodesic orbit of homogeneous Finler space with different metrics.

3.2 Geodesic orbit of homogeneous Finsler space with Matsumoto metric

Our study in this section explores the Finsler geodesic orbit for the Matsumoto metric. On N, a metric $F = \dfrac{\alpha^2}{\alpha - \beta}$ is constructed by combining a G-invariant Riemannian metric α with a smooth G-invariant 1-form β.

Consider a homogeneous Finsler space (N, F). Thus, a Minkowski norm is induced on $\mathfrak{l}$ by the invariant metric F such that

$$F(\eta) = \frac{\langle \eta, \eta \rangle}{\sqrt{\langle \eta, \eta \rangle} - \langle u, \eta \rangle}, \forall \, \eta \in \mathfrak{l}. \tag{3.2.1}$$

The geodesic vectors in homogeneous Finsler space $(G/H, F)$ with $F = \dfrac{\alpha^2}{\alpha - \beta}$ can be described as follows:

Theorem 3.2.1. *Let N be a homogeneous Finsler space with Matsumoto metric $F = \dfrac{\alpha^2}{\alpha - \beta}$. Then a vector $\eta (\neq 0) \in \mathfrak{g}$ is a geodesic vector if and only if*

$$\big\langle [\eta, \xi]_{\mathfrak{l}}, |\eta_{\mathfrak{l}}|\eta_{\mathfrak{l}} - 2\langle u, \eta_{\mathfrak{l}} \rangle \eta_{\mathfrak{l}} + |\eta_{\mathfrak{l}}|^2 u \big\rangle = 0, \tag{3.2.2}$$

holds for every $\xi \in \mathfrak{l}$.

Proof. From Eq. (3.2.1), we can write

$$F^2(\eta+s\zeta+t\omega)=\frac{\langle\eta+s\zeta+t\omega,\eta+s\zeta+t\omega\rangle^2}{\left(\sqrt{\langle\eta+s\zeta+t\omega,\eta+s\zeta+t\omega\rangle}-\langle u,\eta+s\zeta+t\omega\rangle\right)^2}.$$

From Eq. (2.2.2), we get

$$g_\eta(\zeta,\omega)=\frac{1}{2}\frac{\partial^2}{\partial s\partial t}\left[\frac{\langle\eta+s\zeta+t\omega,\eta+s\zeta+t\omega\rangle^2}{\left(\sqrt{\langle\eta+s\zeta+t\omega,\eta+s\zeta+t\omega\rangle}-\langle u,\eta+s\zeta+t\omega\rangle\right)^2}\right]\Bigg|_{(t=s=0)},$$

differentiate $F^2(\eta+s\zeta+t\omega)$ with respect to s and t partially, we obtain

$$\begin{aligned}g_\eta(\zeta,\omega)=&\frac{4\langle\eta,\zeta\rangle\langle\eta,\omega\rangle+2\langle\eta,\eta\rangle\langle\zeta,\omega\rangle}{\mu^2}-\frac{4\langle\eta,\eta\rangle\langle\eta,\omega\rangle}{\mu^3}\left(\frac{\langle\eta,\zeta\rangle}{\sqrt{\langle\eta,\eta\rangle}}-\langle u,\zeta\rangle\right)\\&-\left[\frac{4\langle\eta,\eta\rangle\langle\eta,\zeta\rangle}{\mu^3}-\frac{3\langle\eta,\eta\rangle^2}{\mu^4}\left(\frac{\langle\eta,\zeta\rangle}{\sqrt{\langle\eta,\eta\rangle}}-\langle u,\zeta\rangle\right)\right]\\&\times\left\{\frac{\langle\eta,\omega\rangle}{\sqrt{\langle\eta,\eta\rangle}}-\langle u,\omega\rangle\right\}-\frac{\langle\eta,\eta\rangle^2}{\mu^3}\left[\frac{\langle\zeta,\omega\rangle}{\sqrt{\langle\eta,\eta\rangle}}-\frac{\langle\eta,\zeta\rangle\langle\eta,\omega\rangle}{\langle\eta,\eta\rangle^{\frac{3}{2}}}\right],\end{aligned}\tag{3.2.3}$$

where, $\left(\sqrt{\langle\eta,\eta\rangle}-\langle u,\eta\rangle\right)=\mu$.

Taking $\zeta=\eta$ in the above equation

$$\begin{aligned}g_\eta(\eta,\omega)=&\frac{2\langle\eta,\eta\rangle\langle\eta,\omega\rangle}{\mu^2}-\frac{\langle\eta,\eta\rangle^2}{\mu^3}\left\{\frac{\langle\eta,\omega\rangle}{\sqrt{\langle\eta,\eta\rangle}}-\langle u,\omega\rangle\right\}\\=&\frac{F(\eta)}{\mu^2}\left[\langle\eta,\omega\rangle\sqrt{\langle\eta,\eta\rangle}-2\langle\eta,\omega\rangle\langle u,\eta\rangle+\langle\eta,\eta\rangle\langle u,\omega\rangle\right]\\g_\eta(\eta,\omega)=&\frac{F(\eta)}{\mu^2}\langle|\eta|\eta-2\langle u,\eta\rangle\eta+|\eta|^2u,\omega\rangle.\end{aligned}\tag{3.2.4}$$

From Eq. (3.2.4), we obtain

$$g_{\eta_{\mathfrak{l}}}(\eta_{\mathfrak{l}},[\eta,\xi]_{\mathfrak{l}})=\frac{F(\eta_{\mathfrak{l}})}{\mu^2}\left\langle[\eta,\xi]_{\mathfrak{l}},|\eta_{\mathfrak{l}}|\eta_{\mathfrak{l}}-2\langle u,\eta\rangle\eta_{\mathfrak{l}}+|\eta_{\mathfrak{l}}|^2u\right\rangle.\tag{3.2.5}$$

From the lemma 1.6.1, we get

$$\left\langle[\eta,\xi]_{\mathfrak{l}},|\eta_{\mathfrak{l}}|\eta_{\mathfrak{l}}-2\langle u,\eta_{\mathfrak{l}}\rangle\eta_{\mathfrak{l}}+|\eta_{\mathfrak{l}}|^2u\right\rangle=0.$$

Thus, when Eq. (3.2.2) holds for the metric $F=\dfrac{\alpha^2}{\alpha-\beta}$, a vector $\eta(\neq 0)\in\mathfrak{g}$ is a geodesic vector, and conversely. □

Corollary 3.2.2. *A homogeneous Finsler space (N, F) with $F = \dfrac{\alpha^2}{\alpha - \beta}$ characterized by $\langle \cdot, \cdot \rangle$ and $\tilde{u}$ with $\tilde{u}(H) = u$, u is a vector field. A vector $\eta(\neq 0) \in \mathfrak{g}$ with $\langle u, [\eta, \xi]_{\mathfrak{l}} \rangle = 0$ is a geodesic vector for both (N, F) and $(N, \langle \cdot, \cdot \rangle)$ together.*

Proof. Considering the above theorem 3.2.1, the necessary and sufficient condition of a geodesic vector $\eta(\neq 0) \in \mathfrak{g}$ is

$$\big\langle [\eta, \xi]_{\mathfrak{l}}, |\eta_{\mathfrak{l}}|\eta_{\mathfrak{l}} - 2\langle u, \eta_{\mathfrak{l}} \rangle \eta_{\mathfrak{l}} + |\eta_{\mathfrak{l}}|^2 u \big\rangle = 0.$$

Since $\langle u, [\eta, \xi]_{\mathfrak{l}} \rangle = 0$, we get $\langle \eta_{\mathfrak{l}}, [\eta, \xi]_{\mathfrak{l}} \rangle = 0$.

Thus, a geodesic vector $\eta \in \mathfrak{g}$ of (N, F) is also a geodesic vector of $(N, \langle \cdot, \cdot \rangle)$ and vice versa. □

Theorem 3.2.3. *For a homogeneous Finsler space (N, F), there exists at least one homogeneous geodesic, with metric $F = \dfrac{\alpha^2}{\alpha - \beta}$.*

Proof. Let (N, F) be a homogeneous Finsler space, and a connected isometry group $I(N, F)$ of G acts transitively on N, at fixed point $p \in N$, an isotropy group of G is H. For G and H, $\mathfrak{g}$ and $\mathfrak{h}$ be the respective Lie algebras. A Killing form of $\mathfrak{g}$ can be represented by K and its null space by $\operatorname{rad} K$. According to Kowalski and Szenthe's study [40], in the case of a compact H; on $\mathfrak{h}$, K is nondegenerate. For K, let $\mathfrak{h}$ and $\mathfrak{l}$ be orthogonal to each other, which can be written as

$$\mathfrak{l} = \mathfrak{h}^{\perp} = \{z \in \mathfrak{g} : K(z; \xi) = 0; \forall \xi \in \mathfrak{h}\}.$$

As a result, the reductive decomposition is $\mathfrak{g} = \mathfrak{l} + \mathfrak{h}$ and $\operatorname{rad} K \subset \mathfrak{l}$. Thus, from the theorem 1.1 of [84], we get the result. □

3.3 Geodesic orbit of homogeneous Finsler space with metric $F = \alpha e^{\frac{\beta}{\alpha}} + \frac{\beta^2}{\alpha}$

This section discusses the Finsler geodesic orbit for the special metric $F = \alpha e^{\frac{\beta}{\alpha}} + \frac{\beta^2}{\alpha}$, established from a G-invariant Riemannian metric α and a smooth G-invariant 1-form β on N. Let N be a homogeneous Finsler space. The invariant metric F induces a Minkowski norm on $\mathfrak{l}$ such that

$$F(\eta) = \sqrt{\langle \eta, \eta \rangle} \exp\left(\frac{\langle u, \eta \rangle}{\sqrt{\langle \eta, \eta \rangle}}\right) + \frac{\langle u, \eta \rangle^2}{\sqrt{\langle \eta, \eta \rangle}}, \ \forall \eta \in \mathfrak{l}. \tag{3.3.1}$$

The geodesic vectors in homogeneous Finsler space (N, F) with $F = \alpha e^{\frac{\beta}{\alpha}} + \frac{\beta^2}{\alpha}$ can be described as follows:

Theorem 3.3.1. *Let (N, F) be a homogeneous Finsler space with special metric $F = \alpha e^{\frac{\beta}{\alpha}} + \frac{\beta^2}{\alpha}$. Then a non-zero vector $\eta \in \mathfrak{g}$ is a geodesic vector if and only if*

$$\begin{aligned} \Bigg\langle [\eta, \xi]_{\mathfrak{l}}, \Bigg(\exp\left(\frac{\langle u, \eta_{\mathfrak{l}} \rangle}{|\eta_{\mathfrak{l}}|}\right) - \frac{\langle u, \eta_{\mathfrak{l}} \rangle}{|\eta_{\mathfrak{l}}|} \exp\left(\frac{\langle u, \eta_{\mathfrak{l}} \rangle}{|\eta_{\mathfrak{l}}|}\right) - \frac{\langle u, \eta_{\mathfrak{l}} \rangle^2}{|\eta_{\mathfrak{l}}|^2} \Bigg) \eta_{\mathfrak{l}} \\ + \left(|\eta_{\mathfrak{l}}| \exp\left(\frac{\langle u, \eta_{\mathfrak{l}} \rangle}{|\eta_{\mathfrak{l}}|}\right) + 2\langle u, \eta_{\mathfrak{l}} \rangle \right) u \Bigg\rangle = 0, \end{aligned} \tag{3.3.2}$$

holds for every $\xi \in \mathfrak{l}$.

Proof. From Eq. (3.3.1) the Finsler metric $F = \alpha e^{\frac{\beta}{\alpha}} + \frac{\beta^2}{\alpha}$ can be written as

$$F(\eta) = \sqrt{\langle \eta, \eta \rangle} \exp\left(\frac{\langle u, \eta \rangle}{\sqrt{\langle \eta, \eta \rangle}}\right) + \frac{\langle u, \eta \rangle^2}{\sqrt{\langle \eta, \eta \rangle}},$$

thus

$$\begin{aligned} F^2(\eta + s\zeta + t\omega) = \Bigg(& \sqrt{\langle \eta + s\zeta + t\omega, \eta + s\zeta + t\omega \rangle} \exp\left(\frac{\langle u, \eta + s\zeta + t\omega \rangle}{\sqrt{\langle \eta + s\zeta + t\omega, \eta + s\zeta + t\omega \rangle}}\right) \\ & + \frac{\langle u, \eta + s\zeta + t\omega \rangle^2}{\sqrt{\langle \eta + s\zeta + t\omega, \eta + s\zeta + t\omega \rangle}} \Bigg)^2. \end{aligned}$$

Using Eq. (2.2.2), we get

$$g_\eta(\zeta,\omega) = \frac{1}{2}\frac{\partial^2}{\partial s\partial t}\left(\sqrt{\langle \eta+s\zeta+t\omega, \eta+s\zeta+t\omega\rangle}\exp\left(\frac{\langle u, \eta+s\zeta+t\omega\rangle}{\sqrt{\langle \eta+s\zeta+t\omega, \eta+s\zeta+t\omega\rangle}}\right)\right.$$
$$\left.+\frac{\langle u, \eta+s\zeta+t\omega\rangle^2}{\sqrt{\langle \eta+s\zeta+t\omega, \eta+s\zeta+t\omega\rangle}}\right)^2\Bigg|_{(t=s=0)},$$

differentiate $F^2(\eta+s\zeta+t\omega)$ with respect to s and t partially, we obtain

$$\begin{aligned}
g_\eta(\zeta,\omega) =& \langle\zeta,\omega\rangle\lambda_1 + 2\langle\eta,\zeta\rangle\lambda_1\left\{\frac{\langle u,\omega\rangle}{\sqrt{\langle\eta,\eta\rangle}} - \frac{\langle\eta,\omega\rangle\langle u,\eta\rangle}{\langle\eta,\eta\rangle^{\frac{3}{2}}}\right\} + 2\langle\eta,\omega\rangle\lambda_1\\
&\times\left\{\frac{\langle u,\zeta\rangle}{\sqrt{\langle\eta,\eta\rangle}} - \frac{\langle\eta,\zeta\rangle\langle u,\eta\rangle}{\langle\eta,\eta\rangle^{\frac{3}{2}}}\right\} + \left\{2\langle\eta,\eta\rangle\lambda_1 + \langle u,\eta\rangle^2\lambda_2\right\}\\
&\times\left\{\frac{\langle u,\zeta\rangle}{\sqrt{\langle\eta,\eta\rangle}} - \frac{\langle\eta,\zeta\rangle\langle u,\eta\rangle}{\langle\eta,\eta\rangle^{\frac{3}{2}}}\right\}\left\{\frac{\langle u,\omega\rangle}{\sqrt{\langle\eta,\eta\rangle}} - \frac{\langle\eta,\omega\rangle\langle u,\eta\rangle}{\langle\eta,\eta\rangle^{\frac{3}{2}}}\right\}\\
&+\left\{\langle\eta,\eta\rangle\lambda_1 + \langle u,\eta\rangle^2\lambda_2\right\}\left\{\frac{3\langle\eta,\zeta\rangle\langle\eta,\omega\rangle\langle u,\eta\rangle}{\langle\eta,\eta\rangle^{\frac{5}{2}}}\right.\\
&\left.-\frac{\langle\eta,\zeta\rangle\langle u,\omega\rangle + \langle\eta,\omega\rangle\langle u,\zeta\rangle + \langle\zeta,\omega\rangle\langle u,\eta\rangle}{\langle\eta,\eta\rangle^{\frac{3}{2}}}\right\} + 2\langle u,\zeta\rangle\langle u,\omega\rangle\lambda_2\\
&+2\langle u,\eta\rangle\langle u,\omega\rangle\lambda_2\left\{\frac{\langle u,\zeta\rangle}{\sqrt{\langle\eta,\eta\rangle}} - \frac{\langle\eta,\zeta\rangle\langle u,\eta\rangle}{\langle\eta,\eta\rangle^{\frac{3}{2}}}\right\} + 2\langle u,\eta\rangle\langle u,\zeta\rangle\lambda_2\\
&\times\left\{\frac{\langle u,\omega\rangle}{\sqrt{\langle\eta,\eta\rangle}} - \frac{\langle\eta,\omega\rangle\langle u,\eta\rangle}{\langle\eta,\eta\rangle^{\frac{3}{2}}}\right\} + \frac{6\langle u,\eta\rangle^2\langle u,\zeta\rangle\langle u,\omega\rangle}{\langle\eta,\eta\rangle} - \frac{\langle\zeta,\omega\rangle\langle u,\eta\rangle^4}{\langle\eta,\eta\rangle^2}\\
&-\frac{4\langle\eta,\zeta\rangle\langle u,\eta\rangle^3\langle u,\omega\rangle}{\langle\eta,\eta\rangle^2} - \frac{4\langle\eta,\omega\rangle\langle u,\eta\rangle^3\langle u,\zeta\rangle}{\langle\eta,\eta\rangle^2} + \frac{4\langle\eta,\zeta\rangle\langle\eta,\omega\rangle\langle u,\eta\rangle^4}{\langle\eta,\eta\rangle^3},
\end{aligned} \tag{3.3.3}$$

where $\lambda_1 = \exp\left(\frac{2\langle u,\eta\rangle}{\sqrt{\langle\eta,\eta\rangle}}\right)$ and $\lambda_2 = \exp\left(\frac{\langle u,\eta\rangle}{\sqrt{\langle\eta,\eta\rangle}}\right)$.

Taking $\zeta=\eta$ in the above equation, we have

$$\begin{aligned}
g_\eta(\eta,\omega) = &\left\{\exp\left(\frac{\langle u,\eta\rangle}{|\eta|}\right) + \frac{\langle u,\eta\rangle^2}{|\eta|^2}\right\}\left\langle\left(\exp\left(\frac{\langle u,\eta\rangle}{|\eta|}\right) - \frac{\langle u,\eta\rangle}{|\eta|}\exp\left(\frac{\langle u,\eta\rangle}{|\eta|}\right)\right.\right.\\
&\left.\left.-\frac{\langle u,\eta\rangle^2}{|\eta|^2}\right)\eta + \left(|\eta|\exp\left(\frac{\langle u,\eta\rangle}{|\eta|}\right) + 2\langle u,\eta\rangle\right)u, \omega\right\rangle.
\end{aligned} \tag{3.3.4}$$

From Eq. (3.3.4), we obtain

$$\begin{aligned} g_{\eta_{\mathfrak{l}}}(\eta_{\mathfrak{l}}, [\eta,\xi]_{\mathfrak{l}}) = &\left\{\exp\left(\frac{\langle u,\eta_{\mathfrak{l}}\rangle}{|\eta_{\mathfrak{l}}|}\right) + \frac{\langle u,\eta_{\mathfrak{l}}\rangle^2}{|\eta_{\mathfrak{l}}|^2}\right\}\Big\langle [\eta,\xi]_{\mathfrak{l}}, \left(\exp\left(\frac{\langle u,\eta_{\mathfrak{l}}\rangle}{|\eta_{\mathfrak{l}}|}\right)\right.\\ &\left. - \frac{\langle u,\eta_{\mathfrak{l}}\rangle}{|\eta_{\mathfrak{l}}|}\exp\left(\frac{\langle u,\eta_{\mathfrak{l}}\rangle}{|\eta_{\mathfrak{l}}|}\right) - \frac{\langle u,\eta_{\mathfrak{l}}\rangle^2}{|\eta_{\mathfrak{l}}|^2}\right)\eta \\ &+ \left(|\eta_{\mathfrak{l}}|\exp\left(\frac{\langle u,\eta_{\mathfrak{l}}\rangle}{|\eta_{\mathfrak{l}}|}\right) + 2\langle u,\eta_{\mathfrak{l}}\rangle\right)u\Big\rangle. \end{aligned} \tag{3.3.5}$$

From the lemma 1.6.1, the condition (2.2.2) holds if and only if, the following equation holds:

$$\begin{aligned} \Big\langle [\eta,\xi]_{\mathfrak{l}}, &\left(\exp\left(\frac{\langle u,\eta_{\mathfrak{l}}\rangle}{|\eta_{\mathfrak{l}}|}\right) - \frac{\langle u,\eta_{\mathfrak{l}}\rangle}{|\eta_{\mathfrak{l}}|}\exp\left(\frac{\langle u,\eta_{\mathfrak{l}}\rangle}{|\eta_{\mathfrak{l}}|}\right) - \frac{\langle u,\eta_{\mathfrak{l}}\rangle^2}{|\eta_{\mathfrak{l}}|^2}\right)\eta_{\mathfrak{l}} \\ &+ \left(|\eta_{\mathfrak{l}}|\exp\left(\frac{\langle u,\eta_{\mathfrak{l}}\rangle}{|\eta_{\mathfrak{l}}|}\right) + 2\langle u,\eta_{\mathfrak{l}}\rangle\right)u\Big\rangle = 0. \end{aligned}$$

Thus, when Eq. (3.3.2) holds for the metric $F = \alpha e^{\frac{\beta}{\alpha}} + \dfrac{\beta^2}{\alpha}$, a non-zero vector $\eta \in \mathfrak{g}$ is a geodesic vector, and conversely. □

Corollary 3.3.2. *A homogeneous Finsler space (N,F) with $F = \alpha e^{\frac{\beta}{\alpha}} + \dfrac{\beta^2}{\alpha}$ characterised by $\langle\cdot,\cdot\rangle$ and $\tilde{u}$ with $\tilde{u}(H) = u$, u is vector field. A vector $\eta(\neq 0) \in \mathfrak{g}$ with $\langle u, [\eta,\xi]_{\mathfrak{l}}\rangle = 0$ is a geodesic vector for both (N,F) and $(N,\langle\cdot,\cdot\rangle)$ together.*

Proof. Considering the above theorem 3.3.1, the necessary and sufficient condition of a geodesic vector $\eta(\neq 0) \in \mathfrak{g}$ is

$$\begin{aligned} \Big\langle [\eta,\xi]_{\mathfrak{l}}, &\left(\exp\left(\frac{\langle u,\eta_{\mathfrak{l}}\rangle}{|\eta_{\mathfrak{l}}|}\right) - \frac{\langle u,\eta_{\mathfrak{l}}\rangle}{|\eta_{\mathfrak{l}}|}\exp\left(\frac{\langle u,\eta_{\mathfrak{l}}\rangle}{|\eta_{\mathfrak{l}}|}\right) - \frac{\langle u,\eta_{\mathfrak{l}}\rangle^2}{|\eta_{\mathfrak{l}}|^2}\right)\eta_{\mathfrak{l}} \\ &+ \left(|\eta_{\mathfrak{l}}|\exp\left(\frac{\langle u,\eta_{\mathfrak{l}}\rangle}{|\eta_{\mathfrak{l}}|}\right) + 2\langle u,\eta_{\mathfrak{l}}\rangle\right)u\Big\rangle = 0, \end{aligned}$$

Since $\langle u, [\eta,\xi]_{\mathfrak{l}}\rangle = 0$, we get $\langle \eta_{\mathfrak{l}}, [\eta,\xi]_{\mathfrak{l}}\rangle = 0$.

Thus, a geodesic vector $\eta \in \mathfrak{g}$ of (N,F) is also a geodesic vector of $(N,\langle\cdot,\cdot\rangle)$ and vice versa. □

Theorem 3.3.3. *For a homogeneous Finsler space (N, F), there exits at least one homogeneous geodesic, with metric $F = \alpha e^{\frac{\beta}{\alpha}} + \frac{\beta^2}{\alpha}$.*

Proof. Let (N, F) be a homogeneous Finsler space and a connected isometry group $I(N, F)$ of G acts transitively on N, at fixed point $p \in N$, an isotropy group of G is H. For G and H, $\mathfrak{g}$ and $\mathfrak{h}$ be the respective Lie algebras. A Killing forms of $\mathfrak{g}$ can be represented by K and its null space by $\operatorname{rad} K$. According to Kowalski and Szenthe's study [40], in the case of a compact H; on $\mathfrak{h}$, K is nondegenerate. For K, let $\mathfrak{h}$ and $\mathfrak{l}$ be the orthogonal to each other, which can be written as

$$\mathfrak{l} = \mathfrak{h}^{\perp} = \{z \in \mathfrak{g} : K(z; \xi) = 0; \forall \xi \in \mathfrak{h}\}.$$

As a result, the reductive decomposition is $\mathfrak{g} = \mathfrak{l} + \mathfrak{h}$ and $\operatorname{rad} K \subset \mathfrak{l}$. Thus, from the theorem 1.1 of [84], we get the result. □

3.4 Geodesic vector of invariant (α, β)-metrics on nilpotent Lie groups of dimension five

Using lemma 1.6.1, and results of section 3.2 and 3.3 in a two-step nilpotent Lie group of dimension five with left invariant Finsler (α, β)-metrics, we prove the conditions for a non-zero vector to be a geodesic vector.

3.4.1 Geodesic vector of Matsumoto metrics on nilpotent Lie groups of dimension five

Lie algebras with one-dimensional center

Here, we consider a simply connected nilpotent Lie groups of dimension five with left-invariant Matsumoto metric and has a one-dimensional center. Let e_5 be a unit vector in $\mathfrak{h}$ and let $\mathfrak{l}$ be the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$, here $\mathfrak{g}$ denoted as a five-dimensional 2-step nilpotent Lie algebra with 1-dimensional center $\mathfrak{h}$. In [31], they have shown that there exists an orthonormal basis $\{e_1, e_2, e_3, e_4, e_5\}$ of $\mathfrak{g}$ such that

$$[e_1, e_2] = \lambda e_5, \qquad [e_3, e_4] = \mu e_5, \tag{3.4.1}$$

where $\{e_5\}$ is a basis for the center of $\mathfrak{g}$, $\lambda \geq \mu > 0$ and all other commutators are zero.

Riemannian metric $\tilde{\alpha} = \widetilde{\langle \cdot, \cdot \rangle}$ defines F as a left invariant (α, β)-metric on a simply connected two-step nilpotent Lie group N and the vector field $u = \sum_{i=1}^{5} u_i e_i$.

The following results can be obtained by using the relation (3.2.4),

$$g_{\eta_\mathfrak{l}}(\eta_\mathfrak{l}, [\eta, \xi]_\mathfrak{l}) = \frac{|\eta_\mathfrak{l}|^2 F(\eta_\mathfrak{l})}{\mu^2} \left\langle \mathfrak{B}\eta_\mathfrak{l} + u, [\eta, \xi]_\mathfrak{l} \right\rangle. \tag{3.4.2}$$

where $\mathfrak{B} = \dfrac{|\eta| - 2\langle u, \eta\rangle}{|\eta|^2}\eta$. From the lemma 1.6.1 and Eq. (3.4.2), a vector $\eta = \sum_{i=1}^{5} \eta_i e_i$ of $\mathfrak{g}$ is a geodesic vector if and only if

$$\left\langle \mathfrak{B} \sum_{i=1}^{5} \eta_i e_i + \sum_{i=1}^{5} u_i e_i, \left[\sum_{i=1}^{5} \eta_i e_i, e_j \right] \right\rangle = 0, \tag{3.4.3}$$

for each $j = 1,2,3,4,5$. Therefore we obtain

$$\left.\begin{aligned} \lambda\eta_1(\mathfrak{B}\eta_5 + u_5) = 0, \\ \lambda\eta_2(\mathfrak{B}\eta_5 + u_5) = 0, \\ \lambda\eta_3(\mathfrak{B}\eta_5 + u_5) = 0, \\ \lambda\eta_4(\mathfrak{B}\eta_5 + u_5) = 0, \end{aligned}\right\} \tag{3.4.4}$$

Thus, if $u = \sum_{i=1}^{5} u_i e_i$ of $\mathfrak{g}$ is a geodesic vector if and only if

- $\eta \in span\{e_1, e_2, e_3, e_4\}$, or
- $\eta = \beta e_5$ for $\beta \neq 0$.

Corollary 3.4.1. *Let* $F = \dfrac{\alpha^2}{\alpha - \beta}$ *be the Finsler* (α,β)*-metric defined by an invariant Riemannian metric* $\tilde{a}$ *and the left invariant vector field* $u = \sum_{i=1}^{5} u_i e_i$ *on simply connected two-step nilpotent Lie group of dimension five with one-dimensional center. Then geodesic vectors depend only on* u_5*.*

Theorem 3.4.2. *Let* F *be the Matsumoto metric defined by an invariant Riemannian metric* $\tilde{a}$ *and the left invariant vector field* $u = \sum_{i=1}^{5} u_i e_i$ *on simply connected two-step nilpotent Lie group of dimension five with one-dimensional center. Then* $\eta \in \mathfrak{g}$ *is a geodesic vector of* $(N, \tilde{a})$ *if and only if* η *is a geodesic vector of* (N, F)*.*

Proof. By using theorem 3.2.1, completes the proof. □

Lie algebras with two-dimensional center

This section considers Lie algebra $\mathfrak{g}$ having a two-dimensional center. In [31], they have shown that there exists an orthonormal basis $\{e_1, e_2, e_3, e_4, e_5\}$ of $\mathfrak{g}$ such that

$$[e_1, e_2] = \lambda e_4, \qquad [e_1, e_3] = \mu e_5, \tag{3.4.5}$$

where $\{e_4, e_5\}$ is a basis for the center $\mathfrak{g}$, $\lambda \geq \mu > 0$ and all other commutators are zero.

From the lemma 1.6.1 and Eq. (3.4.2), a vector $\eta = \sum_{i=1}^{5} \eta_i e_i$ of $\mathfrak{g}$ is a geodesic vector if and only if

$$\left\langle \mathfrak{B} \sum_{i=1}^{5} \eta_i e_i + \sum_{i=1}^{5} u_i e_i, \left[\sum_{i=1}^{5} \eta_i e_i, e_j \right] \right\rangle = 0,$$

for each $j = 1, 2, 3, 4, 5$. Therefore we obtain

$$\left.\begin{aligned} &\lambda\eta_2(\mathfrak{B}\eta_4 + u_4) + \mu\eta_3(u_5 + \mathfrak{B}\eta_5) = 0, \\ &\lambda\eta_1(\mathfrak{B}\eta_4 + u_4) = 0, \\ &\mu\eta_1(\mathfrak{B}\eta_5 + u_5) = 0. \end{aligned}\right\} \tag{3.4.6}$$

Corollary 3.4.3. *Let F be the Matsumoto metric defined by an invariant Riemannian metric $\tilde{a}$ and the left invariant vector field $\eta = \sum_{i=1}^{5} \eta_i e_i$ on simply connected two-step nilpotent Lie group of dimension five with two-dimensional center. Then the geodesics depend only on λ, μ, u_4 and u_5.*

Theorem 3.4.4. *Let F be the Matsumoto metric defined by an invariant Riemannian metric $\tilde{a}$ and the left invariant vector field $\eta = \sum_{i=1}^{5} \eta_i e_i$ on simply connected two-step nilpotent Lie group of dimension five with two-dimensional center. Then $\eta \in \mathfrak{g}$ is a geodesic vector of $(N, \tilde{a})$ if and only if η is a geodesic vector of (N, F).*

Proof. Using theorem 3.2.1, completes the proof. □

Lie algebras with three-dimensional center

This section considers simply connected nilpotent Lie groups of dimension five with left-invariant Matsumoto metric and has a three-dimensional center. In [31], they have shown that there exists an orthonormal basis $\{e_1, e_2, e_3, e_4, e_5\}$ of $\mathfrak{g}$ such that

$$[e_1, e_2] = \lambda e_3, \tag{3.4.7}$$

where $\{e_3, e_4, e_5\}$ is a basis for the center of $\mathfrak{g}$, $\lambda > 0$ and all other commutators are zero.

Let F be a left invariant Matsumoto metric on simply connected two-step nilpotent Lie groups of dimension five defined by the Riemannian metric $\tilde{a}$ and the vector field $u = \sum_{i=1}^{5} u_i e_i$. From the lemma 1.6.1 and Eq. (3.4.2), a vector $\eta = \sum_{i=1}^{5} \eta_i e_i$ of $\mathfrak{g}$ is a geodesic vector if and only if

$$\left\langle \mathfrak{B} \sum_{i=1}^{5} \eta_i e_i + \sum_{i=1}^{5} u_i e_i, \left[\sum_{i=1}^{5} \eta_i e_i, e_j \right] \right\rangle = 0,$$

for each $j = 1, 2, 3, 4, 5$. Therefore we obtain

$$\left. \begin{aligned} \lambda \eta_1 (\mathfrak{B} \eta_3 + u_3) = 0, \\ \lambda \eta_3 (\mathfrak{B} \eta_3 + u_3) = 0, \end{aligned} \right\} \tag{3.4.8}$$

Corollary 3.4.5. *Let F be the Matsumoto metric defined by an invariant Riemannian metric $\tilde{a}$ and the left invariant vector field $\eta = \sum_{i=1}^{5} \eta_i e_i$ on simply connected two-step nilpotent Lie group of dimension five with three-dimensional center. Then the geodesics depend only on u_3.*

Theorem 3.4.6. *Let F be the Matsumoto metric defined by an invariant Riemannian metric $\tilde{a}$ and the left invariant vector field $\eta = \sum_{i=1}^{5} \eta_i e_i$ on simply connected two-step nilpotent Lie group of dimension five with three-dimensional center. Then $\eta \in \mathfrak{g}$ is a geodesic vector of $(N, \tilde{a})$ if and only if η is a geodesic vector of (N, F).*

Proof. Using theorem 3.2.1, completes the proof. □

3.4.2 Geodesic vectors of $F = \alpha e^{\frac{\beta}{\alpha}} + \frac{\beta^2}{\alpha}$ metric on nilpotent Lie groups of dimension five

Lie algebras with one-dimensional center

This section considers a simply connected nilpotent Lie groups of dimension five with left-invariant special (α, β)-metric and has a one-dimensional center. Let e_5 be a unit vector in $\mathfrak{h}$ and let $\mathfrak{l}$ be the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$, here $\mathfrak{g}$ is denoted as a five-dimensional 2-step nilpotent Lie algebra with 1-dimensional center $\mathfrak{h}$. In [31], they have shown that there exists an orthonormal basis $\{e_1, e_2, e_3, e_4, e_5\}$ of $\mathfrak{g}$ as defined in Eq. (3.4.1).

Riemannian metric $\tilde{\alpha} = \widetilde{\langle \cdot, \cdot \rangle}$ defines F as a left invariant (α, β)-metric on a simply connected two-step nilpotent Lie group N and the vector field $u = \sum_{i=1}^{5} u_i e_i$.

The following results is obtained from the relation (3.3.4),

$$g_{\eta_{\mathfrak{l}}}(\eta_{\mathfrak{l}}, [\eta, \xi]_{\mathfrak{l}}) = \frac{\left\{ \exp\left(\frac{\langle u, \eta \rangle}{|\eta|}\right) + \frac{\langle u, \eta \rangle^2}{|\eta|^2} \right\}}{|\eta| \exp\left(\frac{\langle u, \eta \rangle}{|\eta|}\right) + 2\langle u, \eta \rangle} \langle \mathfrak{B}\eta_{\mathfrak{l}} + u, [\eta, \xi]_{\mathfrak{l}} \rangle . \tag{3.4.9}$$

where $\mathfrak{B} = \dfrac{\left\{ \exp\left(\frac{\langle u,\eta \rangle}{|\eta|}\right) - \frac{\langle u,\eta \rangle}{|\eta|} \exp\left(\frac{\langle u,\eta \rangle}{|\eta|}\right) - \left(\frac{\langle u,\eta \rangle}{|\eta|}\right)^2 \right\} \eta}{|\eta| \exp\left(\frac{\langle u,\eta \rangle}{|\eta|}\right) + 2\langle u, \eta \rangle}$.

From the lemma 1.6.1 and Eq. (3.4.9), a vector $\eta = \sum_{i=1}^{5} \eta_i e_i$ of $\mathfrak{g}$ is a geodesic vector if and only if

$$\left\langle \mathfrak{B} \sum_{i=1}^{5} \eta_i e_i + \sum_{i=1}^{5} u_i e_i, \left[\sum_{i=1}^{5} \eta_i e_i, e_j \right] \right\rangle = 0, \tag{3.4.10}$$

for each $j = 1, 2, 3, 4, 5$. Therefore we obtain

$$\left.\begin{aligned} \lambda\eta_1(\mathfrak{B}\eta_5 + u_5) &= 0, \\ \lambda\eta_2(\mathfrak{B}\eta_5 + u_5) &= 0, \\ \lambda\eta_3(\mathfrak{B}\eta_5 + u_5) &= 0, \\ \lambda\eta_4(\mathfrak{B}\eta_5 + u_5) &= 0. \end{aligned}\right\} \tag{3.4.11}$$

Thus, if $u = \sum_{i=1}^{5} u_i e_i$ of $\mathfrak{g}$ is a geodesic vector if and only if

- $\eta \in span\{e_1, e_2, e_3, e_4\}$, or
- $\eta = \beta e_5$ for $\beta \neq 0$.

Corollary 3.4.7. *Let $F = \alpha e^{\frac{\beta}{\alpha}} + \frac{\beta^2}{\alpha}$ be the (α, β)-metric defined by an invariant Riemannian metric $\tilde{a}$ and the left invariant vector field $u = \sum_{i=1}^{5} u_i e_i$ on simply connected two-step nilpotent Lie group of dimension five with one-dimensional center. Then geodesic vectors depend only on u_5.*

Theorem 3.4.8. *Let $F = \alpha e^{\frac{\beta}{\alpha}} + \frac{\beta^2}{\alpha}$ be the special (α, β)-metric defined by an invariant Riemannian metric $\tilde{a}$ and the left invariant vector field $u = \sum_{i=1}^{5} u_i e_i$ on simply connected two-step nilpotent Lie group of dimension five with one-dimensional center. Then $\eta \in \mathfrak{g}$ is a geodesic vector of $(N, \tilde{a})$ if and only if η is a geodesic vector of (N, F).*

Proof. Using Theorem 3.3.1, completes the proof. □

Lie algebras with two-dimensional center

This section considers Lie algebra $\mathfrak{g}$ having a two-dimensional center. In [31], they have shown that there exists an orthonormal basis $\{e_1, e_2, e_3, e_4, e_5\}$ of $\mathfrak{g}$ is as in Eq. (3.4.5)

From the lemma 1.6.1 and Eq. (3.4.9), a vector $\eta = \sum_{i=1}^{5}\eta_i e_i$ of $\mathfrak{g}$ is a geodesic vector if and only if

$$\left\langle \mathfrak{B}\sum_{i=1}^{5}\eta_i e_i + \sum_{i=1}^{5}u_i e_i, \left[\sum_{i=1}^{5}\eta_i e_i, e_j\right]\right\rangle = 0,$$

for each $j = 1,2,3,4,5$. Therefore we obtain

$$\left.\begin{aligned} &\lambda\eta_2(\mathfrak{B}\eta_4 + u_4) + \mu\eta_3(u_5 + \mathfrak{B}\eta_5) = 0,\\ &\lambda\eta_1(\mathfrak{B}\eta_4 + u_4) = 0,\\ &\mu\eta_1(\mathfrak{B}\eta_5 + u_5) = 0. \end{aligned}\right\} \tag{3.4.12}$$

Corollary 3.4.9. *Let $F = \alpha e^{\frac{\beta}{\alpha}} + \frac{\beta^2}{\alpha}$ be the (α,β)-metric defined by an invariant Riemannian metric $\tilde{a}$ and the left invariant vector field $\eta = \sum_{i=1}^{5}\eta_i e_i$ on simply connected two-step nilpotent Lie group of dimension five with two-dimensional center. Then the geodesics depend only on λ, μ, u_4 and u_5.*

Theorem 3.4.10. *Let $F = \alpha e^{\frac{\beta}{\alpha}} + \frac{\beta^2}{\alpha}$ be the special (α,β)-metric defined by an invariant Riemannian metric $\tilde{a}$ and the left invariant vector field $\eta = \sum_{i=1}^{5}\eta_i e_i$ on simply connected two-step nilpotent Lie group of dimension five with two-dimensional center. Then $\eta \in \mathfrak{g}$ is a geodesic vector of $(N,\tilde{a})$ if and only if η is a geodesic vector of (N,F).*

Proof. Using theorem 3.3.1, completes the proof. □

Lie algebras with three-dimensional center

This section considers simply connected nilpotent Lie groups of dimension five with left-invariant metric $F = \alpha e^{\frac{\beta}{\alpha}} + \frac{\beta^2}{\alpha}$ and has a three-dimensional center. In [31], they have shown that there exists an orthonormal basis $\{e_1, e_2, e_3, e_4, e_5\}$ of $\mathfrak{g}$ is as in Eq. (3.4.7).

Let F be a left invariant metric $F = \alpha e^{\frac{\beta}{\alpha}} + \frac{\beta^2}{\alpha}$ on simply connected two-step nilpotent Lie groups of dimension five defined by the Riemannian metric $\tilde{a}$ and the vector field $u = \sum_{i=1}^{5} u_i e_i$.

From the lemma 1.6.1 and Eq. (3.4.9), a vector $\eta = \sum_{i=1}^{5} \eta_i e_i$ of $\mathfrak{g}$ is a geodesic vector if and only if

$$\left\langle \mathfrak{B} \sum_{i=1}^{5} \eta_i e_i + \sum_{i=1}^{5} u_i e_i, \left[\sum_{i=1}^{5} \eta_i e_i, e_j \right] \right\rangle = 0,$$

for each $j = 1, 2, 3, 4, 5$. Therefore we obtain

$$\left.\begin{aligned} \lambda\eta_1(\mathfrak{B}\eta_3 + u_3) = 0, \\ \lambda\eta_3(\mathfrak{B}\eta_3 + u_3) = 0. \end{aligned}\right\} \qquad (3.4.13)$$

Corollary 3.4.11. *Let $F = \alpha e^{\frac{\beta}{\alpha}} + \frac{\beta^2}{\alpha}$ be the (α, β)-metric defined by an invariant Riemannian metric $\tilde{a}$ and the left invariant vector field $\eta = \sum_{i=1}^{5} \eta_i e_i$ on simply connected two-step nilpotent Lie group of dimension five with three-dimensional center. Then the geodesics depend only on u_3.*

Theorem 3.4.12. *Let $F = \alpha e^{\frac{\beta}{\alpha}} + \frac{\beta^2}{\alpha}$ be the special (α, β)-metric defined by an invariant Riemannian metric $\tilde{a}$ and the left invariant vector field $\eta = \sum_{i=1}^{5} \eta_i e_i$ on simply connected two-step nilpotent Lie group of dimension five with three-dimensional center. Then $\eta \in \mathfrak{g}$ is a geodesic vector of $(N, \tilde{a})$ if and only if η is a geodesic vector of (N, F).*

Proof. Using theorem 3.3.1, completes the proof. □

3.5 Conclusion

We have concluded this chapter by following results:

- Let N be a homogeneous Finsler space with Matsumoto metric $F = \dfrac{\alpha^2}{\alpha-\beta}$. Then a vector $\eta(\neq 0) \in \mathfrak{g}$ is a geodesic vector if and only if

$$\big\langle [\eta,\xi]_{\mathfrak{l}}, |\eta_{\mathfrak{l}}|\eta_{\mathfrak{l}} - 2\langle u,\eta_{\mathfrak{l}}\rangle\eta_{\mathfrak{l}} + |\eta_{\mathfrak{l}}|^2 u\big\rangle = 0,$$

  holds for every $\xi \in \mathfrak{l}$.

- In a homogeneous Finsler space (N,F), there exists at least one homogeneous geodesic, with metric $F = \dfrac{\alpha^2}{\alpha-\beta}$.

- Let (N,F) be a homogeneous Finsler space with special metric $F = \alpha e^{\frac{\beta}{\alpha}} + \dfrac{\beta^2}{\alpha}$. Then a non-zero vector $\eta \in \mathfrak{g}$ is a geodesic vector if and only if

$$\begin{aligned}\Bigg\langle [\eta,\xi]_{\mathfrak{l}}, \Bigg(\exp\left(\frac{\langle u,\eta_{\mathfrak{l}}\rangle}{|\eta_{\mathfrak{l}}|}\right) - \frac{\langle u,\eta_{\mathfrak{l}}\rangle}{|\eta_{\mathfrak{l}}|}\exp\left(\frac{\langle u,\eta_{\mathfrak{l}}\rangle}{|\eta_{\mathfrak{l}}|}\right) - \frac{\langle u,\eta_{\mathfrak{l}}\rangle^2}{|\eta_{\mathfrak{l}}|^2}\Bigg)\eta_{\mathfrak{l}} \\ + \left(|\eta_{\mathfrak{l}}|\exp\left(\frac{\langle u,\eta_{\mathfrak{l}}\rangle}{|\eta_{\mathfrak{l}}|}\right) + 2\langle u,\eta_{\mathfrak{l}}\rangle\right)u\Bigg\rangle = 0,\end{aligned}$$

  holds for every $\xi \in \mathfrak{l}$.

- For a homogeneous Finsler space (N,F), there exists at least one homogeneous geodesic, with metric $F = \alpha e^{\frac{\beta}{\alpha}} + \dfrac{\beta^2}{\alpha}$.

- Using above results, we discuss the geodesic vectors for a two-step nilpotent Lie group of dimension five with left-invariant (α,β)-metrics.

Chapter 4

The study of S-curvature of a homogeneous Finsler space with Randers-Matsumoto metric

In this chapter, we have considered the Randers-Matsumoto metric $F = \dfrac{\alpha^2}{\alpha - \beta} + \beta$ to study the S-curvature on homogeneous Finsler metric. It discusses the invariant vector fields, S-curvature and an isotropic S-curvature of homogeneous Finsler space with Randers-Matsumoto metric. This presents a new approach to analyse the curvature properties of homogenous Finsler space using the concept of change of metrics.

4.1 Introduction

Matsumoto proposed (α, β)-metric of the form $F = \dfrac{\alpha^2}{\alpha - \beta}$, $\alpha = \sqrt{a_{ij}(x)\eta^i\eta^j}$, and $\beta = b_i(x)\eta^i$, is said to be a slope of a mountain metric as well as Matsumoto metric [48]. With the use of this metric, Finsler geometry has been enhanced, and researchers have a useful working tool [78]. Randers change of metric is a change of Finsler metric i.e., $F(x,\eta) \to F(x,\eta) = F(x,\eta) + b_i(x)\eta^i$. Matsumoto advanced the notion of a Randers change, Hashiguchi and Ichijyo [29] named it and Shibata studied it from a detailed perspective [71]. An (α, β)-metric $F(x, \eta) = \dfrac{\alpha^2}{\alpha - \beta} + \beta$ is said to be Randers-Matsumoto metric. The characteristics of a Finsler space with the Randers-Matsumoto metric were recently discussed by Nagaraja and Pradeep Kumar [53]. Matsumoto [46] presented the theory of Finslerian hypersurface. The geometrical characteristics of hypersurfaces in a few unique Finsler spaces have been studied by Gupta and Pandey [27, 28].

In Finsler geometry, S-curvature is one of the important non-Riemannian curvatures. The structure of S-curvature was first initiated by Shen [73]. S-curvature measures the rate of change of volume form of Finsler space and is subtly related to flag curvature of Finsler metrics. In 2013, Shaoqiang Deng and Zhiguang Hu [23], who have discussed the curvatures of homogeneous Randers space and positive flag curvature. In 2017, Laurian-Ioan Piscoran and Vishnu Narayan Mishra [61] investigated the S-curvature, Landsberg curvature, Cartan torsion and mean Cartan torsion and studied the classes of metrics with bounded Cartan torsion, also obtained the condition for class of a Finsler metrics to be Riemannian or locally Minkowskian. Recently so many authors have worked on this S-curvature [15, 25, 56, 69, 80, 81, 83].

Definition 4.1.1. A smooth manifold G together with an abstract group structure is said to be a Lie group if the mapping $(g_1, g_2) \to g_1 g_2^{(-1)}$ from $G \times G \to G$ is C^∞. Let G be a Lie group and N, a smooth manifold. If there exists a C^∞ mapping $\Psi : G \times N \to N$ which satisfies,

1. $\Psi(e, g_1) = g_1$, where e, the identity of G, $\forall g_1 \in G$,

2. $\Psi(g_1, \Psi(g_2, x)) = \Psi(g_1 g_2, x), \forall x \in N$ and $\forall g_1, g_2 \in G$,

then G is said to act smoothly on N and G is a Lie transformation group of N.

4.2 Invariant vector field on homogeneous Finsler space

Here we prove the theorem for the existence of an invariant vector field corresponding to the 1-form β for the homogeneous Finlser space with (α,β)-metric $F = \dfrac{\alpha^2}{\alpha-\beta} + \beta$. For that, we need the following lemmas:

Lemma 4.2.1. *A Riemannian space (N,α) and $\beta = b_i\eta^i$ be a 1-form, with $\|\beta\| = \sqrt{b_ib^i} < 1$. On N, corresponding to β $\exists$ a smooth vector field u with $\alpha(u|_x) < 1, \forall\ x \in N$ such that the metric F of a Finsler space (N,F) can be described through α along with u as*

$$F(x,\eta) = \frac{\alpha^2(x,\eta)}{\alpha(x,\eta) - \langle u|_x, \eta\rangle} + \langle u|_x, \eta\rangle, x \in N, \eta \in T_xN, \tag{4.2.1}$$

where $\langle\cdot,\cdot\rangle$ is the inner product extracted from the Riemannian metric α.

Proof. As we know, an inner product is used to constrain a Riemannian metric to a tangent space. Therefore, the bilinear form $\langle\xi,\eta\rangle = a_{ij}\xi^i\eta^j$, $\forall\,\xi,\eta \in T_xN$ is an inner product on T_xN for $x \in N$, and this inner product induces an inner product on T_x^*N, the cotangent space of N at x which gives us $\langle dx_i, dx_j\rangle = a^{ij}$. An existence of linear isomorphism between T_x^*N and T_xN, considering this inner product, is defined. It follows that the 1-form β corresponds to a smooth vector field u on N, which can be written as $u|_x = b^i\dfrac{\partial}{\partial x^i}$, where $b^i = a^{ij}b_j$.

Then, for $\eta \in T_xN$, we have $\langle u|_x,\eta\rangle = \left\langle b^i\dfrac{\partial}{\partial x^i}, \eta^j\dfrac{\partial}{\partial x^j}\right\rangle = b^i\eta^ja_{ij} = b_j\eta^j = \beta(\eta)$. Also, we have $\alpha^2(x,\eta) = a_{ij}\eta^i\eta^j$,

$\Longrightarrow\ \alpha^2(u|_x) = a_{ij}b^ib^j = \|\beta\|^2 < 1$, equivalent to $\alpha(u|_x) < 1$. □

Lemma 4.2.2. *Let (N,F) be a Finsler space with the Randers-Matsumoto metric $F = \dfrac{\alpha^2}{\alpha-\beta} + \beta$. Let φ be an isometry of (N,F). Then φ is an isometry of (N,α) if and only if $\langle u|_x,\eta\rangle = \langle u|_{\varphi(x)}, d\varphi_x(\eta)\rangle$.*

Proof. Let $x \in N$ and $\varphi : (N, F) \to (N, F)$ be an isometry. Consequently,

$$F(x, \eta) = F(\varphi(x), d\varphi_x(\eta)), \forall \eta \in T_x N. \tag{4.2.2}$$

By the lemma (4.2.1), we get

$$\frac{\alpha^2(x,\eta)}{\alpha(x,\eta) - \langle u|_x, \eta\rangle} + \langle u|_x, \eta\rangle = \frac{\alpha^2(\varphi(x), d\varphi_x(\eta))}{\alpha(\varphi(x), d\varphi_x(\eta)) - \langle u|_{\varphi(x)}, d\varphi_x(\eta)\rangle} + \langle u|_{\varphi(x)}, d\varphi_x(\eta)\rangle. \tag{4.2.3}$$

Replacing η by $-\eta$ in Eq. (4.2.3), we get

$$\frac{\alpha^2(x,\eta)}{\alpha(x,\eta) + \langle u|_x, \eta\rangle} - \langle u|_x, \eta\rangle = \frac{\alpha^2(\varphi(x), d\varphi_x(\eta))}{\alpha(\varphi(x), d\varphi_x(\eta)) + \langle u|_{\varphi(x)}, d\varphi_x(\eta)\rangle} - \langle u|_{\varphi(x)}, d\varphi_x(\eta)\rangle. \tag{4.2.4}$$

Adding Eqs. (4.2.3) and (4.2.4),

$$\frac{\alpha^3(x,\eta)}{\alpha^2(x,\eta) - \langle u|_x, \eta\rangle^2} = \frac{\alpha^3(\varphi(x), d\varphi_x(\eta))}{\alpha^2(\varphi(x), d\varphi_x(\eta)) - \langle u|_{\varphi(x)}, d\varphi_x(\eta)\rangle^2},$$

$$\frac{\alpha^3(x,\eta) - \alpha^3(\varphi(x), d\varphi_x(\eta))}{\alpha^3(\varphi(x), d\varphi_x(\eta))} = \frac{\alpha^2(x,\eta) - \langle u|_x, \eta\rangle^2 - \alpha^2(\varphi(x), d\varphi_x(\eta)) + \langle u|_{\varphi(x)}, d\varphi_x(\eta)\rangle^2}{\alpha^2(\varphi(x), d\varphi_x(\eta)) - \langle u|_{\varphi(x)}, d\varphi_x(\eta)\rangle^2}.$$

If $\alpha(x,\eta) = \alpha(\varphi(x), d\varphi_x(\eta))$, then we have

$$\langle u_x, \eta\rangle^2 = \langle u|_{\varphi(x)}, d\varphi_x(\eta)\rangle^2,$$

implies

$$\langle u|_x, \eta\rangle = \langle u|_{\varphi(x)}, d\varphi_x(\eta)\rangle, \tag{4.2.5}$$

and thus $d\varphi(u|_x) = u|_{\varphi(x)}$.

Now, subtracting Eq. (4.2.4) from Eq. (4.2.3),

$$\frac{\alpha^2(x,\eta)\langle u_x, \eta\rangle}{\alpha^2(x,\eta) - \langle u_x, \eta\rangle^2} + \langle u_x, \eta\rangle = \frac{\alpha^2(\varphi(x), d\varphi_x(\eta))\langle u|_{\varphi(x)}, d\varphi_x(\eta)\rangle}{\alpha^2(\varphi(x), d\varphi_x(\eta)) - \langle u|_{\varphi(x)}, d\varphi_x(\eta)\rangle^2} + \langle u|_{\varphi(x)}, d\varphi_x(\eta)\rangle.$$

Again, if $\langle u_x, \eta \rangle = \langle u|_{\varphi(x)}, d\varphi_x(\eta) \rangle$, then

$$\frac{\alpha^2(x,\eta)}{\alpha^2(x,\eta) - \langle u|_x, \eta \rangle^2} = \frac{\alpha^2(\varphi(x), d\varphi_x(\eta))}{\alpha^2(\varphi(x), d\varphi_x(\eta)) - \langle u|_x, \eta \rangle^2}$$

implies $\alpha^2(x,\eta)\alpha^2(\varphi(x), d\varphi_x(\eta)) - \alpha^2(x,\eta)\langle u|_x, \eta \rangle^2$

$$= \alpha^2(x,\eta)\alpha^2(\varphi(x), d\varphi_x(\eta)) - \alpha^2(\varphi(x), d\varphi_x(\eta)) - \langle u|_x, \eta \rangle^2, \tag{4.2.6}$$

thus

$$\alpha(x,\eta) = \alpha(\varphi(x), d\varphi_x(\eta)). \tag{4.2.7}$$

Hence, we proved. □

Lemma 4.2.3. *Let (N, F) be a Finsler space with the metric $F = \frac{\alpha^2}{(\alpha - \beta)} + \beta$. Let $I(N, F)$ be a group of isometries of (N, F) and $I(N, \alpha)$ be that of Riemannian space (N, α). Then $I(N, F)$ is a closed subgroup of $I(N, \alpha)$ if and only if $\langle u|_x, \eta \rangle = \langle u|_{\varphi(x)}, d\varphi_x(\eta) \rangle$.*

Proof. Let $I(N, F)$ be a closed subgroup of $I(N, \alpha)$. Thus, if φ is an isometry of (N, F), φ is an isometry of (N, α). Then from the lemma (4.2.2), we have $\langle u|_x, \eta \rangle = \langle u|_{\varphi(x)}, d\varphi_x(\eta) \rangle$.

On the other hand, suppose φ is an isometry of the space (N, F) and satisfies $\langle u|_x, \eta \rangle = \langle u|_{\varphi(x)}, d\varphi_x(\eta) \rangle$, then from lemma (4.2.2) φ is an isometry of (N, α). Thus $I(N, F)$ is a closed subgroup of $I(N, \alpha)$. Hence the proof. □

From the above lemma, we determine that if (N, F) is a homogeneous Finsler space with the metric $F = \frac{\alpha^2}{(\alpha - \beta)} + \beta$, then (N, α) is homogeneous.

Therefore, a homogeneous Finsler space with the Randers-Matsumoto metric can be stated as the coset space of a connected Lie group with metric F. Considering the metric $F = \frac{\alpha^2}{(\alpha - \beta)} + \beta$ as G-invariant Finsler metric on N.

Theorem 4.2.4. *Let* $F = \dfrac{\alpha^2}{(\alpha-\beta)} + \beta$ *be a* G*-invariant Randers-Matsumoto metric on* G/H*,* u *the vector field corresponding to* β*. Then* α *is a* G*-invariant Riemannian metric if and only if the vector field* u *is also* G*-invariant.*

Proof. As F is a G-invariant metric on G/H, we have

$$F(\eta) = F(\mathrm{Ad}(h)\eta), \ \ \forall\, h \in H, \, \eta \in \mathfrak{l}.$$

By the lemma 4.2.1, we get

$$\frac{\alpha^2(\eta)}{\alpha(\eta) - \langle u, \eta\rangle} + \langle u, \eta\rangle = \frac{\alpha^2(\mathrm{Ad}(h)\eta)}{\alpha(\mathrm{Ad}(h)\eta) - \langle u, \mathrm{Ad}(h)\eta\rangle} + \langle u, \mathrm{Ad}(h)\eta\rangle. \tag{4.2.8}$$

Replacing η by $-\eta$ in Eq. (4.2.8), we get

$$\frac{\alpha^2(\eta)}{\alpha(\eta) + \langle u, \eta\rangle} - \langle u, \eta\rangle = \frac{\alpha^2(\mathrm{Ad}(h)\eta)}{\alpha(\mathrm{Ad}(h)\eta) + \langle u, \mathrm{Ad}(h)\eta\rangle} - \langle u, \mathrm{Ad}(h)\eta\rangle. \tag{4.2.9}$$

Adding Eqs. (4.2.8) and (4.2.9) , implies that

$$\frac{\alpha^3(\eta)}{\alpha^2(\eta) - \langle u, \eta\rangle^2} = \frac{\alpha^3(\mathrm{Ad}(h)(\eta))}{\alpha^2(\mathrm{Ad}(h)(\eta)) - \langle u, \mathrm{Ad}(h)(\eta)\rangle^2},$$

$$\frac{\alpha^3(\eta) - \alpha^3(u, \mathrm{Ad}(h)(\eta))}{\alpha^3(\mathrm{Ad}(h)(\eta))} = \frac{\alpha^2(x,\eta) - \langle u, \eta\rangle^2 - \alpha^2(u, \mathrm{Ad}(h)(\eta)) + \langle u, \mathrm{Ad}(h)(\eta)\rangle^2}{\alpha^2(\mathrm{Ad}(h)(\eta)) - \langle u, \mathrm{Ad}(h)(\eta)\rangle^2},$$

If $\alpha(\eta) = \alpha(\mathrm{Ad}(h)\eta)$

$$\langle u, \eta\rangle^2 = \langle u, \mathrm{Ad}(h)(\eta)\rangle^2,$$

implies

$$\langle u, \eta\rangle = \langle u, \mathrm{Ad}(h)(\eta)\rangle. \tag{4.2.10}$$

Now, subtracting Eq. (4.2.9) from Eq. (4.2.8), we get

$$\frac{\alpha^2(\eta)\langle u, \eta\rangle}{\alpha^2(\eta) - \langle u, \eta\rangle^2} + \langle u, \eta\rangle = \frac{\alpha^2(\mathrm{Ad}(h)(\eta))\langle u, \mathrm{Ad}(h)(\eta)\rangle}{\alpha^2(\mathrm{Ad}(h)(\eta)) - \langle u, \mathrm{Ad}(h)(\eta)\rangle^2} + \langle u, \mathrm{Ad}(h)(\eta)\rangle.$$

Again, if $\langle u,\eta\rangle = \langle u, \mathrm{Ad}(h)(\eta)\rangle$, then

$$\frac{\alpha^2(\eta)}{\alpha^2(\eta) - \langle u,\eta\rangle^2} = \frac{\alpha^2(u,\mathrm{Ad}(h)(\eta))}{\alpha^2(u,\mathrm{Ad}(h)(\eta)) - \langle u,\eta\rangle^2},$$

implies $\alpha^2(\eta)\alpha^2(u,\mathrm{Ad}(h)(\eta)) - \alpha^2(\eta)\langle u,\eta\rangle^2$

$$= \alpha^2(\eta)\alpha^2(u,\mathrm{Ad}(h)(\eta)) - \alpha^2(u,\mathrm{Ad}(h)(\eta))\langle u,\eta\rangle^2,$$

thus $\alpha(\eta) = \alpha(u,\mathrm{Ad}(h)(\eta)).$ (4.2.11)

Therefore, α is a G-invariant Riemannian metric if and only if $\mathrm{Ad}(h)u = u$. □

4.3 S-curvature of Randers-Matsumoto metric

Now, we derive the equation for the S-curvature of a homogeneous Finsler space with $F = \dfrac{\alpha^2}{\alpha-\beta} + \beta$.

Theorem 4.3.1. *[24] Let $F = \alpha\phi(s)$ be a G-invariant (α,β)-metric on the reductive homogeneous manifold G/H with $\mathfrak{g} = \mathfrak{h} + \mathfrak{l}$. Then the S-curvature formula for F is of the form*

$$S(H,\eta) = \frac{\Phi}{2\alpha\Delta^2}\Big(\langle [u,\eta]_{\mathfrak{l}}, \eta\rangle + \alpha Q\langle [u,\eta]_{\mathfrak{l}}, u\rangle\Big), \tag{4.3.1}$$

where $u \in \mathfrak{l}$ corresponds to the 1-form β and $\mathfrak{l}$ is identified with the tangent space $T_H(G/H)$ of G/H at the origin H.

Now, with metric $F = \dfrac{\alpha^2}{\alpha-\beta} + \beta$, we derive a formula for S-curvature of a homogeneous Finsler spaces.

Theorem 4.3.2. *Consider G/H as a reductive homogeneous Finsler space with Lie algebra decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{l}$, and $F = \dfrac{\alpha^2}{\alpha-\beta} + \beta$ be a G-invariant Randers-Matsumoto* *metric on G/H. Then S-curvature is written as,*

$$S(H,\eta)=\left[\frac{\begin{array}{r}6s^4-(9n+15)s^3+(6b^2n+27n+21)s^2-(16b^2n+10b^2+14n+14)s\\ +4b^2n+10b^2+2n+2\end{array}}{2(s^2-s-1)(2b^2-3s+1)^2}\right]$$
$$\times\left(\frac{s^2-2s+2}{1-2s}\langle[u,\eta]_{\mathfrak{l}},u\rangle+\frac{1}{\alpha}\langle[u,\eta]_{\mathfrak{l}},\eta\rangle\right), \tag{4.3.2}$$

where $u\in\mathfrak{l}$ corresponds to the 1-form β, $\mathfrak{l}$ is verified with the tangent space $T_H(G/H)$ of G/H at the origin H.

Proof. For a metric $F=\alpha\phi(s)$, where $\phi(s)=\dfrac{1}{1-s}+s$, the components of Eq. (1.2.1) take the values as follows:

$$Q=\frac{-s^2+2s-2}{2s-1},\quad Q'=\frac{-2s^2+2s+2}{(2s-1)^2},\quad Q''=\frac{-10}{(2s-1)^3},$$
$$\Delta=-\frac{(s^2-s-1)(2b^2-3s+1)}{(2s-1)^2},$$
$$\Phi=\frac{(s^2-s-1)\left(\begin{array}{r}6s^4-3(3n+5)s^3+3(2nb^2+9n+7)s^2-((16n+10)b^2\\ +14(n+1))s+(4n+10)b^2+2(n+1)\end{array}\right)}{(2s-1)^4}.$$

When these values are substituted in (4.3.1), we obtain a required S-curvature formula as shown in (4.3.2). □

Corollary 4.3.3. *Let G/H be reductive homogeneous Finsler space with Lie algebra decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{l}$ and $F=\dfrac{\alpha^2}{\alpha-\beta}+\beta$ be a G-invariant Randers-Matsumoto metric on G/H. Then $(G/H,F)$ has an isotropic S-curvature if and only if it has vanishing S-curvature.*

Proof. Consider G/H has an isotropic S-curvature, then $S(x,\eta)=(n+1)c(x)F(\eta)$, $x\in G/H$, $\eta\in T_x(G/H)$.

Taking $x = H$ and $\eta = u$ in Eq. (4.3.2), we get $c(H) = 0$. Consequently, $S(H, \eta) = 0, \ \forall \eta \in T_H(G/H)$. Since F is a homogeneous metric, we have $S = 0$ everywhere, and sufficient condition is obvious. □

4.4 Mean Berwald Curvature

The mean Berwald curvature [76], a quantity associated with S-curvature, is given by

$$E_{ij} = \frac{1}{2}\frac{\partial^2}{\partial\eta^i\partial\eta^j}S(x,\eta) = \frac{1}{2}\frac{\partial^2}{\partial\eta^i\partial\eta^j}\left(\frac{\partial G^m}{\partial\eta^m}\right)(x,\eta),$$

where G^m are the spray coefficients. On $TN \setminus \{0\}$, $E := E_{ij}dx^i \otimes dx^j$ is tensor, which is E tensor. A group of symmetric forms of E tensor considered as $E_\eta : T_xN \times T_xN \to R$ defined as $E_\eta(\xi,\zeta) = E_{ij}(x,\eta)\xi^i\zeta^j$, where $\xi = \xi^i\frac{\partial}{\partial x^i}\Big|_x$, $\zeta = \zeta^i\frac{\partial}{\partial x^i}\Big|_x \in T_xN$. Then the collection $\left\{E_\eta : \eta \in TN \setminus \{0\}\right\}$ is said to be E-curvature or mean Berwald curvature. With $F = \dfrac{\alpha^2}{\alpha-\beta}+\beta$, we determine the mean Berwald curvature of a homogeneous Finsler space in this section. For this, the following is required:

At the origin, $a_{ij} = \delta^i_j$, therefore, $\eta_i = a_{ij}\eta^j = \delta^i_j\eta^j = \eta^i$, $\alpha_{\eta^i} = \dfrac{\eta_i}{\alpha}$, $\beta_{\eta^i} = b_i$,

$$s_{\eta^i} = \frac{\partial}{\partial\eta^i}\left(\frac{\beta}{\alpha}\right) = \frac{b_i\alpha - s\eta_i}{\alpha^2},$$

$$s_{\eta^i\eta^j} = \frac{\partial}{\partial\eta^j}\left(\frac{b_i\alpha - s\eta_i}{\alpha^2}\right) = \frac{\alpha^2\left[b_i\frac{\eta_j}{\alpha} - \left(\frac{b_j\alpha - s\eta_j}{\alpha^2}\right)\eta_i - s\delta^i_j\right] - (b_i\alpha - s\eta_i)2\eta_j}{\alpha^4},$$

$$= \frac{-(b_i\eta_j + b_j\eta_i)\alpha + 3s\eta_i\eta_j - \alpha^2 s\delta^i_j}{\alpha^4}.$$

In $S(H,\eta)$, we are assuming

$$A = \left[\frac{6s^4 - (9n+15)s^3 + (6b^2n + 27n + 21)s^2 - (16b^2n + 10b^2 + 14n + 14)s + 4b^2n + 10b^2 + 2n + 2}{2(s^2 - s - 1)(2b^2 - 3s + 1)^2}\right],$$

then

$$\frac{\partial A}{\partial \eta^j} = \frac{k_1 s^5 + k_2 s^4 + k_3 s^3 + k_4 s^2 + k_5 s + k_6}{2(s^2 - s - 1)^2 (2b^2 - 3s + 1)^3} s_{\eta^j},$$

and

$$\begin{aligned}\frac{\partial^2 A}{\partial \eta^i \partial \eta^j} =& \frac{l_1 s^7 + l_2 s^6 + l_3 s^5 + l_4 s^4 + l_5 s^3 + l_6 s^2 + l_7 s + l_8}{(s^2 - s - 1)^3 (2b^2 - 3s + 1)^4} s_{\eta^i} s_{\eta^j} \\ &+ \frac{k_1 s^5 + k_2 s^4 + k_3 s^3 + k_4 s^2 + k_5 s + k_6}{2(s^2 - s - 1)^2 (2b^2 - 3s + 1)^3} s_{\eta^i \eta^j},\end{aligned}$$

where,

$k_1 = 24b^2 - 27n - 15, \quad k_2 = (18n - 66)b^2 + 153n + 129,$

$k_3 = -((126n + 78)b^2 + 216n + 228),$

$k_4 = 20(n+1)b^4 + (182n + 266)b^2 + 122n + 146,$

$k_5 = -(40(n+1)b^4 + (124n + 172)b^2 + 34n + 22),$

$k_6 = 40(n+1)b^4 + (28n - 8)b^2 + 4n + 4, \quad l_1 = 72b^2 - 81n - 45,$

$l_2 = 24b^4 + (54n - 300)b^2 + 675n + 573,$

$l_3 = -(72b^4 + (702n + 288)b^2 - 1566n - 1584),$

$l_4 = 240(n+1)b^4 + (1860n + 2220)b^2 + 1815n + 1995,$

$l_5 = -(40(n+1)b^6 + (780n + 660)b^4 + (2730n + 3420)b^2 + 1175n + 1145),$

$l_6 = 120(n+1)b^6 + 1440(n+1)b^4 + (1890n + 1674)b^2 + 438n + 330,$

$l_7 = -(240(n+1)b^6 + (990n + 972)b^4 + (432n + 90)b^2 + 75n + 129),$

$l_8 = 120(n+1)b^6 + (60n + 36)b^4 + (6n + 144)b^2 + 3n - 3.$

Theorem 4.4.1. *Let G/H be a reductive homogeneous Finsler space with Lie algebra decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{l}$, and a G-invariant metric $F = \dfrac{\alpha^2}{(\alpha - \beta)} + \beta$ on G/H. Then for* *the homogeneous Finsler space with Randers-Matsumoto metric, mean Berwald curvature is given by*

$$E_{ij}(H,\eta) = \frac{1}{2}\frac{\partial^2 S}{\partial\eta^i\partial\eta^j} = \frac{1}{2}\left(\frac{\partial^2\phi_1}{\partial\eta^i\partial\eta^j} + \frac{\partial^2\Psi_1}{\partial\eta^i\partial\eta^j}\right),$$

where $\frac{\partial^2\phi_1}{\partial\eta^i\partial\eta^j}$ *and* $\frac{\partial^2\Psi_1}{\partial\eta^i\partial\eta^j}$ *are as in Eqs.* (4.4.1) *and* (4.4.2) *respectively.*

Proof. From the equation $S(H,\eta)$, we can write S-curvature at the origin as follows:

$$S(H,\eta) = \phi_1 + \Psi_1,$$

where $$\phi_1 = \frac{A}{\alpha}\langle [u,\eta]_{\mathfrak{l}}, \eta\rangle \quad \text{and} \quad \Psi_1 = \frac{(s^2-2s+2)A}{1-2s}\langle [u,\eta]_{\mathfrak{l}}, u\rangle.$$

Therefore, mean Berwald curvature is

$$E_{ij} = \frac{1}{2}\frac{\partial^2 S}{\partial\eta^i\partial\eta^j} = \frac{1}{2}\left(\frac{\partial^2\phi_1}{\partial\eta^i\partial\eta^j} + \frac{\partial^2\Psi_1}{\partial\eta^i\partial\eta^j}\right),$$

where $\frac{\partial^2\phi_1}{\partial\eta^i\partial\eta^j}$ and $\frac{\partial^2\Psi_1}{\partial\eta^i\partial\eta^j}$ are calculated as follows:

$$\begin{aligned}
\frac{\partial\phi_1}{\partial\eta^j} =& \frac{\partial}{\partial\eta^j}\left(\frac{A}{\alpha}\langle [u,\eta]_{\mathfrak{l}}, \eta\rangle\right)\\
=& \left(\frac{1}{\alpha}\frac{\partial A}{\partial\eta^j} - \frac{A}{\alpha^2}\frac{\eta_j}{\alpha}\right)\langle [u,\eta]_{\mathfrak{l}}, \eta\rangle + \frac{A}{\alpha}\Big(\langle [u,u_j]_{\mathfrak{l}}, \eta\rangle + \langle [u,\eta]_{\mathfrak{l}}, u_j\rangle\Big).\\
\frac{\partial^2\phi_1}{\partial\eta^i\partial\eta^j} =& \frac{\partial}{\partial\eta^i}\left(\left(\frac{1}{\alpha}\frac{\partial A}{\partial\eta^j} - \frac{A}{\alpha^2}\frac{\eta_j}{\alpha}\right)\langle [u,\eta]_{\mathfrak{l}}, \eta\rangle + \frac{A}{\alpha}\left(\langle [u,u_j]_{\mathfrak{l}}, \eta\rangle + \langle [u,\eta]_{\mathfrak{l}}, u_j\rangle\right)\right),\\
=& \left(\frac{1}{\alpha}\frac{\partial^2 A}{\partial\eta^i\partial\eta^j} - \frac{\eta_i}{\alpha^3}\frac{\partial A}{\partial\eta^j} - \frac{\eta_j}{\alpha^3}\frac{\partial A}{\partial\eta^i} - \frac{A}{\alpha^3}\delta_i^j + \frac{3A}{\alpha^5}\eta_i\eta_j\right)\langle [u,\eta]_{\mathfrak{l}}, \eta\rangle\\
&+ \left(\frac{1}{\alpha}\frac{\partial A}{\partial\eta^j} - \frac{A\eta_j}{\alpha^3}\right)\left(\langle [u,u_i]_{\mathfrak{l}}, \eta\rangle + \langle [u,\eta]_{\mathfrak{l}}, u_i\rangle\right)\\
&+ \left(\frac{1}{\alpha}\frac{\partial A}{\partial\eta^i} - \frac{A}{\alpha^3}\eta_i\right)\left(\langle [u,u_j]_{\mathfrak{l}}, \eta\rangle + \langle [u,\eta]_{\mathfrak{l}}, u_j\rangle\right) + \frac{A}{\alpha}\left(\langle [u,u_j]_{\mathfrak{l}}, u_i\rangle\right.\\
&\left.+\langle [u,u_i]_{\mathfrak{l}}, u_j\rangle\right).
\end{aligned} \tag{4.4.1}$$

and

$$\begin{aligned}\frac{\partial\Psi_1}{\partial\eta^j} =& \frac{\partial}{\partial\eta^j}\left(\frac{(s^2-2s+2)A}{1-2s}\langle[u,\eta]_{\mathfrak{l}},u\rangle\right),\\ =& \left(\frac{s^2-2s+2}{1-2s}\frac{\partial A}{\partial\eta^j}-\frac{2s^2-2s-2}{(2s-1)^2}As_{\eta^j}\right)\langle[u,\eta]_{\mathfrak{l}},u\rangle+\frac{(s^2-2s+2)A}{1-2s}\\ &\times\langle[u,u_j]_{\mathfrak{l}},u\rangle.\end{aligned}$$

$$\begin{aligned}\frac{\partial^2\Psi_1}{\partial\eta^i\partial\eta^j} =& \frac{\partial}{\partial\eta^i}\left[\left(\frac{s^2-2s+2}{1-2s}\frac{\partial A}{\partial\eta^j}-\frac{2s^2-2s-2}{(2s-1)^2}As_{\eta^j}\right)\langle[u,\eta]_{\mathfrak{l}},u\rangle\right.\\ &\left.+\frac{(s^2-2s+2)A}{1-2s}\langle[u,u_j]_{\mathfrak{l}},u\rangle\right],\\ =& \left[\frac{s^2-2s+2}{1-2s}\frac{\partial^2 A}{\partial\eta^i\partial\eta^j}-\frac{2s^2-2s-2}{(2s-1)^2}s_{\eta^i}\frac{\partial A}{\partial\eta^j}-\frac{(2s^2-2s-2)}{(2s-1)^2}s_{\eta^j}\frac{\partial A}{\partial\eta^i}\right.\\ &\left.-\frac{10A}{(2s-1)^3}s_{\eta^i}s_{\eta^j}-\frac{2s^2-2s-2}{(2s-1)^2}As_{\eta^i\eta^j}\right]\langle[u,\eta]_{\mathfrak{l}},u\rangle\\ &+\left(\frac{s^2-2s+2}{1-2s}\frac{\partial A}{\partial\eta^j}-\frac{2s^2-2s-2}{(2s-1)^2}As_{\eta^j}\right)\langle[u,u_i]_{\mathfrak{l}},u\rangle\\ &+\left(\frac{s^2-2s+2}{1-2s}\frac{\partial A}{\partial\eta^i}-\frac{2s^2-2s-2}{(2s-1)^2}As_{\eta^i}\right)\langle[u,u_j]_{\mathfrak{l}},u\rangle. \qquad (4.4.2)\end{aligned}$$

Substituting Eqs. (4.4.1) and (4.4.2) in E_{ij}, we get the formula $E_{ij}(H,\eta)$. □

4.5 Conclusion

We have obtained the following results:

- Let G/H be reductive homogeneous Finsler space with a decomposition of the Lie algebra $\mathfrak{g}=\mathfrak{h}+\mathfrak{l}$, and $F=\dfrac{\alpha^2}{(\alpha-\beta)}+\beta$ be a G-invariant Randers-Matsumoto metric on G/H. Then the S-curvature is given by

$$\begin{aligned}S(H,\eta) =& \left[\frac{\begin{array}{r}6s^4-(9n+15)s^3+(6b^2n+27n+21)s^2-(16b^2n+10b^2+14n\\ +14)s+4b^2n+10b^2+2n+2\end{array}}{2(s^2-s-1)(2b^2-3s+1)^2}\right]\\ &\times\left(\frac{s^2-2s+2}{1-2s}\langle[u,\eta]_{\mathfrak{l}},u\rangle+\frac{1}{\alpha}\langle[u,\eta]_{\mathfrak{l}},\eta\rangle\right), \qquad (4.5.1)\end{aligned}$$

where $u \in \mathfrak{l}$ corresponds to the 1-form β, $\mathfrak{l}$ is verified with the tangent space $T_H(G/H)$ of G/H at the origin H.

- Let G/H be reductive homogeneous Finsler space with a decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{h} + \mathfrak{l}$ and $F = \dfrac{\alpha^2}{(\alpha - \beta)} + \beta$ be a G-invariant Randers-Matsumoto metric on G/H. Then $(G/H, F)$ has isotropic S-curvature if and only if it has vanishing S-curvature.

- Let G/H be a reductive homogeneous Finsler space with a decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{h} + \mathfrak{l}$, and $F = \dfrac{\alpha^2}{(\alpha - \beta)} + \beta$ be a G-invariant special metric on G/H. Then the mean Berwald curvature E_{ij} of the homogeneous Finsler space with special (α, β)-metric is also derived.

Chapter 5

Ricci curvature of a homogeneous Finsler space with special (α, β)-metric

In this chapter, we have obtained the formula for the Ricci curvature of homogeneous Finsler space with special metric, $F = \alpha + \sqrt{\alpha^2 + \beta^2}$. We have discussed the conditions for vanishing S-curvature for the space $(G/H, F)$.

5.1 Introduction

For (α, β)-metric Riemannian and Ricci curvatures are given by Zhou [86]. The Ricci curvature of a Finsler metric F on a manifold is a scalar function $Ric : TN \rightarrow R$ with the homogeneity $\mathrm{Ric}(\lambda u) = \lambda^2 \,\mathrm{Ric}(u)$. A Finsler metric F on an n-dimensional manifold N is called an Einstein metric if there is a scalar function $\mu = \mu(x)$ on N such that $\mathrm{Ric} = (n-1)\mu F^2$ [85],[12]. In [17], Cheng et al. have shown that the formulae given in [86] are incorrect. Later, they have also given the corrected formulae for Ricci curvature and Riemannian curvature for (α, β)-metrics. Curvature properties of homogeneous Finsler spaces with (α, β)-metrics are among the most significant topics in Finsler geometry.

5.2 Ricci curvature of Finsler space

Theorem 5.2.1. *[17] On Finsler space N, let F be an (α,β)-metric. Then*

$$\mathrm{Ric}(Z) = \mathrm{Ric}^{\alpha}(Z) + RT^q_q \tag{5.2.1}$$

is its given Ricci curvature and Ric^{α} *be Ricci curvature of* α*, where*

$$\begin{aligned}
RT^q_q = & \frac{1}{\alpha^2}\left\{(n-1)K_1 + K_2\right\} r_{00}^2 \\
& + \frac{1}{\alpha}\Big[\big((n-1)K_3 + K_4\big) r_{00} s_0 + \big((n-1)K_5 + K_6\big) r_{00} r_0 \\
& + \big((n-1)K_7 + K_8\big) r_{00;0}\Big] + \big((n-1)K_9 + K_{10}\big) s_0^2 + (r r_{00} - r_0^2) K_{11} \\
& + \big((n-1)K_{12} + K_{13}\big) r_0 s_0 + \big(r_{00} r^q_q - r_{0q} r^q_0 + r_{00;q} b^q - r_{0q;0} b^q\big) K_{14} \\
& + \big((n-1)K_{15} + K_{16}\big) r_{0q} s^q_0 + \big((n-1)K_{17} + K_{18}\big) s_{0;0} + s_{0q} s^q_0 K_{19} \\
& + \big((n-1)K_{17} + K_{18}\big) s_{0;0} + s_{0q} s^q_0 K_{19} + \alpha\Big[r s_0 K_{20} + \big((n-1)K_{21} + K_{22}\big) s_q s^q_0\Big] \\
& + \alpha\big((3 s_q r^q_0 - 2 s_0 r^q_q + 2 r_q s^q_0 - 2 s_{0;q} b^q + s_{q;0} b^q) K_{23} + s^q_{0;q} K_{24}\big) \\
& + \alpha^2\big(s_q s^q K_{25} + s^i_q s^q_i K_{26}\big),
\end{aligned}$$

where

$$K_1 = 2\Psi\Theta_s(B - s^2) - 2s\Psi\Theta + \Theta^2 - \Theta_s,$$

$$K_2 = 2\Psi\Psi_{ss}(B - s^2)^2 - (6s\Psi\Psi_s + \Psi_{ss})(B - s^2) + 2s\Psi_s,$$

$$K_3 = -4(2Q\Theta_s + Q_s\Theta)\Psi(B - s^2) + 4Q\Theta_s + 2Q_s\Theta + 4Q\Theta(s\Psi - \Theta) - 2\Theta_B,$$

$$\begin{aligned}
K_4 = & -4\Psi(2Q\Psi_{ss} + Q_s\Psi_s + Q_{ss}\Psi_s^2)(B - s^2)^2 + \big(-4\Psi^2(Q - sQ_s) \\
& + 4Q_{ss}\Psi + 2Q_s\Psi_s + 4Q\Psi_{ss} - 2\Psi_{sB} + 20sQ\Psi\Psi_s\big)(B - s^2) + 2\Psi(Q - sQ_s) \\
& - 4\Psi_s - Q_{ss} - 10sQ\Psi_s,
\end{aligned}$$

$$K_5 = 4\Psi\Theta - 2\Theta_B, \qquad K_6 = 2(2\Psi\Psi_s - \Psi_{sB})(B - s^2) - 2\Psi_s,$$

$$K_7 = -\Theta, \qquad K_8 = -\Psi_s(B - s^2),$$

$$K_9 = 8Q\Psi(Q\Theta_s + Q_S\Theta)(B - s^2) + 4Q^2(\Theta^2 - \Theta_s) + 4Q(\Theta_B - Q_s),$$

$$K_{10} = \big(4\Psi^2(2QQ_{ss} - Q_s^2) + 8Q\Psi(Q\Psi_{ss} + Q_s\Psi_s) - 4Q^2\Psi_s^2\big)(B - s^2)^2 + \big(-16sQ\Psi(Q\Psi_s + Q_s\Psi) - 4\Psi(2QQ_{ss} - Q_s^2) - 4Q(Q\Psi_{ss} + Q_s\Psi_s) + 4Q\Psi_{sB} + 4Q_s\Psi_B\big)(B - s^2) - 4s^2Q^2\Psi^2 + 4(2 + 3sQ)(Q\Psi_s + Q_s\Psi) - 8Q^2\Psi + 2QQ_{ss} - Q_s^2 + 4sQ\Psi_B,$$

$$K_{11} = 4\Psi^2 + 4\Psi_B, \qquad K_{12} = 4Q(-2\Psi\Theta + \Theta_B),$$

$$K_{13} = \Big(8\Psi(Q_s\Psi - Q\Psi_s) + 4Q\Psi_{sB} + 4Q_s\Psi_B\Big)(B - s^2) + 8sQ\Psi^2 + 4Q\Psi_s - 4(1 - sQ)\Psi_B,$$

$$K_{14} = 2\Psi, \qquad K_{15} = 4Q\Theta,$$

$$K_{16} = 4(Q\Psi_s - Q_s\Psi)(B - s^2) + 2Q_s - 2(1 + 2sQ)\Psi, \qquad K_{17} = 2Q\Theta,$$

$$K_{18} = 2(Q\Psi_s + Q_s\Psi)(B - s^2) + 2sQ\Psi - Q_s, \qquad K_{19} = 2(1 + sQ)Q_s - 2Q^2,$$

$$K_{20} = -8(\Psi^2 + \Psi_B)Q, \qquad K_{21} = -4Q^2\Theta, \qquad B = b^2,$$

$$K_{22} = 2Q\Psi - 4Q^2\Psi_s(B - s^2), \qquad K_{23} = 2Q\Psi, \qquad K_{24} = 2Q,$$

$$K_{25} = -4Q^2\Psi, \qquad K_{26} = -Q^2, \qquad Q = \frac{\phi'}{\phi - s\phi'},$$

$$\Psi = \frac{\phi''}{2(\phi - s\phi' + (B - s^2)\phi'')}, \qquad \Theta = \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi(\phi - s\phi' + (B - s^2)\phi'')}, \qquad .$$

5.3 Ricci curvature of special (α, β)-metric on a homogeneous Finsler space

Now we calculate the Ricci curvature of a homogeneous Finsler space. Let us consider u to be a G-invariant vector field in $\mathfrak{l}$ corresponding to 1-form β of length c and with respect to the restriction $\langle \cdot, \cdot \rangle$ of Riemannian metric α of an orthonormal basis $\left\{u_1, u_2, \ldots, u_n = \frac{u}{c}\right\}$. Let Γ^k_{ij} be the Christoffel symbols given by $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k}$. The structure constants of $\mathfrak{g}$ is $C^k_{ij} = \langle [u_i, u_j]_{\mathfrak{l}}, u_k \rangle$, $(1 \leq i, j, k \leq n)$ where $[u_i, u_j]_{\mathfrak{l}}$ denotes the projection of $[u_i, u_j]$ to $\mathfrak{l}$, and $f(k, i)$ can be defined by

$$f(k,i) := \begin{cases} 1, & k < i \\ 0, & k \geq i. \end{cases}$$

As a prerequisite to the rest of the calculation, we need the following lemma:

Lemma 5.3.1. *[23] At the origin $o = eH$, we have the following values:*

$$b_i = c\delta_{ni}, \qquad s_{ij} = \frac{c}{2} C^n_{ij}, \qquad s_j = \frac{c^2}{2} C^n_{nj},$$

$$r_{ij} = -\frac{c}{2}(C^j_{ni} + C^i_{nj}), \quad s_{i;j} = c\big(s_{ni;j} + \frac{c}{2} C^n_{qi} \Gamma^q_{nj}\big), \quad r_{ij;k} = s_{ij;k} + b_{j;i;k},$$

$$s_{ij;k} = \frac{c}{2}\Big(\frac{1}{2} C^n_{qj}(C^q_{ki} + C^k_{iq} + C^i_{kq}) + \frac{1}{2} C^n_{iq}(C^q_{kj} + C^k_{jq} + C^j_{kq}) + C^q_{ji} C^n_{kq}\Big),$$

$$\frac{1}{c} b_{i;j;k} = -\Gamma^p_{nj} \langle \nabla_{\hat{u_k}} \hat{u_i}, \hat{u_p} \rangle - \Gamma^i_{np} \langle \nabla_{\hat{u_k}} \hat{u_j}, \hat{u_p} \rangle + C^p_{kn} \langle \nabla_{\hat{u_p}} \hat{u_j}, \hat{u_i} \rangle + \hat{u_k} \langle \nabla_{\hat{u_n}} \hat{u_j}, \hat{u_i} \rangle,$$

$$\Gamma^k_{ij} = f(i,j) C^k_{ij} + \langle \nabla_{\hat{u_i}} \hat{u_j}, \hat{u_k} \rangle,$$

$$\langle \nabla_{\hat{u_i}} \hat{u_j}, \hat{u_k} \rangle \big|_0 = -\frac{1}{2}\Big(C^i_{jk} + C^j_{ik} + C^k_{ij}\Big),$$

$$\langle \nabla_{\hat{u_i}} \hat{u_j}, \hat{u_k} \rangle \hat{u_k} \big|_0 = \frac{1}{2}\Big(C^k_{ha} C^a_{ij} + C^j_{ha} C^a_{ik} + C^i_{ha} C^a_{jk} + C^p_{ij} C^p_{kl} + C^p_{ik} C^p_{hj} + C^p_{jk} C^p_{hi}\Big),$$

$$\frac{\partial \Gamma^k_{ij}}{\partial x^h}\bigg|_0 = \langle \nabla_{\hat{u_i}} \hat{u_j}, \hat{u_k} \rangle \hat{u_h} + f(h,j) C^a_{hj} \langle \nabla_{\hat{u_i}} \hat{u_a}, \hat{u_k} \rangle + f(h,i) C^p_{hi} \langle \nabla_{\hat{u_p}} \hat{u_j}, \hat{u_k} \rangle$$
$$- (\Gamma^k_{hp} + \langle \nabla_{\hat{u_h}} \hat{u_k}, \hat{u_p} \rangle) \Gamma^p_{ij}, \qquad i \geq j.$$

As a result of lemma 5.3.1, we calculate the quantities involved in the statement of theorem 5.2.1 at the origin of homogeneous Finsler spaces.

$$r_{00} = cC^0_{0n}, \qquad s_0 = \frac{c^2}{2}C^n_{n0}, \qquad r_0 = -\frac{c^2}{2}C^n_{n0},$$
$$r_{00;0} = -cC^0_{0p}(C^0_{np} + C^p_{n0}), \quad r = 0, \qquad s_{0;0} = \frac{c^2}{2}C^n_{np}C^0_{0p},$$
$$r^q_q = -cC^q_{nq}, \qquad r_{0q}r^q_0 = \frac{c^2}{4}(C^q_{n0} + C^0_{nq})^2,$$
$$r_{oq}s^q_0 = -\frac{c^2}{4}C^n_{q0}(C^q_{n0} + C^0_{nq}), \qquad s_{oq}s_0 = -\frac{c^2}{4}(C^n_{q0})^2,$$
$$s_q s^q_0 = \frac{c^3}{4}C^n_{nq}C^n_{q0}, \qquad s_q r^q_0 = -\frac{c^3}{4}C^n_{nq}(C^0_{nq} + C^q_{n0}),$$
$$r_q s^q_0 = -\frac{c^3}{4}C^n_{nq}C^n_{q0}, \qquad s_q s^q = \frac{c^4}{4}(C^n_{nq})^2,$$
$$s_{0;q}b^q = \frac{c^3}{4}C^n_{np}(C^p_{0n} + C^n_{0p} + C^0_{np}), \qquad s_{q;0}b^q = \frac{c^3}{4}C^n_{np}(C^p_{n0} + C^n_{0p} + C^0_{np}),$$
$$s^q_{0;q} = \frac{c}{2}C^n_{p0}C^q_{qp} + \frac{c}{4}C^n_{qp}(C^0_{qp} + C^q_{0p} + C^p_{0q}), \qquad s^i_q s_i = -\frac{c^2}{4}(C^n_{qi})^2,$$
$$r_{00;q}b^q = -\frac{c^2}{2}(C^p_{n0} + C^0_{np})(-C^p_{n0} + C^0_{np} + C^n_{op}),$$
$$r_{oq;0}b^q = \frac{c^2}{2}\left[C^n_{np}C^0_{p0} + \frac{1}{2}(C^p_{n0} + C^0_{np})(C^p_{0n} + C^n_{p0} + C^0_{pn})\right].$$

For special metric $F = \alpha + \sqrt{\alpha^2 + \beta^2}$, K_1 to K_{26} can be calculated as follows:

$$\phi = 1 + \sqrt{s^2+1}, \qquad \phi' = \frac{s}{\sqrt{s^2+1}} \qquad \phi'' = \frac{1}{(s^2+1)^{\frac{3}{2}}},$$
$$K_1 = -\frac{\begin{pmatrix} 3s^2((B+s)s^4 + (6B+14)s^2 + 4B + 8)\sqrt{s^2+1} + (-s^4 + 2s^2 + 4B + 8) \\ \times (s^2+1)^2 \end{pmatrix}}{4\phi^2\sqrt{s^2+1}(s^2\sqrt{s^2+1} + B + \phi)^3},$$
$$K_2 = \frac{\begin{pmatrix} 3\sqrt{s^2+1}(s^6 - (7B+6)s^4 + (2B^2 - 3B - 6)s^2 + B^2 + 2B)\sqrt{s^2+1} + 2s^8 \\ + s^6(-4B+1) - (12B+9)s^4 + (3B^2 - 2B - 6)s^2 + B^2 + 2B \end{pmatrix}}{2(s^2\sqrt{s^2+1} + B + \phi)^4},$$
$$K_3 = \frac{\begin{pmatrix} ((3B+19)s^4 + (20B+52)s^2 + 16B + 32)\sqrt{s^2+1} + (s^2+1) \\ \times (3s^2+4)(-s^4 + 3s^2 + 4B + 8) \end{pmatrix} s^3}{\phi^3\sqrt{s^2+1}(s^2\sqrt{s^2+1} + B + \phi)^3},$$

$$
\begin{aligned}
K_4 =& \frac{-1}{(s^2+1)^{\frac{3}{2}}\phi^2(s^2\sqrt{s^2+1}+B+\phi)^5}\Bigg(\Bigg(\Bigg(3s^{14}-(101+72B)s^{12}-(683+12B^2\\
&+458B)s^{10}-2(64B^2+676B+939)s^8+2(15B^3-127B^2-1032B-1351)s^6\\
&+(-2133-171B^2+74B^3-1664B)s^4+(-672B-876-29B^2+54B^3-B^4)s^2\\
&+2(-72+15B^3+12B^2+B^5+\frac{9B^4}{2}-48B)\Bigg)\sqrt{s^2+1}+6s^{16}-(7+24B)s^{14}\\
&-2(151+116B)s^{12}-(1378+52B^2+989B)s^{10}-\frac{1}{2}(4478B-24B^3+418B^2\\
&+6005)s^8-\frac{1}{2}(7327+5678B-90B^3+620B^2)s^6+\frac{1}{2}(-5106-359B^2+12B^4\\
&+164B^3-3994B)s^4+\frac{1}{2}(-1440B-1896-25B^2+138B^3+20B^4)s^2-144\\
&+30B^3+24B^2+B^5+9B^4-96B\Bigg)s\Bigg),
\end{aligned}
$$

$$
K_5 = \frac{2s^3}{\phi(s^2\sqrt{s^2+1}+B+\phi)^2}, \qquad K_6 = \frac{\phi(s^2+1)+2(B-s^2)}{\phi(s^2\sqrt{s^2+1}+B+\phi)^3},
$$

$$
K_7 = \frac{-s^3}{2\phi(s^2\sqrt{s^2+1}+B+\phi)}, \qquad K_8 = \frac{3s\sqrt{s^2+1}(B-s^2)}{2\phi(s^2\sqrt{s^2+1}+B+\phi)^2},
$$

$$
\begin{aligned}
K_9 =& -\frac{1}{\phi^4\sqrt{s^2+1}(s^2\sqrt{s^2+1}+B+\phi)^3}\Bigg(s\Bigg(\Bigg(4s^{10}-2s^9+24s^8+(3B+17)s^7\\
&+4(6B+20)s^6+(18B+52)s^5+4(3B^2+24B+37)s^4+36(B+2)^2s^2\Bigg)\\
&\times\sqrt{s^2+1}-3s^{11}+8s^{10}+2s^9+4(3B+13)s^8+(39+8B)s^7+4(35+15B)s^6\\
&+4(17+6B)s^5+4(6B^2+39B+53)s^4-2(B+2)(B-8)\phi s^3\\
&+4(B+11)(B+2)^2s^2+8(B+2)^3\phi\Bigg)\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
K_{10} =& \frac{1}{\phi^2\sqrt{s^2+1}(s^2\sqrt{s^2+1}+B+\phi)^4}\Bigg(\Bigg(\Big(17s^{10}+(142+56B)s^8+(144B+299\\
&-3B^2)s^6+(215+88B-17B^2)s^4+(24-12B-9B^2+B^3)s^2\Big)+2(s^{10}\\
&+12(B+3)s^8+(40B+104)s^6+(2B^2+41B+97)s^4+(-B^2+2B+18)s^2)\\
&\times\sqrt{s^2+1}-(B+3)(B+2)^2(s^2+\phi)\Bigg)\Bigg),
\end{aligned}
$$

$$K_{11} =\frac{-1}{(s^2\sqrt{s^2+1}+B+\phi)^2}, \qquad K_{12} = \frac{-4s^4}{\phi^2(s^2\sqrt{s^2+1}+B+\phi)^2},$$

$$K_{13} =\frac{2s^2(-3s^4-14s^2+6B-1)+2(B+2)\phi}{\phi(s^2\sqrt{s^2+1}+B+\phi)^3}, \qquad K_{14} = \frac{1}{s^2\sqrt{s^2+1}+B+\phi},$$

$$K_{15} =\frac{2s^4}{\phi^2(s^2\sqrt{s^2+1}+B+\phi)},$$

$$K_{16} =\frac{s^2\sqrt{s^2+1}(s^2+2)+6s^6+(-6B+7)s^4+(-5B+3)s^2+(B+2)\phi}{\phi\sqrt{s^2+1}(s^2\sqrt{s^2+1}+B+\phi)^2},$$

$$K_{17} =\frac{s^4}{\phi^2(s^2\sqrt{s^2+1}+B+\phi)},$$

$$K_{18} =\frac{-(s^4+2s^2)\sqrt{s^2+1}-(B+2)\phi+3s^6+(-3B+2)s^4-(4B+3)s^2}{\phi\sqrt{s^2+1}(s^2\sqrt{s^2+1}+B+\phi)^2},$$

$$K_{19} =\frac{-2(s^2\sqrt{s^2+1}-s^2-\phi)}{\phi^2\sqrt{s^2+1}}, \qquad K_{20} = \frac{2s}{\phi(s^2\sqrt{s^2+1}+B+\phi)^2},$$

$$K_{21} =\frac{-2s^5}{\phi^3(s^2\sqrt{s^2+1}+B+\phi)},$$

$$K_{22} =\frac{s\left(\begin{array}{r} s-6s^4\sqrt{s^2+1}+6Bs^2\sqrt{s^2+1}+s^4+s^2\sqrt{s^2+1}+B\sqrt{s^2+1}+2s^2 \\ +B+2\sqrt{s^2+1}+2\end{array}\right)}{\phi(\sqrt{s^2+1}(s^2+1)+B+1)^2},$$

$$K_{23} =\frac{s}{\phi(\sqrt{s^2+1}(s^2+1)+B+1)}, \qquad K_{24} = \frac{2s}{\phi},$$

$$K_{25} =\frac{-2s^2}{\phi^2(\sqrt{s^2+1}(s^2+1)+B+1)}, \qquad K_{26} = \frac{-s^2}{\phi^2}, \qquad Q = \frac{s}{\phi},$$

$$Q_s =\frac{1}{\phi\sqrt{s^2+1}}, \qquad Q_{ss} = \frac{-(\sqrt{s^2+1}+\phi)s}{(s^2+1)^{\frac{3}{2}}\phi^2},$$

$$\Psi =\frac{1}{2(s^2\sqrt{s^2+1}+B+\phi)}, \qquad \Psi_s = \frac{-3s\sqrt{s^2+1}}{2(s^2\sqrt{s^2+1}+B+\phi)^2},$$

$$\Psi_{ss} =\frac{3((4s^4+3s^2-1)\sqrt{s^2+1}-3(B+1))}{2\sqrt{s^2+1}(s^2\sqrt{s^2+1}+B+\phi)^3}, \qquad \Psi_B = \frac{-1}{2(s^2\sqrt{s^2+1}+B+\phi)^2},$$

$$\Psi_{sB} =\frac{3s\sqrt{s^2+1}}{(s^2\sqrt{s^2+1}+B+\phi)^3}, \qquad \Theta = \frac{s^3}{2\phi\sqrt{s^2+1}(s^2\sqrt{s^2+1}+B+\phi)},$$

$$\Theta_s =-\frac{2(s^2\sqrt{s^2+1}-s^2-\phi)}{\phi^2(\sqrt{s^2+1}(s^2+1)+B+1)^2}, \qquad \Theta_B = \frac{-s^3}{2\phi(s^2\sqrt{s^2+1}+B+\phi)^2},$$

$$\Theta_{sB} =-\frac{((-s^2(4s^2+1)+3(B+2))\sqrt{s^2+1})}{2\phi^2\sqrt{s^2+1}((s^2\sqrt{s^2+1}+B+\phi)^3)}.$$

Using the above computation, we obtain the following result:

Theorem 5.3.2. *A compact homogeneous Finsler space G/H with G-invariant special metric $F = \alpha + \sqrt{\alpha^2 + \beta^2}$. Then the Ricci curvature is given by*

$$
\begin{aligned}
Ric(Z) = {} & Ric^{\alpha}(Z) + \frac{c^2 (C^0_{0n})^2}{\alpha^2(Z)} \Big\{ (n-1)K_1 + K_2 \Big\} + \frac{c^3 C^0_{0n} C^n_{n0}}{2\alpha(Z)} \Big\{ (n-1)(K_3 - K_5) \\
& + K_4 - K_6 \Big\} - \frac{c C^0_{0q}(C^0_{nq} + C^q_{n0})}{\alpha(Z)} \Big\{ (n-1)K_7 + K_8 \Big\} + \frac{c^4 (C^n_{n0})^2}{4} \\
& \times \Big\{ (n-1)(K_9 - K_{12}) + K_{10} - K_{11} - K_{13} \Big\} - \frac{c^2}{4} \Big\{ 4C^0_{0n} C^q_{nq} + (C^q_{n0} + C^0_{nq}) \\
& \times (2C^0_{nq} + C^n_{0q} + 2C^q_{0n}) + 2C^n_{nq} C^0_{q0} \Big\} K_{14} + \frac{c^2}{4} C^n_{0q}(C^q_{n0} + C^0_{nq}) \Big\{ (n-1)K_{15} \\
& + K_{16} \Big\} + \frac{c^2}{2} C^n_{nq} C^0_{0q} \Big\{ (n-1)K_{17} + K_{18} \Big\} - \frac{c^2}{4} (C^n_{q0})^2 K_{19} + \frac{c^3}{4} \alpha(Z) C^n_{q0} C^n_{nq} \\
& \times \Big\{ (n-1)K_{21} + K_{22} \Big\} + \frac{c^3}{4} \alpha(Z) \Big\{ 4C^n_{n0} C^q_{nq} - C^n_{nq}(4C^0_{nq} - C^n_{0q}) \Big\} K_{23} \\
& + \frac{c}{4} \alpha(Z) \Big\{ 2C^n_{m0} C^q_{qm} + C^n_{qm} C^0_{qm} \Big\} K_{24} + \frac{c^2}{4} \alpha^2(Z) \Big\{ c^2 (C^n_{nq})^2 K_{25} - (C^n_{qi})^2 K_{26} \Big\},
\end{aligned}
\tag{5.3.1}
$$

where $Z(\neq 0) \in \mathfrak{l}$.

5.4 Ricci curvature and vanishing S-curvature

In this section we discuss the conditions for vanishing S-curvature for the space $(G/H, F)$. For that we need the following lemma

Lemma 5.4.1. *[83] Suppose $F = \alpha\phi(\beta/\alpha)$ is an invariant metric on the compact connected coset space G/H. Then F has vanishing S-curvature if and only if $\langle [u, x]_{\mathfrak{l}}, x \rangle = 0, \ \forall\, x \in \mathfrak{l}$.*

Theorem 5.4.2. *Let $(N = G/H, F)$ be a compact connected homogeneous Finsler space with G-invariant special metric $F = \alpha + \sqrt{\alpha^2 + \beta^2}$ on G/H. Suppose that (N, F) has* *vanishing S-curvature. Then Ricci curvature is given by*

$$Ric(Z) = Ric^{\alpha}(Z) - \frac{c^2}{4}(C^n_{q0})^2 K_{19} + \frac{c}{4}\alpha(Z)\Big(2C^n_{m0}C^q_{qm} + C^n_{qm}C^0_{qm}\Big)K_{24} - \frac{c^2}{4}\alpha^2(Z)(C^n_{ik})^2 K_{26},$$

where $Z(\neq 0) \in \mathfrak{l}$ *and* K_{19}, K_{24}, K_{26} *are already defined.*

Proof. Assume that $(N = G/H, F)$ has vanishing S-curvature, using lemma 5.4.1

$$\langle [u, Z]_{\mathfrak{l}}, Z\rangle = 0, \forall Z \in \mathfrak{l}.$$

Therefore,

$$C^0_{0n} = \langle Z, [u_n, Z]_{\mathfrak{l}}\rangle = \Big\langle Z, \Big[\frac{u}{c}, Z\Big]_{\mathfrak{l}}\Big\rangle = 0,$$

and

$$\begin{aligned} C^i_{nj} + C^j_{ni} =& \langle u_i, [u_n, u_j]_{\mathfrak{l}}\rangle + \langle u_j, [u_n, u_i]_{\mathfrak{l}}\rangle \\ =& \Big\langle u_i, \Big[\frac{u}{c}, u_j\Big]_{\mathfrak{l}}\Big\rangle + \Big\langle u_j, \Big[\frac{u}{c}, u_i\Big]_{\mathfrak{l}}\Big\rangle \\ =& \frac{1}{c}\Big\{\langle u_i, [u, u_j]_{\mathfrak{l}}\rangle + \langle u_j, [u, u_j]_{\mathfrak{l}}\rangle + \langle u_j, [u, u_i]_{\mathfrak{l}}\rangle + \langle u_i, [u, u_i]_{\mathfrak{l}}\rangle\Big\} \\ =& \frac{1}{c}\langle u_i + u_j, [u, u_j]_{\mathfrak{l}}\rangle + \langle u_i + u_j, [u, u_i]_{\mathfrak{l}}\rangle \\ =& \frac{1}{c}\langle u_i + u_j, [u, u_i + u_j]_{\mathfrak{l}}\rangle \\ =& 0, \qquad \forall 1 \leq i, j \leq n. \end{aligned}$$

also

$$\begin{aligned} C^n_{ni} =& \langle [u_n, u_i]_{\mathfrak{l}}, u_n\rangle \\ =& \Big\langle \Big[\frac{u}{c}, u_i\Big]_{\mathfrak{l}}, \frac{u}{c}\Big\rangle \\ =& \frac{1}{c^2}\Big\{\langle [u, u_i]_{\mathfrak{l}}, u\rangle + \langle [u, u_i]_{\mathfrak{l}}, u_i\rangle\Big\} \\ =& \frac{1}{c^2}\,\langle [u, u]_{\mathfrak{l}} + [u, u_i]_{\mathfrak{l}}, u + u_i\rangle \\ =& \frac{1}{c^2}\,\langle [u, u + u_i]_{\mathfrak{l}}, [u + u_i]\rangle \\ =& 0. \end{aligned}$$

From 5.3.2, hence the required result is obtained. □

5.5 Conclusion

We have obtained the following results:

- A compact homogeneous Finsler space G/H with G-invariant special metric $F = \alpha + \sqrt{\alpha^2 + \beta^2}$. Then the Ricci curvature is given in Eq. (5.3.1).

- Let $(N = G/H, F)$ be a compact connected homogeneous Finsler space with G-invariant special metric $F = \alpha + \sqrt{\alpha^2 + \beta^2}$ on G/H. Suppose that (N, F) has vanishing S-curvature. Then Ricci curvature is given by

$$\begin{aligned}\mathrm{Ric}(Z) =& Ric^{\alpha}(Z) - \frac{c^2}{4}(C^n_{q0})^2 K_{19} + \frac{c}{4}\alpha(Z)\Big(2C^n_{m0}C^q_{qm} + C^n_{qm}C^0_{qm}\Big)K_{24} \\ & - \frac{c^2}{4}\alpha^2(Z)(C^n_{ik})^2 K_{26},\end{aligned}$$

where, $Z(\neq 0) \in \mathfrak{l}$ and $K_{19} = \dfrac{-2(s^2\sqrt{s^2+1} - s^2 - \phi)}{\phi^2\sqrt{s^2+1}}, \quad K_{24} = \dfrac{2s}{\phi}, \quad K_{26} = \dfrac{-s^2}{\phi^2}.$

Chapter 6

Properties of Finsler space with (α, β)-metric

In this chapter, we have discussed the nonholonomic frames, hypersurface of a Finsler space and projective flatness of Finsler space. We have constructed the non-holonomic Finsler frames for the two types of deformed Finsler metrics. We have obtained the some results for the hypersurface of a Finsler space with generalized Matsumoto metric. And we have proved the necessary and sufficient condition for Finsler metric $F = \alpha + \sqrt{\alpha^2 + \beta^2}$.

6.1 Introduction

The theory of gauge transformation has been established in the context of Finsler space by G. S. Asanov and his co-researchers (1985-1989) [8], here interesting thing is that the theory of gauge transformation the Finsler tangent vectors are considered as independent variables are attached to points in space-time. The homogeneous transformations of the tangent space are called gauge transformations. In 1982, P. R. Holland worked on a unified (formalism) field theory that uses a nonholonomic Finsler frame on space-time as a sort of plastic deformation by considering the motion of charged particles in an electromagnetic field [13, 32, 33].

Nonholonomic frames have been studied by so many physicists and geometricians about the motion of charged particles in electromagnetic field theory. In 1951, Y. Katsurada [36] introduced the theory of a nonholonomic system in Finsler geometry. In 1995, R. G. Beil [10, 9] worked on a gauge transformation considered a nonholonomic frame on the tangent bundle of a four-dimensional base manifold. This introduces that there is a unified approach to gravitation and gauge symmetries. I. Bucataru [13], in his research, discussed how the Beil metric is used in the deformation of the Riemannian metric and also in the nonholonomic frame. For finding the nonholonomic frame he considered the most general case of the Beil metric. In this case, the Generalized Lagrange metric (in short, GL-metric) is known as the Beil metric.

The systematic theory of the hypersurface of a Finsler space was built by Matsumoto in 1985, along with this he explained the hyperplane of the first kind, second kind and third kind are the classification of hypersurfaces. Further, many researchers considered these three kinds of hyperplanes in different types of (α, β)-metrics of Finsler spaces and they came to various conclusions. In recent years, H. G. Nagaraja, S. K. Narasimhamurthy, Pradeep Kumar and S. T. Aveesh obtained some results on geometrical properties of Finslerian hypersurfaces with (α, β)-metrics in 2009 [51, 54, 62]. In 2018, K. Vineet and R. K. Gupta worked on some special (α, β)-metrics and in 2020, Brijesh Kumar Tripathi introduced the same aspect with the deformed Berwald-infinite series metric. Projectively flat Finsler spaces with special (α, β)-metric. The condition for a Finsler space to be projectively flat was studies by L. Berwald and this work was completed by M. Matsumoto. Later on many authors worked on projectively flat Finsler spaces [35, 60].

6.2 Nonholonomic frames of Finsler space with (α, β)-metric

Definition 6.2.1. [13] A generalized Lagrange metric is a metric g on the vertical subbundle VTN of the tangent space TN. This means that for every $u \in TN, g_u : V_uTN \times V_uTN \to R$ is bilinear, symmetric, of rank n and of constant signature. A pair $GL^n = (N, g)$, with g a GL-metric is called a Generalized Lagrange space. If $(\pi^{-1}(U), \phi = (x^i, \eta^i))$ is an induced local chart at $u = (x, \eta) \in TN$, we denote by $g_{ij}(u) = g_u \left(\frac{\partial}{\partial \eta^i}\Big|_u, \frac{\partial}{\partial \eta^j}\Big|_u \right)$. Then a GL-metric may be given by a collection of functions $g_{ij}(x, \eta)$ such that we have: $rank(g_{ij}) = n$, $g_{ij}(x, \eta) = g_{ji}(x, \eta)$; the quadratic form $g_{ij}(x, \eta)\xi^i\xi^j$ has a constant signature on TN; if another local chart $(\pi^{-1}(V), \phi = (\tilde{x}^i, \tilde{\eta}^i))$ at $u \in TN$ is given and $\tilde{g}_{kl}(x, \eta) = g_u \left(\frac{\partial}{\partial \tilde{\eta}^k}\Big|_u, \frac{\partial}{\partial \tilde{\eta}^l}\Big|_u \right)$ then g_{ij} and are related by

$$g_{ij} = \frac{\partial \tilde{x}^k}{\partial \eta^i} \frac{\partial \tilde{x}^l}{\partial \eta^j} \tilde{g}_{kl}.$$

Further consider $g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial \eta^i \partial \eta^j}$ the fundamental tensor of the Randers space (N, F). Taking into account the homogeneity of α and F we have the following formulae:

$$\left. \begin{array}{llll} p^i = \frac{1}{\alpha}\eta^i = a^{ij}\frac{\partial \alpha}{\partial \eta^j}, & P_i = a_{ij}p^j = \frac{\partial \alpha}{\partial \eta^i}, & & l^i = \frac{1}{F}\eta^i = g^{ij}\frac{\partial l}{\partial \eta^j}, \\ l_i = g_{ij}l^j = \frac{\partial F}{\partial \eta^i} = p_i + b_i l^i = \frac{1}{F}p^i, & & & l^i l_i = p^i p_i = 1, \\ l^i p_i = \frac{\alpha}{F}, \quad p^i l_i = \frac{F}{\alpha}, & b_i P^i = \frac{\beta}{\alpha}, & & b_i l^i = \frac{\beta}{F}, \end{array} \right\} \tag{6.2.1}$$

with respect to these notations, the metric tensors a_{ij} and g_{ij} are related by [47],

$$g_{ij}(x, \eta) = \frac{F}{\alpha}a_{ij} + b_i p_j + p_i b_j + b_i b_j - \frac{\beta}{\alpha}p_i p_j = \frac{F}{\alpha}(a_{ij} - p_i p_j) + l_i l_j. \tag{6.2.2}$$

Theorem 6.2.1. *[14] For a Finsler space (N, F) consider the metric with the entries:*

$$Y^i_j = \sqrt{\frac{\alpha}{F}}\left(\delta^i_j - l^i l_j + \sqrt{\frac{\alpha}{F}}p^i p_j\right); \tag{6.2.3}$$

defined on TN. Then $Y_j = Y_j^i\left(\frac{\partial}{\partial\eta^i}\right)$, $j \in 1,\ 2,\ 3, \ldots, n$ is a nonholonomic frame.

Theorem 6.2.2. *[2] With respect to frame the holonomic components of the Finsler metric tensor $a_{\alpha\beta}$ are the Randers metric g_{ij}, i.e.,*

$$g_{ij} = Y_i^{\alpha} Y_j^{\beta} a_{\alpha\beta}. \tag{6.2.4}$$

Throughout this section we shall rise and lower indices only with the Riemannian metric $a_{ij}(x)$ that is $\eta_i = a_{ij}\eta^j$, $\beta^i = a^{ij}b_j$, and so on. For a Finsler space with (α,β)-metric $F^2(x,\eta) = F\{\alpha(x,\eta),\ \beta(x,\eta)\}$ we have the Finsler invariants [47].

$$\left.\begin{aligned} \rho &= \frac{1}{2\alpha}\frac{\partial F}{\partial\alpha}, & \rho_0 &= \frac{1}{2}\frac{\partial^2 F}{\partial\beta^2}, \\ \rho_{-1} &= \frac{1}{2\alpha}\frac{\partial^2 F}{\partial\alpha\partial\beta}, & \rho_{-2} &= \frac{1}{2\alpha^2}\left(\frac{\partial^2 F}{\partial\alpha^2} - \frac{1}{\alpha}\frac{\partial F}{\partial\alpha}\right), \end{aligned}\right\} \tag{6.2.5}$$

where subscripts 1, 0, -1, -2 give us the degree of homogeneity of these invariants.

For a Finsler space with (α,β)-metric we have,

$$\rho_{-1}\beta + \rho_{-2}\alpha^2 = 0, \tag{6.2.6}$$

with respect to the notations we have that the metric tensor g_{ij} of a Finsler space with (α,β)-metric is given by [47].

$$g_{ij}(x,\eta) = \rho a_{ij}(x) + \rho_0 b_i(x) + \rho_{-1}\{b_i(x)\eta_j + b_j(x)\eta_i\} + \rho_{-2}\eta_i\eta_j. \tag{6.2.7}$$

From Eq. (6.2.7) we can see that g_{ij} is the result of two Finsler deformations:

1. $a_{ij} \to h_{ij} = \rho a_{ij} + \frac{1}{\rho_{-2}}(\rho_{-1}b_i + \rho_{-2}\eta_i)(\rho_{-1}b_j + \rho_{-2}\eta_j)$
2. $h_{ij} \to g_{ij} = h_{ij} + \frac{1}{\rho_{-2}}\left(\rho_0\rho_{-1} - \rho_{-1}^2\right) b_i b_j.$

The nonholonomic Finsler frame that corresponds to the theorem (7.9.1) in [14], given by,

$$X_j^i = \sqrt{\rho}\delta_j^i - \frac{1}{B^2}\left\{\sqrt{\rho} + \sqrt{\rho + \frac{B^2}{\rho_{-2}}}\right\}\left(\rho_{-1}b^i + \rho_{-2}\eta^i\right)\left(\rho_{-1}b_j + \rho_{-2}\eta_j\right), \qquad (6.2.8)$$

where $B^2 = a_{ij}\left(\rho_{-1}b^i + \rho_{-2}\eta^i\right)\left(\rho_{-1}b_j + \rho_{-2}\ \eta_j\right) = \rho_{-1}^2 b^2 + \beta\rho_{-1}\rho_{-2}$.

This metric tensor a_{ij} and h_{ij} are related by,

$$h_{ij} = X_i^k X_j^l a_{kl}. \qquad (6.2.9)$$

Again the frame that corresponds to the II deformation of the Eq. (6.2.8) is given by,

$$Y_j^i = \delta_j^i - \frac{1}{C^2}\left\{1 \pm \sqrt{1 + \frac{\rho_{-2}C^2}{\rho_0\rho_{-2} - \rho_{-1}^2}}\right\} b^i b_j, \qquad (6.2.10)$$

where $C^2 = h_{ij}b^i b^j = \rho b^2 + \dfrac{1}{\rho_{-2}}\left(\rho_{-1}b^2 + \rho_{-2}\beta\right)^2$.

The metric tensor h_{ij} and g_{ij} are related by the formula;

$$g_{mn} = Y_m^i Y_n^j h_{ij}. \qquad (6.2.11)$$

Theorem 6.2.3. *Let* $F^2(x,\eta) = F(\alpha(x,\eta), \beta(x,\eta))$ *be the metric function of a Finsler space with* (α,β)*-metric for which the condition* (6.2.6) *is true. Then*

$$V_j^i = X_k^i Y_j^k$$

is a nonholonomic Finsler frame with X_k^i *and* Y_j^k *are given by Eqs.* (6.2.9) *and* (6.2.11) *respectively.*

Here we have two kinds of metrics combinations, such as the product of Randers and Matsumoto metric, another one is the product of Randers and the first approximate of Matsumoto metric.

6.2.1 Nonholonomic frame for $F(\alpha,\beta) = (\alpha+\beta)\left(\dfrac{\alpha^2}{\alpha-\beta}\right)$

In the first case, for a Finsler space with the fundamental function $F = (\alpha+\beta)\left(\dfrac{\alpha^2}{\alpha-\beta}\right)$ the Finsler invariants Eq. (6.2.5) are given by

$$\left.\begin{array}{lll} \rho = \dfrac{\alpha^2-\alpha\beta-\beta^2}{(\alpha-\beta)^2}, & \rho_0 = \dfrac{2\alpha^3}{(\alpha-\beta)^2}, & \rho_{-1} = \dfrac{\alpha(\alpha-3\beta)}{(\alpha-\beta)^3}, \\ \rho_{-2} = \dfrac{\beta(3\beta-\alpha)}{\alpha(\alpha-\beta)^3}, & B^2 = \dfrac{(\alpha-3\beta)^2(\alpha^2b^2-\beta^2)}{(\alpha-\beta)^6}. & \end{array}\right\} \tag{6.2.12}$$

$$\begin{aligned} X^i_j = & \sqrt{\frac{\alpha^2-\alpha\beta-\beta^2}{(\alpha-\beta)^2}}\delta^i_j - \frac{\alpha^2}{\alpha^2b^2-\beta^2}\left(\sqrt{\frac{\alpha^2-\alpha\beta-\beta^2}{(\alpha-\beta)^2}}\right. \\ & \left.\pm\sqrt{\frac{(\alpha^2-\alpha\beta-\beta^2)(\alpha\beta-\beta^2)+\alpha(3\beta-\alpha)(\alpha^2b^2-\beta^2)}{\beta(\alpha-\beta)^3}}\right)\left(b^i-\frac{\beta}{\alpha^2}\eta^i\right)\left(b_j-\frac{\beta}{\alpha^2}\eta_j\right). \end{aligned} \tag{6.2.13}$$

Again using Eq. (6.2.12) in Eq. (6.2.11) we have,

$$Y^i_j = \delta^i_j - \frac{1}{C^2}\left(1\pm\sqrt{1+\frac{\beta(\alpha-\beta)^2C^2}{\alpha^3}}\right)b^ib_j, \tag{6.2.14}$$

where $C^2 = \left(\dfrac{\alpha^2-\alpha\beta-\beta^2}{(\alpha-\beta)^2}\right)b^2 - \dfrac{(\alpha-3\beta)(\alpha^2b^2-\beta^2)^2}{\alpha\beta(\alpha-\beta)^3}$. Thus we state:

Theorem 6.2.4. *Let* $F = (\alpha+\beta)\left(\dfrac{\alpha^2}{\alpha-\beta}\right) = \dfrac{\alpha^2(\alpha+\beta)}{\alpha-\beta}$ *be the metric function of a Finsler space with* (α,β)*-metric for which the condition* (6.2.6) *is true. Then*

$$V^i_j = X^i_kY^k_j$$

is a nonholonomic Finsler frame with X^i_k *and* Y^k_j *are given by Eqs.* (6.2.13) *and* (6.2.14) *respectively.*

6.2.2 Nonholonomic frame for $F = (\alpha+\beta)\left(\alpha+\beta+\dfrac{\beta^2}{\alpha}\right)$

In the second case, for a Finsler space with the fundamental function $F = (\alpha+\beta)$ $\times\left(\alpha+\beta+\dfrac{\beta^2}{\alpha}\right)$ be the product of Randers metric and first approximate Matsumoto metric the Finsler invariants Eq. (6.2.5) are given by

$$\left.\begin{aligned} \rho = 1+\frac{\beta}{\alpha}-\frac{\beta^3}{2\alpha^3}, \qquad \rho_0 = 2+\frac{3\beta}{\alpha}, \qquad \rho_{-1} = \frac{1}{\alpha}-\frac{3\beta^2}{2\alpha^3}, \\ \rho_{-2} = \frac{3\beta^3-2\alpha^2\beta}{2\alpha^5}, \qquad B^2 = \frac{\left(2\alpha^2-3\beta^2\right)^2\left(\alpha^2b^2-\beta^2\right)}{4\alpha^8}. \end{aligned}\right\} \tag{6.2.15}$$

Using Eq. (6.2.15) in Eq. (6.2.9) we have,

$$\begin{aligned} X_j^i =&\sqrt{1+\frac{\beta}{\alpha}-\frac{\beta^3}{2\alpha^3}}\delta_j^i - \frac{\alpha^2}{\alpha^2b^2-\beta^2}\left(\sqrt{1+\frac{\beta}{\alpha}-\frac{\beta^3}{2\alpha^3}}\right. \\ &\left.\pm\sqrt{\frac{(3\beta^2-2\alpha^2)(\alpha^2b^2-2\beta^2)+2\beta(\alpha^3+\beta^3)}{2\alpha^3\beta}}\right)\times\left(b^i-\frac{\beta}{\alpha^2}\eta^i\right)\left(b_j-\frac{\beta}{\alpha^2}\eta_j\right), \end{aligned} \tag{6.2.16}$$

again using Eq. (6.2.12) in Eq. (6.2.11) we have,

$$Y_j^i = \delta_j^i - \frac{1}{C^2}\left(1\pm\sqrt{1+\frac{2\alpha\beta C^2}{2\alpha^2+4\alpha\beta+3\beta^2}}\right)b^ib_j,$$

$$\text{where} \qquad C^2 = \left(\frac{2\alpha^3+2\alpha^2\beta-\beta^3}{2\alpha^3}\right)b^2+\frac{3\beta^2-2\alpha^2}{2\alpha^5}\left(\alpha^2b^2-\beta^2\right)^2. \tag{6.2.17}$$

Thus we state:

Theorem 6.2.5. *Let* $F=(\alpha+\beta)\left(\alpha+\beta+\frac{\beta^2}{\alpha}\right)$ *be the metric function of a Finsler space with* (α,β)*-metric for which the condition* (6.2.6) *is true. Then*

$$V_j^i = X_k^iY_j^k$$

is a nonholonomic Finsler frame with X_k^i *and* Y_j^k *are given by Eqs.* (6.2.16) *and* (6.2.17) *respectively.*

6.3 Hypersurface of Finsler space with generalized Matsumoto metric

In this section, we have expressed certain geometrical properties of the hypersurface of Finsler spaces and we have discussed the different kinds of hyperplanes with generalized

Matsumoto metric, $F = \dfrac{\alpha^{m+1}}{(\alpha - \beta)^m}$. We have derived the necessary and sufficient condition for the hypersurface of generalized Matsumoto metric satisfies the conditions of hyperplane of first, second and but not third kind for the above metric. We have considered an n-dimensional Finsler space with smooth manifold N^n assigned with F i.e., $F^n = (N^n, F(\alpha, \beta))$, where α-Riemannian metric and β-differential 1–form. Here

$$F = \frac{\alpha^{m+1}}{(\alpha - \beta)^m}. \tag{6.3.1}$$

Now, differentiate Eq. (6.3.1) partially with respect to α and β, we get,

$$\left.\begin{aligned}
F_\alpha &= \frac{\alpha^m(\alpha - m\beta - \beta)}{(\alpha - \beta)^{m+1}}, \quad F_\beta = \frac{m\alpha^{m+1}}{(\alpha - \beta)^{m+1}},\\
F_{\alpha\alpha} &= -2m(m+1)\left(\frac{\alpha^m}{(\alpha - \beta)^{m+1}} - \frac{\alpha^{m-1}}{2(\alpha - \beta)^m} - \frac{\alpha^{m+1}}{2(\alpha - \beta)^{m+2}}\right),\\
F_{\beta\beta} &= m(m+1)\frac{\alpha^{m+1}}{(\alpha - \beta)^{m+2}},\\
F_{\alpha\beta} &= m(m+1)\left(\frac{\alpha^m}{(\alpha - \beta)^{m+1}} - \frac{\alpha^{m+1}}{(\alpha - \beta)^{m+2}}\right),
\end{aligned}\right\} \tag{6.3.2}$$

where $F_\alpha = \dfrac{\partial F}{\partial \alpha}, \quad F_\beta = \dfrac{\partial F}{\partial \beta}, \quad F_{\alpha\alpha} = \dfrac{\partial F_\alpha}{\partial \alpha}, \quad F_{\beta\beta} = \dfrac{\partial F_\beta}{\partial \beta}, \quad F_{\alpha\beta} = \dfrac{\partial F_\alpha}{\partial \beta}$.

The normalized element with supporting element $l_i = \dot{\partial}_i F$ and angular metric tensor $h_{ij} = F^{-1}\dot{\partial}_i\dot{\partial}_j F$ are given by,

$$\left.\begin{aligned}
l_i &= \alpha^{-1}F_\alpha Y_i + F_\beta b_i,\\
h_{ij} &= Pa_{ij} + Q_0 b_i b_j + Q_1(b_i Y_j + b_j Y_i) + Q_2 Y_i Y_j,
\end{aligned}\right\} \tag{6.3.3}$$

where $Y_i = a_{ij}\eta^j$, $\dot{\partial}_i = \dfrac{\partial}{\partial \eta^i}$ and

$$P = FF_\alpha\alpha^{-1}, \quad Q_0 = FF_{\beta\beta}, \quad Q_1 = FF_{\alpha\beta}\alpha^{-1}, \quad Q_2 = F\alpha^{-2}(F_{\alpha\alpha} - F_\alpha\alpha^{-1}). \tag{6.3.4}$$

The constant quantities of fundamental function Eq. (6.3.1) are given below by using Eq.

(6.3.4), we get

$$\left.\begin{aligned} P &= \frac{\alpha^{2m}(m+1)}{(\alpha-\beta)^{2m}} - \frac{m\alpha^{2m+1}}{(\alpha-\beta)^{2m+1}}, \\ Q_0 &= m(m+1)\frac{\alpha^{2m+2}}{(\alpha-\beta)^{2m+2}}, \\ Q_1 &= m(m+1)\left(\frac{\alpha^{2m}}{(\alpha-\beta)^{2m+1}} - \frac{\alpha^{2m+1}}{(\alpha-\beta)^{2m+2}}\right), \\ Q_2 &= \frac{m(m+1)\alpha^{2m}}{(\alpha-\beta)^{2m+2}} + \frac{(m^2-1)\alpha^{2m-2}}{(\alpha-\beta)^{2m}} - \frac{2m(m+\frac{1}{2})\alpha^{2m-1}}{(\alpha-\beta)^{2m+1}}. \end{aligned}\right\} \tag{6.3.5}$$

Now the fundamental metric tensor $g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2$ is defined as,

$$g_{ij} = Pa_{ij} + P_0 b_i b_j + P_1(b_i Y_j + b_j Y_i) + Q_2 Y_i Y_j, \tag{6.3.6}$$

where, $$P_0 = Q_0 + F_\beta^2, \qquad P_1 = Q_1 + F^{-1}PF_\beta, \qquad P_2 = Q_2 + P^2F^{-2}. \tag{6.3.7}$$

In addition, the reciprocal of tensor g_{ij} is g^{ij} given by

$$g^{ij} = P^{-1}a^{ij} - S_0 b^i b^j - S_1(b^i\eta^j + b^j\eta^i) - S_2\eta^i\eta^j, \tag{6.3.8}$$

where,

$$\left.\begin{aligned} & b^i = a^{ij}b_j, && S_0 = \frac{PP_0 + (P_0P_2 - P_1^2)\alpha^2}{\mu P}, \\ & S_1 = \frac{PP_1 + (P_0P_2 - P_1^2)\beta}{\mu P}, && S_2 = \frac{PP_2 + (P_0P_2 - P_1^2)b^2}{\mu P}, \\ & \mu = P(P + P_0b^2 + P_1\beta) + (P_0P_2 - P_1^2)(\alpha^2b^2 - \beta^2). \end{aligned}\right\} \tag{6.3.9}$$

The Cartan tensor is given by,

$$2PC_{ijk} = P_1(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + \gamma_1 m_i m_j m_k, \tag{6.3.10}$$

where,

$$\left.\begin{aligned} & C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}, \qquad \gamma_1 = P\left(\frac{\partial P_0}{\partial\beta}\right) - 3P_1Q_0, \\ & m_i = b_i - \alpha^{-2}\beta Y_i. \end{aligned}\right\} \tag{6.3.11}$$

The associated Riemannian space contains the components of Christoffel's symbol $\left\{ {i \atop jk} \right\}$ and the covariant derivative ∇_k with respect to x^k corresponding to this Christoffel's symbols.

We have the following components of the symmetric and skew-symmetric tensors respectively,

$$E_{ij} = \frac{b_{ij} + b_{ji}}{2}, \qquad F_{ij} = \frac{b_{ij} - b_{ji}}{2}, \tag{6.3.12}$$

where $b_{ij} = \nabla_j b_i$.

The difference tensor $D^i_{jk} = \Gamma^{*i}_{jk} - \Gamma^i_{jk}$ of the special Finsler space F^n is given by

$$\left.\begin{aligned} D^i_{jk} = \quad & B^i E_{jk} + F^i_k B_j + F^i_j B_k + B^i_j b_{0k} + B^i_k b_{0j} - b_{0m} g^{im} B_{jk} - C^i_{jm} A^m_k - C^i_{km} A^m_j \\ & + C_{jkm} A^m_s g^{is} + \lambda^s (C^i_{jm} C^m_{sk} + C^i_{km} C^m_{sj} - C^m_{jk} C^i_{ms}), \end{aligned}\right\} \tag{6.3.13}$$

where,

$$\left.\begin{aligned} & B_k = P_0 b_k + P_1 Y_k, \;\; B^i = g^{ij} B_j, \qquad F^k_i = g^{kj} F_{ji}, \;\; B_0 = b_i \eta^i, \\ & B_{ij} = \frac{\left\{ P_1 (a_{ij} - \alpha^{-2} Y_i Y_j) + \frac{\partial P_0}{\partial \beta} m_i m_j \right\}}{2}, \\ & A^m_k = B^m_k E_{00} + B^m E_{k0} + B_k F^m_0 + B_0 F^m_k, \\ & B^k_i = g^{kj} B_{ji}, \;\; \lambda^m = B^m E_{00} + 2 B_0 F^m_0. \end{aligned}\right\} \tag{6.3.14}$$

Here and also for the following, we denote 0 as contraction with η^i except for the quantities P_0, Q_0 and S_0.

6.3.1 Hypersurface F^{n-1} of Finsler space

Let us consider a Finsler space F^n with the (α,β)- metric $F = \dfrac{\alpha^{m+1}}{(\alpha-\beta)^m}$, where, $\alpha = \sqrt{a_{ij}(x)\eta^i\eta^j}$ is a Riemannian metric, $\beta = b_i(x)\eta^i$ is 1-form metric and $b_i(x) = \dfrac{\partial b}{\partial x^i}$ is a gradient of scalar function $b(x)$. In this section, we prove the necessary and sufficient condition of the hypersurface of a Finsler space to be a hyperplane which satisfies the 1^{st}, 2^{nd} and but not the 3^{rd} kind. We have,

$$b_i B^i_\alpha = 0, \qquad b_i \eta^i = \beta = 0. \tag{6.3.15}$$

Accordingly, the induced metric $F(u, v)$ of hypersurface $F^{n-1}$is Riemannian metric given by

$$F(u,v) = \sqrt{a_{\alpha\beta}(u) v^\alpha v^\beta}, \qquad a_{\alpha\beta} = a_{ij} B^i_\alpha B^j_\beta. \tag{6.3.16}$$

Thus for $F^{n-1}(c)$, from the Eqs. (6.3.5), (6.3.7) and (6.3.9), we have

$$\left.\begin{aligned} &P = 1, \quad Q_0 = m(m+1), \quad Q_1 = 0, \quad Q_2 = -\frac{1}{\alpha^2}, \quad P_0 = m(2m+1),\\ &P_1 = \frac{m}{\alpha}, \quad P_2 = 0, \quad \mu = 1 + (m^2+m)b^2, \quad S_0 = \frac{m^2+m}{1+(m^2+m)b^2},\\ &S_1 = \frac{m}{\alpha(1+(m^2+m)b^2)}, \quad S_2 = -\frac{m^2b^2}{\alpha^2(1+(m^2+m)b^2)}. \end{aligned}\right\} \tag{6.3.17}$$

wherefore (6.3.8) yields,

$$\begin{aligned} g^{ij} =& a^{ij} - \frac{(m^2+m)}{1+(m^2+m)b^2} b^i b^j - \frac{m}{\alpha(1+(m^2+m)b^2)}(b^i\eta^j + b^j\eta^i)\\ &+ \frac{m^2b^2}{\alpha^2(1+(m^2+m)b^2)}\eta^i\eta^j. \end{aligned} \tag{6.3.18}$$

Operating $b_i b_j$ and using Eqs. (6.3.15) and (6.3.18) we get

$$g^{ij} b_i b_j = \frac{b^2}{1+(m^2+m)b^2}. \tag{6.3.19}$$

Therefore, we obtain

$$b_i(x(u)) = \sqrt{\frac{b^2}{1+(m^2+m)b^2}} N_i, \qquad b^2 = a^{ij} b_i b_j, \tag{6.3.20}$$

where b is the length of vector b^i. From Eqs. (6.3.18) and (6.3.20), we get

$$b^i = a^{ij} b_j = \sqrt{b^2\left((m^2+m)b^2+1\right)} N^i + \frac{mb^2}{\alpha}\eta^i. \tag{6.3.21}$$

Thus we state:

Theorem 6.3.1. *Let F^{n-1} be a hypersurface of Finsler space F^n with metric $F = \dfrac{\alpha^{m+1}}{(\alpha-\beta)^m}$ with scalar function $b_i(x) = \partial_i b(x)$ given by (6.3.19) and (6.3.20) and a hypersurface F^{n-1} of F^n. Then the induced Riemannian metric is lead by (6.3.15).*

Further, h_{ij} and g_{ij} are identified by (6.3.3) and (6.3.6) into (6.3.17),

$$h_{ij} = a_{ij} + m(m+1)b_i b_j - \frac{1}{\alpha^2} Y_i Y_j, \tag{6.3.22}$$

$$g_{ij} = a_{ij} + m(2m+1)b_i b_j + \frac{m}{\alpha}(b_i \eta_j + b_j \eta_i), \tag{6.3.23}$$

where, h_{ij} is an angular metric tensor and g_{ij} is a metric tensor.

Along hypersurface $F^{n-1}(c)$ with $h^{(a)}_{\alpha\beta}$ which represents the angular metric tensor of Riemannian $a_{ij}(x)$, then $h_{\alpha\beta} = h^{(a)}_{\alpha\beta}$. Accordingly from (6.3.7), as well as $F^{n-1}(c)$, $\dfrac{\partial P_0}{\partial \beta} = 2m(m+1)(2m+1)\dfrac{\alpha^{2m+2}}{(\alpha-\beta)^{2m+3}}$.

Therefore, (6.3.11) yields $\gamma_1 = \dfrac{m(m+1)(m+2)}{\alpha}, m_i = b_i$.

Now from (6.3.10), we get

$$C_{ijk} = \frac{m}{2\alpha}(h_{ij}b_k + h_{jk}b_i + h_{ki}b_j) + \left(\frac{m(m+1)(m+2)}{2\alpha}\right) b_i b_j b_k. \tag{6.3.24}$$

Using (6.3.14) and (6.3.4) and (6.3.24), we obtain

$$M_{\alpha\beta} = \frac{m}{2\alpha}\sqrt{\frac{b^2}{1+(m^2+m)b^2}}\, h_{\alpha\beta}, \quad M_\alpha = 0. \tag{6.3.25}$$

Thus from Eqs. (6.3.25) and (1.4.12) thereby $H_{\alpha\beta} = H_{\beta\alpha}$.

Thus, we state

Theorem 6.3.2. *The second fundamental v-tensor of the hypersurface of a Finsler space with the metric $F = \dfrac{\alpha^{m+1}}{(\alpha-\beta)^m}$ vanishes, and the second fundamental h-tensor is symmetric.*

Proof. Subsequently, from Eq. (6.3.15) we get

$$b_{i|\beta}B^i_\alpha + b_i B^i_{\alpha|\beta} = 0. \tag{6.3.26}$$

Therefore from Eqs. (1.4.15) and (1.4.16), we get

$$b_{i|j}B^i_\alpha B^j_\beta + b_{i|j}B^i_\alpha N^j H_\beta + b_i H_{\alpha\beta}N^i = 0. \tag{6.3.27}$$

Along $b_{i}\big|_{j} = -b_s C^s_{ij}$, becomes $b_{i}\big|_{j} B^i_\alpha N^j = -\sqrt{\frac{b^2}{1+(m^2+m)b^2}}M_\alpha = 0$. Including $b_{i|j}$ is symmetric, from Eq. (6.3.27) we obtain

$$\sqrt{\frac{b^2}{1+(m^2+m)b^2}}H_{\alpha\beta} + b_{i|j}B^i_\alpha B^j_\beta = 0. \tag{6.3.28}$$

Now, contracting the Eq. (6.3.28) with v^β and again that with v^α, yields

$$\sqrt{\frac{b^2}{1+(m^2+m)b^2}}H_\alpha + b_{i|j}B^i_\alpha \eta^j = 0, \tag{6.3.29}$$

$$\sqrt{\frac{b^2}{1+(m^2+m)b^2}}H_0 + b_{i|j}\eta^i\eta^j = 0. \tag{6.3.30}$$

In the context of lemma (1.4.1), the hypersurface $F^{n-1}(c)$ is a hyperplane of the first kind if and only if $H_0 = 0$ and also on the other hand $b_{i|j}\eta^i\eta^j = 0$. The covariant derivative $b_{i|j}$ depends on η^i. □

Thus Eq. (6.3.12) becomes $E_{ij} = b_{ij},\ F_{ij} = 0$ and $F^i_j = 0$.

Hence Eq. (6.3.13) reduces to

$$\left.\begin{aligned} D^i_{jk} &= B^i b_{jk} + B^i_j b_{0k} + B^i_k b_{0j} - b_{0m}g^{im}B_{jk} - C^i_{jm}A^m_k - C^i_{km}A^m_j + C_{jkm}A^m_s g^{is} \\ &\quad + \lambda^s(C^i_{jm}C^m_{sk} + C^i_{km}C^m_{sj} - C^m_{jk}C^i_{ms}). \end{aligned}\right\} \tag{6.3.31}$$

In the context of Eqs. (6.3.17) and (6.3.18), the relations in Eq. (6.3.14) consists of

$$\left.\begin{aligned}
B_i =&\quad m(2m+1)b_i + \frac{m}{\alpha}Y_i, \qquad B^i = \frac{m^2+m}{1+(m^2+m)b^2}b^i + \frac{m}{\alpha(1+(m^2+m)b^2)}\eta^i,\\
B_{ij} =&\quad \frac{m}{2\alpha}a_{ij} - \frac{m}{2\alpha^3}Y_iY_j + \frac{m(m+1)(2m+1)}{\alpha}b_ib_j,\\
B^i_j =&\quad g^{ik}B_{kj} = \frac{m}{2\alpha}(\delta^i_j - \alpha^{-2}\eta^iY_j) + \frac{m(m+1)(3m+2)}{2\alpha(1+(m^2+m)b^2)}b^ib_j\\
&-\frac{m^2+2m^2(m+1)(2m+1)b^2}{2\alpha^2(1+(m^2+m)b^2)}\eta^ib_j,\\
A^m_k =&\quad B^m_kb_{00} + B^mb_{k0}, \qquad \lambda^m = B^mb_{00}.
\end{aligned}\right\}\tag{6.3.32}$$

and

$$\left.\begin{aligned}
D^i_{j0} &= B^ib_{j0} + B^i_jb_{00} - B^mC^i_{jm}b_{00},\\
D^i_{00} &= B^ib_{00} = \left(\frac{m^2+m}{1+(m^2+m)b^2}b^i + \frac{m}{\alpha(1+(m^2+m)b^2)}\eta^i\right)b_{00}.
\end{aligned}\right\}\tag{6.3.33}$$

By the relation (6.3.15), we obtain

$$b_iD^i_{j0} = \frac{(m^2+m)b^2}{1+(m^2+m)b^2}b_{j0} + \frac{m+3m(m+1)^2b^2}{2\alpha(1+(m^2+m)b^2)}b_jb_{00} - \frac{(m^2+m)b^m}{1+(m^2+m)b^2}b_iC^i_{jm}b_{00}.\tag{6.3.34}$$

$$b_iD^i_{00} = \frac{(m^2+m)b^2}{1+(m^2+m)b^2}b_{00}.\tag{6.3.35}$$

Hence $b_{i|j} = b_{ij} - b_sD^s_{ij}$ with Eqs. (6.3.34) and (6.3.35) yields

$$b_{i|j}\eta^i\eta^j = \frac{1}{1+(m^2+m)b^2}b_{00}.\tag{6.3.36}$$

Subsequently Eqs. (6.3.29) and (6.3.30) are expressed as

$$\frac{b}{\sqrt{1+(m^2+m)b^2}}H_\alpha + b_{i0}B^i_\alpha = 0.\tag{6.3.37}$$

$$\frac{b}{\sqrt{1+(m^2+m)b^2}}H_0 + \frac{1}{1+(m^2+m)b^2}b_{00} = 0.\tag{6.3.38}$$

As consequence of that the condition $H_0 = 0$ for an induced metric which is equivalent to $b_{00} = 0$, using Eq. (6.3.15) which can be written as $b_{ij}\eta^i\eta^j = (b_i\eta^i)(c_j\eta^j)$ for some $c_j(x)$, therefore

$$b_{ij} = \frac{b_i c_j + b_j c_i}{2}. \tag{6.3.39}$$

Thus, we state

Theorem 6.3.3. *The necessary and sufficient condition for a hypersurface $F^{n-1}(c)$ of a Finsler space with the generalized Matsumoto metric $F = \dfrac{\alpha^{m+1}}{(\alpha-\beta)^m}$ to be a hyperplane of first kind $b_{ij} = \dfrac{b_i c_j + b_j c_i}{2}$ holds.*

And also $b_{00} = 0,\ b_{ij}B^i_\alpha B^j_\beta = 0,\ b_{ij}B^i_\alpha \eta^j = 0$. Here Eq. (6.3.38) gives $H_\alpha = 0$, and from Eqs. (6.3.32) and (6.3.39) we obtain

$$b_{i0}b^i = \frac{c_0 b^2}{2},\ \lambda^m = 0,\ A^i_j B^j_\beta = 0 \text{ and } B_{ij}B^i_\alpha B^j_\beta = 0.$$

With the help of Eqs. (1.4.14), (6.3.18), (6.3.21), (6.3.25) and (6.3.31), we get

$$b_s D^s_{ij} B^i_\alpha B^j_\beta = -\frac{c_0 b^2 m}{4\alpha(1+(m^2+m)b^2)} h_{\alpha\beta},$$

henceforth, Eq. (6.3.28) reduces to

$$\frac{b}{\sqrt{1+(m^2+m)b^2}} H_{\alpha\beta} + \frac{c_0 b^2 m}{4\alpha(1+(m^2+m)b^2)} h_{\alpha\beta} = 0. \tag{6.3.40}$$

From eqs. (1.4.13),(6.3.24), (6.3.32) and (6.3.30) and the theorem 6.3.2. Hence the result.

Theorem 6.3.4. *The necessary and sufficient condition for the hypersurface $F^{n-1}(c)$ of Finslerian space equipped with metric $F = \dfrac{\alpha^{m+1}}{(\alpha-\beta)^m}$, to be hyperplane of the 1^{st} kind is (6.3.39) and in this case the 2^{nd} fundamental h-tensor of hypersurface $F^{n-1}(c)$, is proportional to its angular metric tensor.*

By lemma 1.4.2, the hypersurface $F^{n-1}(c)$ is a hyperplane of the 2^{nd} kind if and only if $H_{\alpha\beta} = 0$. Thus from (6.3.40) wet $c_0 = c_i(x)y^i = 0$. Therefore $\exists e(x) \ni c_i(x) = e(x)b_i(x)$.

Hence eq. (6.3.39) gives

$$b_{ij} = eb_i b_j. \tag{6.3.41}$$

Theorem 6.3.5. *The necessary and sufficient condition for the hypersurface $F^{n-1}(c)$ of Finslerian space equipped with metric $F = \dfrac{\alpha^{m+1}}{(\alpha-\beta)^m}$, to be hyperplane of the 2^{nd} kind is* (6.3.41).

In view of Eq. (6.3.25) and lemma 1.4.3, we have

Theorem 6.3.6. *The hypersurface $F^{n-1}(c)$ of Finslerian space equipped with metric $F = \dfrac{\alpha^{m+1}}{(\alpha-\beta)^m}$ is not hyperplane of the 3^{rd} kind.*

6.4 Projectively flat Finsler space with special (α, β)-metric

F is a positively homogeneous function of a Riemannian metric $\alpha = \sqrt{a_{ij}(x)\eta^i\eta^j}$ and a differential 1-form $\beta = b_i\eta^i$ of degree one. The interesting examples of (α, β)-metric are the Randers metric and Kropina metric.

The condition for a Randers space to be projectively flat was given by Hashiguchi-Ichijyo [29] and Matsumoto [49]. The projective flatness of Kropina space was investigated by Matsumoto and Matsumoto space was studied by Aikou-Hashiguchi-Yamauchi [1]. The condition for a Finsler space with a generalized Randers metric F satisfying $F^2 = c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2$, where c_i's are constants, to be projectively flat was given by Park and Choi [59]. Recently, the projective flatness of Finsler spaces with some special metrics have been studied by G. Shanker and R. Yadav [70]. A locally Minkowski space with

(α,β)-metric is called flat-parallel [3] if α is locally flat and β is parallel with respect to α.

In this section, we have considered the projective flatness of Finsler spaces with a special (α,β)-metric $F=\alpha+\sqrt{\alpha^2+\beta^2}$.

In a Finsler space $F^n=(N^n,F)$ with an (α,β)-metric, let $\gamma^i_{jk}(x)$ be Christoffel symbols constructed from a_{ij}, the Riemannian metric. We denote ';' by the covariant differentiation with respect to $\gamma^i_{jk}(x)$. In a Finsler space F^n with (α,β)-metric, we define

$$\gamma_{jhk}=a_{hr}\gamma^r_{jk},\qquad b^2=a^{rs}b_rb_s. \tag{6.4.1}$$

We shall denote the homogeneous polynomials in (η^i) of degree r by $hp(r)$ for brevity. Now the following Matsumoto's theorem [49] is well-known.

Theorem 6.4.1. *A Finsler space F^n with an (α,β)-metric is projectively flat if and only if the space is covered by coordinate neighbourhoods on which γ^i_{jk} satisfies*

$$\frac{1}{2}\left(\gamma^i_{00}-\frac{\gamma_{000}\eta^i}{\alpha^2}\right)+\left(\frac{\alpha F_\beta}{F_\alpha}\right)s^i_0+\left(\frac{F_{\alpha\alpha}}{F_\alpha}\right)\left(C+\frac{\alpha r_{00}}{2\beta}\right)\left(\frac{\alpha^2b^i}{\beta}-\eta^i\right)=0, \tag{6.4.2}$$

where the subscript 0 means a contraction by η^i, and C is given by

$$C+\left(\frac{\alpha^2F_\beta}{\beta F_\alpha}\right)s_0+\left(\frac{\alpha F_{\alpha\alpha}}{\beta^2F_\alpha}\right)(\alpha^2b^2-\beta^2)\left(C+\frac{\alpha r_{00}}{2\beta}\right)=0. \tag{6.4.3}$$

By the homogeneity of F, we know $\alpha^2F_{\alpha\alpha}=\beta^2F_{\beta\beta}$, so the formula (6.4.3) can be rewritten in the following form:

$$\left\{1+\left(\frac{F_{\beta\beta}}{\alpha F_\alpha}\right)(\alpha^2b^2-\beta^2)\right\}\left(C+\frac{\alpha r_{00}}{2\beta}\right)=\left(\frac{\alpha}{2\beta}\right)\left\{r_{00}-\left(\frac{2\alpha F_\beta}{F_\alpha}\right)s_0\right\}. \tag{6.4.4}$$

If $1+\left(\frac{F_{\beta\beta}}{\alpha F_\alpha}\right)(\alpha^2b^2-\beta^2)\neq 0$, then we can eliminate $\left(C+\frac{\alpha r_{00}}{2\beta}\right)$ in Eq. (6.4.2) and it is written as the form:

$$\left\{1 + \frac{F_{\beta\beta}\left(\alpha^2 b^2 - \beta^2\right)}{\alpha F_\alpha}\right\}\left\{\frac{1}{2}\left(\gamma^i_{00} - \frac{\gamma_{000}\eta^i}{\alpha^2}\right) + \left(\frac{\alpha F_\beta}{F_\alpha}\right) s^i_0\right\} + \left(\frac{F_{\alpha\alpha}}{F_\alpha}\right)\left(\frac{\alpha}{2\beta}\right)$$
$$\times \left\{r_{00} - \left(\frac{2\alpha F_\beta}{F_\alpha}\right) s_0\right\}\left(\frac{\alpha^2 b^i}{\beta} - \eta^i\right) = 0. \tag{6.4.5}$$

Thus, we have

Theorem 6.4.2. *If* $1 + \left(\frac{F_{\beta\beta}}{\alpha F_\alpha}\right)(\alpha^2 b^2 - \beta^2) \neq 0$, *then a Finsler space* F^n *with an* (α, β)*-metric is projectively flat if and only if* (6.4.5) *is satisfied.*

It is known [4] that if α^2 contains β as a factor, then the dimension is equal to two, and $b^2 = 0$. Throughout this paper, we assume that the dimension is more than two and $b^2 \neq 0$, that is, $\alpha^2 \not\equiv 0 \pmod{\beta}$.

6.4.1 Projectively flat space

Let F^n be a Finsler space with an (α, β)-metric given by

$$F = \alpha + \sqrt{\alpha^2 + \beta^2}. \tag{6.4.6}$$

It is known that [50] a Finsler space with (α, β)-metric Eq.(6.4.6) is flat-parallel if it is locally Minkowski.

In this section, we find the condition for a Finsler space F^n with Eq. (6.4.6) to be projectively flat.

The partial derivatives with respect to α and β of a metric Eq. (6.4.6) are given by

$$F_\alpha = \frac{\alpha + \sqrt{\alpha^2 + \beta^2}}{\sqrt{\alpha^2 + \beta^2}}, \quad F_\beta = \frac{\beta}{\sqrt{\alpha^2 + \beta^2}},$$
$$F_{\alpha\alpha} = \frac{\beta^2}{(\alpha^2 + \beta^2)^{\frac{3}{2}}}, \quad F_{\beta\beta} = \frac{\alpha^2}{(\alpha^2 + \beta^2)^{\frac{3}{2}}}. \tag{6.4.7}$$

If $1+\left(\frac{F_{\beta\beta}}{\alpha F_\alpha}\right)(\alpha^2 b^2-\beta^2)=0$, then we have $\beta^4-3\alpha^2\beta^2+2b^2\alpha^3\beta+\ 6b^2\alpha^4=0$, which leads a contradiction. Thus the theorem 6.4.2 can be applied.

Substituting Eq. (6.4.7) into Eq. (6.4.5), we get

$$\left(\alpha^3 b^2+\alpha^3+\alpha^2\sqrt{\alpha^2+\beta^2}+\beta^2\sqrt{\alpha^2+\beta^2}\right)\left(2\alpha^3\beta s_0^i+\gamma_{00}^i\alpha^2\sqrt{\alpha^2+\beta^2}+\right.$$
$$\left.\gamma_{00}^i\alpha^3-\gamma_{000}\eta^i\sqrt{\alpha^2+\beta^2}-\gamma_{000}\eta^i\alpha\right)+\ \alpha^3\left(-2\alpha\beta s_0+r_{00}\sqrt{\alpha^2+\beta^2}+r_{00}\alpha\right)$$
$$\left(\alpha^2 b^i-\eta^i\beta\right)=0. \tag{6.4.8}$$

Then the above Eq. (6.4.8) can be rewritten as a polynomial in α as follows:

$$p_6\alpha^6+\ p_4\alpha^4+\ p_2\alpha^2+\ p_0+\alpha\left(p_5\alpha^4+p_3\alpha^2+p_1\right)\sqrt{\alpha^2+\beta^2}=0,$$

where,
$$p_6=\left(2b^2 s_0^i-2b^i s_0+2s_0^i\right)\beta+b^2\gamma_{00}^i+r_{00}b^i+2\gamma_{00}^i,$$
$$p_5=\left(b^2\gamma_{000}+2\beta s_0^i+r_{00}b^i+2\gamma_{00}^i\right),$$
$$p_4=\left(2s_0\eta^i+2\gamma_{00}^i\right)\beta^2-r_{00}\beta\eta^i-\gamma_{000}\left(b^2+2\ \right)\eta^i,$$
$$p_3=2s_0^i\beta^3+\gamma_{00}^i\beta^2-\beta r_{00}\eta^i-\left(b^2+2\right)\gamma_{000}\eta^i,$$
$$p_2=\gamma_{00}^i\beta^4-2\gamma_{000}\beta^2\eta^i,\quad p_1=-\gamma_{000}\beta^2\eta^i,\quad p_0=-\gamma_{000}\beta^4\eta^i.$$

Since $p_6\alpha^6+\ p_4\alpha^4+p_2\alpha^2+\ p_0$ and $p_5\alpha^4+p_3\alpha^2+p_1$ are rational and α is irrational in η^i, thus we have

$$p_6\alpha^6+\ p_4\alpha^4+p_2\alpha^2+\ p_0=0, \tag{6.4.9}$$
$$p_5\alpha^4+p_3\alpha^2+p_1=0. \tag{6.4.10}$$

We observe in Eq. (6.4.10) that the term $-\gamma_{000}\beta^2\eta^i$ must have a factor α^2. Hence, we have 1-form $\mu_0=\mu_i(x)\,\eta^i$ such that

$$\gamma_{000}=\mu_0\alpha^2. \tag{6.4.11}$$

Now, considering Eq. (6.4.10) and using Eq. (6.4.11), we have

$$2s^i_0\beta^2 + \gamma^i_{00}\beta - r_{00}\eta^i - \mu_0\beta\eta^i = -v^i_0\alpha^2, \quad v^i_0 = v^i_j\eta^j.$$

Transvecting above term by η_i,

$$\left(\gamma^i_{00}\eta_i - \mu_0\eta^i\eta_i\right)\beta - r_{00}\eta^i\eta_i = -v^i_0\alpha^2\eta_i,$$

which implies

$$\left(\gamma_{000} - \mu_0\alpha^2\right)\beta - r_{00}\alpha^2 = -v^i_0\alpha^2\eta_i,$$

we get

$$r_{00} = v^i_0\eta_i. \tag{6.4.12}$$

We have

$$\left(b^2\gamma^i_{00} + 2\beta s^i_0 + r_{00}b^i + 2\gamma^i_{00}\right)\alpha^2 - \left(b^2+2\right)\mu_0\alpha^2\eta^i - v^i_0\alpha^2\beta = 0,$$

then we get,

$$s^i_0 - \frac{1}{2}v^i_0 = 0. \tag{6.4.13}$$

Now, using (6.4.11) in (6.4.9), we obtain

$$\left(\left(2b^2s^i_0 - 2b^is_0 + 2s^i_0\right)\beta + b^2\gamma^i_{00} + r_{00}b^i + 2\gamma^i_{00}\right)\alpha^4 + \left(\left(2s_0\eta^i + 2\gamma^i_{00}\right)\beta^2\right.$$
$$\left.-r_{00}\beta\eta^i - \mu_0\alpha^2\left(b^2+2\right)\eta^i\right)\alpha^2 + \gamma^i_{00}\beta^4 - 2\mu_0\alpha^2\beta^2\eta^i - \mu_0\beta^4\eta^i = 0.$$

Thus, the term $(\gamma^i_{00} - \mu_0\eta^i)$ must contain α^2.

Hence, we have $\mu^i = \mu^i(x)$ satisfying

$$\gamma^i_{00} - \mu_0\eta^i = \mu^i\alpha^2. \tag{6.4.14}$$

Transvecting (6.4.14) by η_i, and from (6.4.11), we have, $\mu_i\eta^i = 0$, which implies $\mu^i = 0$.

Thus, we have

$$\gamma^i_{00} = \mu_0 \eta^i, \tag{6.4.15}$$

that is,

$$2\gamma^i_{jk} = \mu_k \delta^i_j + \mu_j \delta^i_k, \tag{6.4.16}$$

which shows that the associated Riemannian space is projectively flat.

Next, substituting (6.4.11) and (6.4.15) into (6.4.9), we get

$$\Big(\left(2b^2 s^i_0 - 2b^i s_0 + 2s^i_0\right)\beta + b^2\gamma^i_{00} + r_{00}b^i + 2\gamma^i_{00}\Big)\alpha^2 + \left(2s_0\eta^i + 2\gamma^i_{00}\right)\beta^2 - r_{00}\beta\eta^i$$
$$- \mu_0\alpha^2\left(b^2+2\right)\eta^i + \mu_0\beta^4\eta^i - 2\mu_0\alpha^2\beta^2\eta^i - \mu_0\beta^4\eta^i = 0. \tag{6.4.17}$$

Transvecting (6.4.17) by b_i, we obtain

$$2s_0\left(\beta^3 - \alpha^2\beta\right) = r_{00}\left(\beta^2 - \alpha^2 b^2\right). \tag{6.4.18}$$

$$\Rightarrow \left(b^2 r_{00} - 2s_0\beta\right)\alpha^2 = \left(r_{00} - 2s_0\beta\right)\beta^2.$$

Therefore, there exists a function $k = k(x)$, such that

$$r_{00} - 2s_0\beta = k\alpha^2, \quad b^2 r_{00} - 2s_0\beta = k\beta^2, \tag{6.4.19}$$

eliminating r_{00} from (6.4.19), we have

$$2\left(b^2-1\right)s_0\beta = k\left(\beta^2 - b^2\alpha^2\right), \tag{6.4.20}$$

that is,

$$\left(b^2-1\right)\left(s_i b_j + s_j b_i\right) = k\left(b_i b_j - b^2 a_{ij}\right). \tag{6.4.21}$$

Transvecting (6.4.21) by a_{ij}, we have $(1-n)\,b^2 k = 0$, which implies $k = 0$.

Assume that $b^2 \neq 1$, then from (6.4.20), $s_0 = 0$ and from (6.4.19) we obtain

$$r_{00} = 0, \quad \text{i.e., } r_{ij} = 0.$$

On the other hand, from $s_i = 0$ and (6.4.11) we have $s_{ij} = 0$. So, we get $b_{i;j} = 0$.

Conversely, it is easy to see that (6.4.8) is a consequence of (6.4.15) and $b_{i;j} = 0$. Thus, we have

Theorem 6.4.3. *A Finsler space F^n $(n > 2)$ with an (α, β)-metric (6.4.6) provided $b^2 \neq 1$ is projectively flat if and only if the associated Riemannian space (N^n, α) is projectively flat and $b_{i;j} = 0$.*

6.5 Conclusion

We have the following results:

- For the deformed Finsler metric $F = (\alpha + \beta)\left(\dfrac{\alpha^2}{\alpha - \beta}\right)$, we have obtained Finsler invariants $\rho, \rho_0, \rho_{-1}, \rho_{-2}$ which satisfies the condition, $\rho_{-1}\beta + \rho_{-2}\alpha^2 = 0$. Then we have constructed a nonholonomic Finsler frames, which is given by,

$$V_j^i = X_k^i Y_j^k,$$

where X_k^i and Y_j^k are given by Eqs. (6.2.13) and (6.2.14) respectively.

- For the deformed Finsler metric $F = (\alpha + \beta)\left(\alpha + \beta + \dfrac{\beta^2}{\alpha}\right)$, we have obtained Finsler invariants $\rho, \rho_0, \rho_{-1}, \rho_{-2}$ which satisfies the condition, $\rho_{-1}\beta + \rho_{-2}\alpha^2 = 0$. Then we have constructed a nonholonomic Finsler frames, which are given by,

$$V_j^i = X_k^i Y_j^k,$$

where X_k^i and Y_j^k are given by Eqs. (6.2.16) and (6.2.17) respectively.

- By considering the hypersurface of a Finsler space with generalized Matsumoto metric, we have obtained the following results:

  (a) The induced metric structure of the generalized Matsumoto metric on the hypersurface F^{n-1} and obtained the scalar function $b(x)$ given by $b_i(x(u)) =$

$\sqrt{\dfrac{b^2}{1+(m^2+m)b^2}}N_i$ and $b^i = \sqrt{b^2\left((m^2+m)b^2+1\right)}N^i + \dfrac{mb^2}{\alpha}\eta^i$, where N_i is a unit normal vector.

(b) For the generalized Matsumoto metric on the Finsler hypersurface F^{n-1} the second fundamental tensor is given $M_{\alpha\beta} = \dfrac{m}{2\alpha}\sqrt{\dfrac{b^2}{1+(m^2+m)b^2}}h_{\alpha\beta}, \quad M_\alpha = 0.$

(c) Further, using Matsumoto's results, we have discussed the properties of hypersurface $F^{n-1}(c)$ that it is a hyperplane of a first and second kind but not of third kind.

- A Finsler space F^n with an (α, β)-metric $F = \alpha + \sqrt{\alpha^2+\beta^2}$ provided $b^2 \neq 1$ is projectively flat if and only if the associated Riemannian space (N^n, α) is projectively flat and $b_{i;j} = 0$.

Bibliography

[1] T. Aikou, M. Hashiguchi and K. Yamauchi, *On Matsumoto's Finsler space with time measure,* Rep. Fac. Sci. Kagoshima Univ. (Math. Phys. Chem.) 23, 1-12, 1990.

[2] P. L. Antonelli and I. Bucataru, *On Holland's frame for Randers space and its applications in Physics,* In: Kozma, L. (ed), Steps in Differential Geometry. Proceedings of the Colloquium on Differential Geometry, Debrecen, Hungary, July 25-30, 2000.

[3] P. L. Antonelli, R. S. Ingarden and M. Matsumoto, *The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology,* Kluwer Academic, Dordrecht, The Netherlands, 1993.

[4] S. Bacso and M. Matsumoto, *Projective changes between Finsler spaces with* (α, β)*-metric,* Tensor (N.S.) 55, no. 3, 252-257, 1994.

[5] P. Bahmandous and D. Latifi, *Naturally reductive homogeneous* (α, β)*-spaces,* Int. J. of Geom. Meth. in Mod. Phy., 2050117 (16 pages), World Scientific Publishing Company, 2020.

[6] D. Bao, S. S. Chern and Z. Shen, *Introduction to Riemann-Finsler Geometry,* GTM-200, Springer-Verlag, 2000.

[7] D. Bao and C. Robles, *Ricci and flag curvatures in Finsler geometry,* A Sampler of Riemann-Finsler Geometry, 50, 197-259, 2004.

[8] R. G. Beil, *Finsler Gauge Transformations and General Relativity,* International Journal of Theoretical Physics, Vol. 31, No. 6, 1992.

[9] R. G. Beil, Finsler and Kaluza-Klein Gauge Theories, Intern. J. Theor. Phys., 32, 6, pp. 1021-1031, 1993.

[10] R. G. Beil, *Comparison of unifed field theories,* Tensors. N. S.,56, pp. 175-183, 1995.

[11] L. Berwald, *Two-dimensional Finsler spaces with rectilinear extremals,* Annals of Mathematics, 42 (2), 84-112, 1941.

[12] A. Besse, *Einstein Manifolds,* Springer-Verlag, 1987.

[13] I. Bucataru, *Nonholonomic Frames in Finsler Geometry,* Balkan Journal of Geometry and Its Applications, Vol.7, No.1, pp. 13-27, 2002.

[14] I. Bucataru and R. Miron, *Finsler-Lagrange Geometry. Applications to dynamical systems,* Bucuresti: Editura Academiei Romane, 2007.

[15] K. Chandru and S. K. Narasimhamurthy, *On curvatures of Homogeneous Finsler-Kropina space,* Gulf Journal of Mathematics, 5, pp. 73-83, 2017.

[16] X. Cheng and Z. Shen, *A class of Finsler metrics with isotropic S-curvature,* Israel Journal of Mathematics, 169, pp. 317–340, 2009.

[17] X. Cheng, Z. Shen and Y. Tian, *A class of Einstein (α, β)-metrics,* Israel J. Math. 192, 221-249, 2012.

[18] J. E. D'Atri and W. Ziller, *Naturally reductive metrics and Einstein metrics on compact Lie groups,* Mem. Amer. Math. Soc., 18(215), 1979.

[19] S. Deng, *Homogeneous Finsler Spaces,* Springer Monographs in Mathematics, Springer New York Heidelberg Dordrecht London, ISBN 978-1-4614-4243-1.

[20] S. Deng and Z. Hou, *The group of isometries of a Finsler space,* Pacific Journal of Mathematics, 207, 149-155, 2002.

[21] S. Deng and Z. Hou, *Invariant Finsler metrics on homogeneous manifolds,* J. Phys. A: Math. Gen., 37, 8245-8253, 2004.

[22] S. Deng and Z. Hou, *Naturally reductive homogeneous Finsler spaces,* Manuscripta Math., 131, 215-229, 2010.

[23] S. Deng and Z. Hu, *Curvatures of homogeneous Randers spaces,* Advances in Mathematics, 240, pp. 194–226, 2013.

[24] S. Deng and X. Wang, *The S-curvature of Homogeneous (α, β)-metrics,* Balkan Journal of Geometry and Its Applications, 15, No. 2, pp. 47-56, 2010.

[25] S. Deng and M. Xu, *Recent progress on homogeneous Finsler spaces with positive curvature,* European Journal of Mathematics, 3, pp. 974–999, 2017.

[26] P. Eberlein, *Geometry of 2-step nilpotent groups with a left invariant metric,* Ann. Sci. Ecole Norm. Sup., 27, 805-828, 1994.

[27] M. K. Gupta and P. N. Pandey, *On hypersurface of a Finsler space with a special metric,* Acta Mathematica Hungarica, 120, no. 1-2, pp. 165–177, 2008.

[28] M. K. Gupta, A. Singh and P. N. Pandey, *On a Hypersurface of a Finsler Space with Randers Change of Matsumoto Metric, Geometry,* Hindawi Publishing Corporation, Article ID: 842573, 6 pages, 2013.

[29] M. Hashiguchi and Y. Ichijyo, *Randers spaces with rectilinear geodesics,* Reports of the Faculty of Science of Kagoshima University, no. 13, pp. 33–40, 1980.

[30] S. Helgason, *Differential geometry, Lie groups and symmetric spaces,* Pure and applied mathematics, a series of monographs and textbooks 80, Academic press, 1978.

[31] S. Homolya and O. Kowalski, *Simply connected two-step homogeneous nilmanifolds of dimension 5,* Note di Matematica, 1, 69-77, 2006.

[32] P. R. Holland, *Electromagnetism, particles and anholonomy,* Phys. Lett., 91A, 275-278, 1982.

[33] P. R. Holland and C. Philippidis, *Anholonomic deformations in the ether: a significance for the electrodynamic potentials, in the volume Quantum Implications,* B. J. Hiley and F. D. Peat (eds.), Routledge and Kegan Paul, London and New York, 295-311, 1987.

[34] M. Hosseini and H. R. S. Moghaddam, *On The Existence Of Homogeneous Geodesics in Homogeneous Kropina Spaces,* Bulletin of the Iranian Mathematical Society, 2019.

[35] V.V. Kajzer, *Conjugate points of left-invariant metrics on Lie groups,* Soviet Math., 34, 32-44, 1990.

[36] Y. Kasturada, *On the theory of nonholonomic system in Finsler Spaces,* Tohoku Mathematical Journal, Second Series, 1951.

[37] O. Kowalski, S. Nikcevic and Z. Vlasek, *Homogeneous geodesics in homogeneous Riemannian manifolds - examples,* Geometry and Topology of Submanifolds X, World Sci., 104-112, 2000.

[38] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry,* Interscience Publishers, New York, Vol. 2, 1969.

[39] O. Kowalski and L. Vanhecke, *Riemannian manifolds with homogeneous geodesics,* Boll. Un. Math. Ital. B(7), 5, 189–246, 1991.

[40] O. Kowalski and Szenthe, *On the existence of homogeneous geodesics in homogeneous Riemannian manifolds,* Geom. Dedicata., 81, 209-214, 2000.

[41] D. Latifi, *Homogeneous geodesics in homogeneous Finsler spaces,* J. of Geom. and Phy., 57, 1421–1433, 2007.

[42] D. Latifi and A. Razavi, *On Homogeneous Finsler Spaces,* Reports on Mathematical Physics, 57, 2006.

[43] D. Latifi and A. Razavi, *Homogeneous geodesics of left invariant Randers metrics on a three-dimensional Lie group,* Int. J. Cont. Math. Sci., 4, 873-881, 2009.

[44] D. Latifi and M. Toomanian, *Invariant naturally reductive Randers metrics on homogeneous spaces,* Math. Sci., 6(63), 2012.

[45] M. Matsumoto, *On C-reducible Finsler spaces*, Tensor, N. S. 24, 29–37, 1972.

[46] M. Matsumoto, *The induced and intrinsic Finsler connections of a hypersurface and Finslerian projective geometry,* Journal of Mathematics of Kyoto University, 25, no. 1, pp. 107–144, 1985.

[47] M. Matsumoto, *Foundations of Finsler geometry and special Finsler spaces,* Kaisheisha Press, Otsu, Japan, 1986.

[48] M. Matsumoto, *A slope of a mountain is a Finsler surface with respect to a time measure,* Journal of Mathematics of Kyoto University, 29, no. 1, pp. 17–25, 1989.

[49] M. Matsumoto, *Projectively flat Finsler spaces with* (α, β)*-metric,* Rep. on Math. Phys., 30, 15-20, 1991.

[50] M. Matsumoto, *A special class of locally Minkowski soaces with* (α, β)*- metric and conformally flat Kropina spaces,* Tensor (N.S.), 50, 202-207, 1991.

[51] M. Matsumoto, *Theory of Finsler spaces with* (α, β)*-metric,* Reports on Mathematical Physics, Vol.31, 1992.

[52] H. R. S. Moghaddam, *On the Randers metrics on two-step homogeneous nilmanifolds of dimension five,* Int. J. Geom. Meth. Mod. Phys., 8(3), 501-510, 2011.

[53] H. G. Nagaraja and P. Kumar, *On Randers change of Matsumoto metric,* Bulletin of Mathematical Analysis and Applications, 4, no. 1, pp. 148–155, 2012.

[54] S. K. Narasimhamurthy, S. T. Aveesh, H. G. Nagaraja and Pradeep Kumar, *On special hypersurface of a Finsler space with the metric* $\alpha + \frac{\beta^{n+1}}{\alpha^n}$*,* Acta Universitatis Apulensis, 17, 129-139, 2009.

[55] K. Nomizu, *Invariant Affine Connections on Homogeneous Spaces,* Am. J. Math. 76, 33-65, 1954.

[56] S. Ohta, *Vanishing* S*-curvature of Randers spaces,* Differential Geometry and its Applications, 29, pp. 174–178, 2011.

[57] M. Parhizkar and D. Latifi, *On the flag curvature of invariant* (α, β)*-metrics,* Int. J. Cont. Math. Sci., 4, 873–881, 2009.

[58] M. Parhizkar and D. Latifi, *Geodesic vectors of Randers metrics on nilpotent Lie groups of dimension five,* Global. J. Adv. Res. Class. Moder. Geom., 7, 92-101, 2018.

[59] H. S. Park and E. S. Choi, *On Finsler space with a special* (α, β)*-metric,* Tensor, N. S. 56, 142-148, 1995.

[60] H. S. Park and I. Y. Lee, *On projectively flat Finsler space with a* (α, β)*-metric,* Comm. Korean Math. Soc., No.02, 14, 373-383, 1999.

[61] L. I. Piscoran and V. N. Mishra, *S-curvature for a new class of* (α, β)*-metrics,* RACSAM, 111, pp. 1187–1200, 2017.

[62] Pradeep Kumar, S. K. Narasimhamurthy, C. S. Bagewadi and S.T. Aveesh, *Special Finsler hypersurfaces admitting a parallel vector field,* SCIENTIA, Series A: Mathematical Science, 20, 123-130, 2010.

[63] T. Puttmann, *Optimal pinching constants of odd dimensional homogeneous spaces,* Invent. Math. 138, 631–684, 1999.

[64] M. Rafie-Rad and B. Rezaei, *On Einstein Matsumoto metrics,* Nonlinear Analysis RWA, 13, no. 2, pp. 882–886, 2012.

[65] G. Randers, *On an asymmetric metric in the four-space of general relativity,* Phys. Rev., 59(2), 195-199, 1941.

[66] W. Rossmann, *Lie Groups: An Introduction Through Linear Groups,* Oxford Graduate Texts in Mathematics, 2006.

[67] H. Rund, *The Differential Geometry of Finsler Spaces,* Springer-Verlag, Berlin-Gottingen-Heidelberg, 1959.

[68] H. R. Salimi Moghaddam, *The flag curvature of invariant (α, β)-metrics of type $F = \frac{(\alpha+\beta)^2}{\alpha}$,* J. of Phy. A: Math. and Theo., IOP publishing, 41, 275206 (6pp), 2008.

[69] G. Shanker and S. Rani, *On S-curvature of a homogeneous Finsler space with square metric,* International Journal of Geometric Methods in Modern Physics, World Scientific Publishing Company, 17, No. 2, 2050019, 2020.

[70] G. Shanker and R. Yadav, *Projectively flat of Finsler spaces with some Special (α, β)-metrics,* Iranian Journal of Mathematical Sciences and Informatics (communicated).

[71] C. Shibata, *On invariant tensors of β-changes of Finsler metrics,* Journal of Mathematics of Kyoto University, 24, no. 1, pp. 163–188, 1984.

[72] H. Shimada and S. V. Sabau, *Introduction to Matsumoto metric,* Nonlinear Analysis, 63, pp. 165–168, 2005.

[73] Z. Shen, *Volume comparison and its applications in Riemann-Finsler geometry*, Advances in Mathematics, 128, 306-328, 1997.

[74] Z. Shen, *Finsler metrics with $K = 0$ and $S = 0$,* Canadian Journal of Mathematics, 55, 112-132, 2003.

[75] Z. Shen, *Finsler manifolds with non-positive flag curvature and constant S-curvature,* Mathematische Zeitschrift, 249 (3), 625-639, 2005.

[76] Y. Shen and Z. Shen, *Introduction to Modern Finsler Geometry,* Higher Education Press Limited Company: World Scientific Publishing Co., ISBN 9789814704908, 2016.

[77] J. L. Synge, *A generalization of the Riemannian line element,* Transactions of the American Mathematical Society, Vol. 27, No. 1, pp. 61-67 (7 pages), Jan., 1925.

[78] A. Tayebi, E. Peyghan and H. Sadeghi, *On Matsumoto-type Finsler metrics,* Nonlinear Analysis, 13, no. 6, pp. 2556– 2561, 2012.

[79] J. H. Taylor, *A generalization of Levi-Civita's parallelism and Frenet formulas,* Transactions of the American Mathematical Society, Vol. 27, No. 2, pp. 246-264 (19 pages), Apr., 1925.

[80] M. Xu and S. Deng, *Killing frames and S-curvature of Homogeneous Finsler spaces,* Glasgow Math. J., 57, pp. 457–464, 2015.

[81] M. Xu and S. Deng, *Homogeneous (α, β)-spaces with positive flag curvature and vanishing S-curvature,* Non-linear Analysis: Theory, Methods and Applications, 127, pp. 45-54, 2015.

[82] Z. Yan and S. Deng, *Finsler spaces whose geodesics are orbits,* Differential Geometry and its Applications, 36, 1-23, 2014.

[83] Z. Yan and S. Deng, *On homogeneous Einstein (α, β)-metrics,* Journal of Geometry and Physics, 103, pp. 20–36, 2016.

[84] Z. Yan and L. Huang, *On the existence of homogeneous geodesic in homogeneous Finsler spaces,* J. Geom. Phys., 124, 264-267, 2018.

[85] Y. Yu and Y. You, *On Einstein m-th root metrics,* Differential Geometry and its Applications, 28, 290–294, 2010.

[86] L. Zhou, *A local classification of a class of* (α, β)*-metrics with constant flag curvature,* Differential Geometry and its Applications, 28, 170-193, 2010.

Publications

1. **Surekha Desai**, Narasimhamurthy S K and Raghavendra R S, Flag curvature and homogeneous geodesics of homogeneous Finsler space with (α, β)-metric, *Palestine Journal of Mathematics*, (Accepted) 2022 .

2. **Surekha Desai**, Narasimhamurthy S K and Raghavendra R S, The study of S-curvature on a homogeneous Finsler space with Randers-Matsumoto metric, *Mapana Journal of Sciences*, Volume 22, No.1, 2023.

3. **Surekha Desai**, Narasimhamurthy S K and Raghavendra R S, Ricci curvature formula for a homogeneous Finsler space with (α, β)-metrics, *J. Int. Acad. Phys. Sci.*, (Accepted) 2023.

4. **Surekha Desai**, Narasimhamurthy S K and Ramesh M, Nonholonomic Frames for Finsler space with Special (α, β)-metrics, *JNNCE Journal of Engineering and Management*, Volume 5, No.2, July-December 2021.

5. **Surekha Desai**, Narasimhamurthy S K and Raghavendra R S, On the Hypersurface of a Finsler space with generalized Matsumoto metric, *International Journal of Innovative Science, Engineering and Technology*, Vol. 09 Issue 08, August 2022.

6. **Surekha Desai**, Narasimhamurthy S K and Raghavendra R S, Homogeneous geodesics of Finsler space with (α, β)-metrics, (Communicated).

7. **Surekha Desai**, Narasimhamurthy S K and Raghavendra R S, Geodesic vectors of invariant (α, β)-metrics on nilpotent Lie groups of dimension five, (Communicated).
8. **Surekha Desai**, Narasimhamurthy S K and Raghavendra R S, Projectively flat of Finsler space with special (α, β)-metrics, (Communicated).

Palestine Journal of Mathematics

Vol. XX(X)(XXXX) , 260–268

Flag curvature and homogeneous geodesics of homogeneous Finsler space with (α, β)-metric

Surekha Desai, Narasimhamurthy S K and Raghavendra R S

Communicated by Zafar Ahsan

MSC 2010 Classifications: Primary 22E60, 22F30; Secondary 53C30, 53C60.

Keywords and phrases: Finsler (α, β)-metric, Flag curvature, Homogeneous geodesics, Naturally reductive, Homogeneous Finsler space.

Abstract The computation of flag curvature of Finsler metrics is very difficult, therefore it is important to find an explicit formula for the flag curvature. In this paper, we have studied the existence of homogeneous geodesics for a naturally reductive homogeneous Finsler space M with metric $F = \alpha + \sqrt{\alpha^2 + \beta^2}$. We have discussed the necessary and sufficient condition for (M, F) to be naturally reductive. Further, by using Puttmann's formula we give the formula for flag curvature of naturally reductive homogeneous Finsler space with (α, β)-metric.

1 Introduction and Definitions

Flag curvature is considered a generalization of sectional curvature from Riemannian manifolds in Finsler geometry. Flag curvature was first introduced by Berwald (1926). It has an important role in characterizing Finsler spaces. The notion of naturally reductive Riemannian metrics was first introduced by Kobayashi and Nomizu [7]. It is well-known that the geodesics of a naturally reductive homogeneous space are the orbits of one-parameter subgroups of isometries [1]. In the field of mechanics homogeneous geodesics has important applications.

In 1979, the naturally reductive metrics and Einstien metrics on compact Lie groups were studied by D. Atri and Ziller [3]. H. R. Salimi Moghaddam gives the formula for flag curvature of invariant metrics of form $F = \frac{(\alpha+\beta)^2}{\alpha}$ such that α is induced by an invariant Riemannian metric g on the homogeneous space and the Chern connection of F coincides with the Levi-Civita connection of g [14]. In recent years, many authors have given the formula for flag curvature of a naturally reductive homogeneous Finsler space with (α, β)-metrics [3, 11, 10, 12], and also discussed the naturally reductiveness of homogeneous Finsler space. We studied the existence of homogeneous geodesics for the homogeneous Finsler space with metric, $F = \alpha + \sqrt{\alpha^2 + \beta^2}$. In this paper, first, we revive some basic concepts related to the homogeneous Finsler space. Next, we deduce the formula for flag curvature of Finsler space (M, F) with metric F and studied the condition for naturally reductiveness of homogeneous space (M, F). Further, we discussed the existence of homogeneous geodesics for the space (M, F). In the last part, we have obtained the formula for flag curvature of naturally reductive homogeneous Finsler space (M, F).

Definition 1.1. A connected Finsler space (M, F) is called a homogeneous Finsler space if $I(M, F)$, the group of isometries of (M, F), acts transitively on M.

The connected homogeneous Finsler space M can be written in the form of $M = G/H$, where G and H are the Lie group of isometries of M and isotropy subgroup of G a point in M respectively. Here homogeneous Finsler space is reductive decomposition if there exists an $Ad(H)$- invariant decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$, here $\mathfrak{h}$ and $\mathfrak{g}$ represents the Lie algebras of H and G, respectively and $\mathfrak{m}$ be the subspace of $\mathfrak{g}$ [9].

Definition 1.2. Let G be a Lie group and M a smooth manifold. If G has a smooth action on M, then G is called a Lie transformation group of M.

Definition 1.3. [7] A homogeneous space G/H of a connected Lie group G is called reductive if the following conditions are satisfied:

The Study of S - Curvature on a Homogeneous Finsler Space with Randers-Matsumoto Metric

Narasimhamurthy
Kuvempu University, Shimoga.

Surekha Desai
Kuvempu University, Shimoga.

Raghavendra
Kuvempu University, Shimoga.

Keywords: Randers-Matsumoto metric, Homogeneous Finsler space, invariant vector field, S-curvature

Abstract

In this article, we have focused on the study of S-curvature of Randers-Matsumoto metric on a homogeneous Finsler space. We have deduced the condition for an isometry of Finsler homogeneous space with Randers-Matsumoto metric to be an isometry of Riemannian homogeneous space and proved that the group of isometries of Finsler space are closed subgroups of that of Riemannian space. We have examined the existence of invariant vector field. Further, we have derived the formula for S -curvature on the reductive homogeneous space, discussed the condition for isotropic S-curvature and derived the S - curvature of the Randers-Matsumoto metric for the homogenous space by using S-curvature formula.

Author Biographies

Narasimhamurthy, Kuvempu University, Shimoga.
Kuvempu University, Shimoga, Karnataka.

Surekha Desai, Kuvempu University, Shimoga.
Kuvempu University, Shimoga, Karnataka.

Raghavendra, Kuvempu University, Shimoga.
Kuvempu University, Shimoga, Karnataka.

Available online @ https://jjem.jnnce.ac.in
https:www.doi.org/10.37314/JJEM.2021.050211
Indexed in International Scientific Indiexing (ISI)
Impact factor: 1.395 for 2021-22
Published on: 31 January 2022

Nonholonomic Frames for Finsler space with Special (α, β)-metrics

Surekha Desai[1], Narasimhamurthy S K[2*], Ramesha M[3]

[1,2*]Department of P.G. Studies and Research in Mathematics, Kuvempu University, Shankaraghatta-577 451, Shivamogga, Karnataka, INDIA.
[3]Department of Mathematics, Jain Institute, Bengaluru, Karnataka, INDIA.

surekhad90@gmail.com, sknmurthy@kuvempu.ac.in, ramfins@gmail.com

Abstract

In the present paper, we determine the nonholonomic Frames for Finsler space with special (α, β)-metrics of type $L(\alpha, \beta) = (\alpha + \beta)\left(\frac{\alpha^2}{\alpha-\beta}\right)$ and $L(\alpha, \beta) = (\alpha + \beta)\left(\alpha + \beta + \frac{\beta^2}{\alpha}\right)$ and also we observed the nonholonomic frames expesses as a Guage Transformation of Finsler metric.

Keywords: *(α, β)-metrics, Nonholonomic frames, Guage transformations, Beil metric, Finsler space.*

1. Introduction

The concept of theory of gauge transformation has been established in the context of Finsler space by G. S. Asanov and his co-researchers (1985-1989) [1], here interesting thing is that the theory of guage transformation the Finsler tangent vectors are considered as independent variables are attached to points in space-time. The homogeneous transformations of the tangent space are called guage transformations. In 1982, P. R. Holland worked on a unified (formalism) field theory that uses a nonholonomic Finsler frame on space-time is a sort of plastic deformation by considering the motion of charged particles in an electromagnetic field[2,3,4]. Nonholonomic frames have studied by so many physicists and geometricians about the motion of charged particles in electromagnetic field theory. In 1951, Y. Katsurada [7] introduced the theory of nonholonomic system in Finsler geometry. In 1995, R. G. Beil [5,6] have worked on a guage transformation considered as a nonholonomic frame on the tangent bundle of a four-dimensional base manifold. This introduces that there is unified approach to gravitation and guage symmetries. I. Bucataru[2], in his research he discussed about the how Beil metric is used in deformation of Riemannian metric and also in nonholonomic frame. For finding the nonholonomic frame he consider the most general case of Beil's metric. In this case, the Generalized Lagrange metric (in short, GL-metric) is known as Beil metric. In this article, evaluated the nonholonomic finslerian deformation with the some distinct special (α, β)-metrics are as follows:

1. $L(\alpha, \beta) = (\alpha + \beta)\left(\frac{\alpha^2}{\alpha-\beta}\right)$, i.e., product of Randers metric and Matsumoto metric.
2. $L(\alpha, \beta) = (\alpha + \beta)\left(\alpha + \beta + \frac{\beta^2}{\alpha}\right)$, i.e., product of Randers metric and first approximate Matsumoto metric.

IJISET - International Journal of Innovative Science, Engineering & Technology, Vol. 09 Issue 08, August 2022
ISSN (Online) 2348 – 7968 | Impact Factor – 6.72
www.ijiset.com

On the Hypersurface of a Finsler space with generalized Matsumoto metric

Surekha Desai[1], Narasimhamurthy S K[2] and Raghavendra R S[3]

Department of P.G. Studies and Research in Mathematics, Kuvempu University,
Shankaraghatta-577 451, Shivamogga, Karnataka, INDIA.
e-mail: surekhad90@gmail.com, nmurthysk@hotmail.com and rsr.smg@gmail.com

Abstract

In this paper, we study some geometrical properties of a Finsler space with the generalized Matsumoto metric $L = \frac{\alpha^{m+1}}{(\alpha-\beta)^m}$. Further, we prove the necessary and sufficient condition for Finsler Hypersurface satisfies the condition of hyperplanes of first, second kinds and but not the hyperplane of the third kind with respect to the above metric.

Keywords: Finsler space, Generalized Matsumoto metric, Hyperplanes, Induced Cartan's connection, $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-metric.

1. Introduction

In 1992,[1] the notion of Finsler spaces with an (α, β)-metric was first proposed by M. Matsumoto named as function of $L(\alpha, \beta)$. After the Matsumoto's accomplishment in the development of Finsler geometry there are lot of contributions were given by several authors they have studied a special form of (α, β)-metrics like Rander's metric, Kropina metric, generalized Kropina metric, Shen's square metric etc. The systematic theory of the hypersurface of a Finsler space was built by Matsumoto in 1985, along with this he explained the hyperplane of the first kind, second kind and third kind are the classification of hypersurfaces. Further, many researchers were considered these three kinds of hyperplanes in different types of (α, β)-metrics of Finsler spaces and they came with various conclusions. Recent years, in 2009, H. G. Nagaraja, S. K. Narasimhamurthy, Pradeep Kumar and S. T. Aveesh obtained some results on geometrical properties of Finslerian hypersurfaces with (α, β)-metrics [3, 4]. In 2018, K. Vineet and R. K. Gupta worked on some special (α, β)-metric. In 2020, Brijesh kumar Tripathi introduced same aspect with deformed Berwald-infinite series metric.

In this paper, our aim is to express certain geometrical properties of hypersurface of a Finsler spaces, and we discussed the different kinds of hyperplanes with generalized Matsumoto metric, $L = \frac{\alpha^{m+1}}{(\alpha-\beta)^m}$. We have derived that the necessary and sufficient condition for hypersurface of generalized Matsumoto metric satisfies the conditions of hyperplane of first, second and but not third kind for above metric.

RICCI CURVATURE FORMULA FOR A HOMOGENEOUS FINSLER SPACE WITH (α,β)-METRICS

Surekha Desai[1], Narasimhamurthy S K[2] and Raghavendra R S[3]

Department of P.G. Studies and Research in Mathematics, Kuvempu University,
Shankaraghatta-577 451, Shivamogga, Karnataka, INDIA.
e-mail: [1]surekhad90@gmail.com, [2]sknmurthy@kuvempu.ac.in, [3]rsr.smg@gmail.com

Abstract: Curvature properties of homogeneous Finsler spaces with (α,β)-metrics are among the most significant topics in Finsler geometry. In this article, first we will discuss Ricci curvatures in Finsler geometry. We have obtained the formula for the Ricci curvature of homogeneous Finsler space with special metric, $F = \alpha + \sqrt{\alpha^2 + \beta^2}$. We have discussed the conditions to have vanishing S-curvature for the space $(G\ /\ H, F)$.

Keywords: Finsler space, Ricci curvature, special (α,β)-metric, vanishing S-curvature, Homogeneous Finsler space.

Mathematics Subject Classification: 53C60, 53C30, 22E60.

1. Introduction

M. Matsumoto introduced the concept of (α,β)-metric in Finsler geometry in 1972 [9]. This is the generalization of Randers metric, introduced by G. Randers [11]. So many authors have worked on this concept [7, 8, 13]. In physics and biology, (α,β)-metrics have several applications. For (α,β)-metrics Riemannian and Ricci curvatures are given by Zhou [15]. A Finsler space (M, F) whose Ricci curvature is written as $Ric(x,y) = \lambda(x)F^2(x,y)$, where λ-smooth function on M, is called an Einstein metric [1]. In [4], Cheng et al. have shown that the formulae given in [15] are incorrect. Later, they have also given the corrected formulae for Ricci curvature and Riemannian curvature for (α,β)-metrics.

Aim of this article is to compute the explicit and applicable formula for the Ricci curvature of homogeneous Finsler space with of special (α,β)-metric $F = \alpha + \sqrt{\alpha^2 + \beta^2}$. This paper is described as in the following manner: In first part, we recall some basic definitions related to the Finsler space and homogeneous Finsler space. along with this we have discussed the condition for positive definiteness of Finsler metric F. In part 2 and part 3, we talk over the definition of Ricci curvature, theorems and calculated some quantities related to the Ricci curvature formulae. At last, we established the explicit formula for Ricci curvature of homogeneous Finsler space with special metric, $F = \alpha + \sqrt{\alpha^2 + \beta^2}$ as well as we have shown that $(G/H, F)$ having vanishing S-curvature.

Definition 1.1. *Let M be a connected smooth manifold. If $\exists$ a continuous function $F: TM \to [0,\infty)$ such that F is smooth on tangent bundle $TM\backslash\{0\}$ and is restricted to the tangent space is*

A THESIS ENTITLED

THE STUDY OF GEODESIC ORBIT IN HOMOGENEOUS FINSLER SPACES

Submitted to the

Faculty of Science and Technology

KUVEMPU **UNIVERSITY**

For the Award of the Degree of

Doctor of Philosophy

in

MATHEMATICS

by

SUREKHA DESAI

Research Supervisor

Dr. S K NARASIMHAMURTHY

Professor

Department of P.G. Studies and Research in Mathematics,

Jnana Sahyadri, Shankaraghatta - 577 451,

Shivamogga, Karnataka, India.

February 2023

Preface

Geometry is a part of Mathematics concerned with questions of size, shape and relative position of figures and with properties of space. Initially a body of practical knowledge concerning lengths, areas and volumes in the third century B.C., geometry was put into a axiomatic form by Euclid, whose treatment is known as Euclidean Geometry. Differential geometry has a long history as a field of Mathematics. The authors Schouten and Van Dantzing in 1930, first tried to transfer the results of differential geometry of spaces with Riemannian metric with affine connection to the case of spaces with complex structure.

The theory of spaces with a generalized metric was initiated by Finsler in 1918 under the influence of geometrization of variation calculus and was developed independently by Synge, Taylor and in particular, Berwald in the middle of 1920's as a generalization of Riemannian geometry. The study of Finsler spaces has important significance in physics. The concept of homogeneity is one of the fundamental notions in geometry although its means must be specified for the concrete situations. Homogeneous Finsler spaces emphasizes the relationship between Lie group and Finsler geometry. Let (N, F) be a connected Finsler space. The group of isometries of (N, F), denoted by $I(N, F)$ is a Lie transformation of N. We say that (N, F) is homogeneous Finsler space if the action of $I(N, F)$ on N is transitive. A homogeneous Finsler space emphasizes the relation between Lie groups and Finsler geometry. A geodesic vector is a non-zero vector that generates a geodesic curve. The non-zero vector of a geodesic orbit in homogeneous Finsler space was first described by Dariush Latifi.

This thesis comprises of six chapters commencing with introduction as **Chapter 1**, which consists of concise history of Finsler geometry, homogeneous Finsler spaces and its applications, definitions and description of significant terms involving formulae. It includes various types of curvatures such as Riemannian curvature, Flag curvature, S-curvature, Ricci curvature etc., and also geodesic orbit spaces, invariant Finsler metric, Projective change, Non-holonomic frames, and hypersurfaces.

Chapter 2 is devoted to study of the explicit formulae for the flag curvature of homogeneous Finsler spaces with some special (α, β)-metrics. First, we discusses a brief review of literature of Flag curvature. Further, by using Puttmann's formula we give the formula for flag curvature of naturally reductive homogeneous Finsler space with (α, β)-metric. Also, we have discussed the existence of homogeneous geodesics for the space (N, F) and obtained the following results:

- Let a compact Lie group G contains a closed subgroup H with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ of G and H respectively. Also an invariant Riemannian metric $\tilde{\alpha}$ on the homogeneous space G/H such that $\langle v, w\rangle = \langle\langle \psi(v), w\rangle\rangle$, where $\psi : \mathfrak{g} \to \mathfrak{g}, \forall\, v, w \in \mathfrak{g}$ is a positive definite endomorphism. Suppose that an invariant vector field $\tilde{u}$ on homogeneous space G/H is parallel with respect to Riemannian metric $\tilde{a}$ and $\tilde{u}_H = u$ and assume that $F = \alpha + \sqrt{\alpha^2 + \beta^2}$ be a special (α, β)-metric arising from $\tilde{\alpha}$ and $\tilde{u}$ such that its Chern connection of F and the Riemannian connection of $\tilde{a}$ are coincides, and a flag $\{P, \eta\}$ in $T_H(G/H)$ such that $\{\zeta, \eta\}$ is an orthonormal basis of P with respect to $\langle\cdot,\cdot\rangle$. Then the flag curvature of the flag $\{P, \eta\}$ is given by
$$K(P,\eta) = \frac{\big\langle \zeta, R(\zeta,\eta)\eta\big\rangle S_1 + \big\langle u,\zeta\big\rangle\big\langle u, R(\zeta,\eta)\eta\big\rangle S_2 + \big\langle \eta, R(\zeta,\eta)\eta\big\rangle S_3}{8 + 4\big\langle u,\eta\big\rangle^2 + 2\big\langle u,\zeta\big\rangle^2 + A_1},$$
where
$$\begin{aligned} S_1 =& 2 + \frac{2 + \langle u,\eta\rangle^2}{\sqrt{1+\langle u,\eta\rangle^2}}, \quad S_2 = \frac{(1+\langle u,\eta\rangle^2)^{\frac{3}{2}} + 1}{(1+\langle u,\eta\rangle^2)^{\frac{3}{2}}}, \quad S_3 = \frac{\langle u,\eta\rangle^3\langle u,\zeta\rangle}{(1+\langle u,\eta\rangle^2)^{\frac{3}{2}}},\\ S_4 =& 8 + 8\langle u,\eta\rangle^2 + 2\langle u,\zeta\rangle^2(1+\langle u,\eta\rangle^2) + \langle u,\eta\rangle^4, \quad S_5 = 2\langle u,\zeta\rangle^2 - \langle u,\eta\rangle^2\langle u,\zeta\rangle^2,\\ A_1 =& \frac{S_4}{\sqrt{1+\langle u,\eta\rangle^2}} + \frac{2\langle u,\zeta\rangle^2}{1+\langle u,\eta\rangle^2} + \frac{S_5}{(1+\langle u,\eta\rangle^2)^{\frac{3}{2}}} - \frac{\langle u,\eta\rangle^2\langle u,\zeta\rangle^2}{(1+\langle u,\eta\rangle^2)^3}. \end{aligned}$$

- Let a homogeneous Finsler space $(G/H, F)$ with (α, β)-metric $F = \alpha + \sqrt{\alpha^2 + \beta^2}$ be defined by an invariant Riemannian metric $\tilde{\alpha}$ and an invariant vector field u such

that the Chern connection of F coincides the Levi-Civita connection of $\tilde{\alpha}$. Then $(G/H, F)$ is naturally reductive if and only if the underlying Riemannian space $(G/H, \tilde{\alpha})$ is naturally reductive.

- Let a homogeneous Finsler space (N, F) with $F = \alpha + \sqrt{\alpha^2 + \beta^2}$ defined by the Riemannian metric $\alpha = a_{ij} dx^i \otimes dx^j$ and the vector field u corresponding to 1-form β. Then the homogeneous Finsler space (N, F) with the origin $p = \{H\}$ and with an Ad(H)-invariant decomposition $\mathfrak{g} = \mathfrak{l} + \mathfrak{h}$ is naturally reductive with respect to this decomposition if and only if for any vector $u \in \mathfrak{l} \setminus \{0\}$, the curve $\gamma(t)$ is geodesic of homogeneous Finsler manifold, here $\gamma(t)$ is $\exp tu(p)$.

Chapter 3 deals with the study of the existence of invariant vector fields of homogeneous Finsler spaces with (α, β)-metrics. The formula for S-curvature of homogeneous Finsler spaces with an (α, β)-metric is obtained. Further, using it, it is shown that these homogeneous Finsler spaces have isotropic S-curvature if and only if they have vanishing S-curvature. In the last section, the formulae for mean Berwald curvature of homogeneous Finsler spaces with (α, β)-metrics are obtained. We have proved the following results:

- Let G/H be reductive homogeneous Finsler space with a decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{h} + \mathfrak{l}$, and $F = \dfrac{\alpha^2}{(\alpha - \beta)} + \beta$ be a G-invariant Randers-Matsumoto metric on G/H. Then the S-curvature is given by
$$S(H, \eta) = \left[\frac{\begin{array}{r} 6s^4 - (9n+15)s^3 + (6b^2n + 27n + 21)s^2 - (16b^2n + 10b^2 + 14n \\ + 14)s + 4b^2n + 10b^2 + 2n + 2 \end{array}}{2(s^2 - s - 1)(2b^2 - 3s + 1)^2} \right] \times \left(\frac{s^2 - 2s + 2}{1 - 2s} \langle [u, \eta]_{\mathfrak{l}}, u \rangle + \frac{1}{\alpha} \langle [u, \eta]_{\mathfrak{l}}, \eta \rangle \right),$$
where $u \in \mathfrak{l}$ corresponds to the 1-form β, $\mathfrak{l}$ is verified with the tangent space $T_H(G/H)$ of G/H at the origin H.

- Let G/H be reductive homogeneous Finsler space with a decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{h} + \mathfrak{l}$ and $F = \dfrac{\alpha^2}{(\alpha - \beta)} + \beta$ be a G-invariant Randers-Matsumoto metric on G/H. Then $(G/H, F)$ has isotropic S-curvature if and only if it has vanishing S-curvature.

- Let G/H be a reductive homogeneous Finsler space with a decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{h} + \mathfrak{l}$, and $F = \dfrac{\alpha^2}{(\alpha - \beta)} + \beta$ be a G-invariant special metric on G/H. Then the mean Berwald curvature E_{ij} of the homogeneous Finsler space with special (α, β)-metric is also derived.

In **Chapter 4**, we discuss the geodesic orbit of homogenous Finsler spaces and we have proved the necessary and sufficient conditions for a non-zero vector in these homogeneous spaces to be a geodesic vector with two different (α, β)-metrics. We have obtained the following results:

- Let N be a homogeneous Finsler space with Matsumoto metric $F = \dfrac{\alpha^2}{\alpha - \beta}$. Then a vector $\eta(\neq 0) \in \mathfrak{g}$ is a geodesic vector if and only if

$$\big\langle [\eta, \xi]_{\mathfrak{l}}, |\eta_{\mathfrak{l}}|\eta_{\mathfrak{l}} - 2\langle u, \eta_{\mathfrak{l}}\rangle \eta_{\mathfrak{l}} + |\eta_{\mathfrak{l}}|^2 u \big\rangle = 0,$$

  holds for every $\xi \in \mathfrak{l}$.

- Let (N, F) be a homogeneous Finsler space with special metric $F = \alpha e^{\frac{\beta}{\alpha}} + \dfrac{\beta^2}{\alpha}$. Then a non-zero vector $\eta \in \mathfrak{g}$ is a geodesic vector if and only if

$$\Bigg\langle [\eta, \xi]_{\mathfrak{l}}, \left(\exp\left(\frac{\langle u, \eta_{\mathfrak{l}}\rangle}{|\eta_{\mathfrak{l}}|} \right) - \frac{\langle u, \eta_{\mathfrak{l}}\rangle}{|\eta_{\mathfrak{l}}|} \exp\left(\frac{\langle u, \eta_{\mathfrak{l}}\rangle}{|\eta_{\mathfrak{l}}|} \right) - \frac{\langle u, \eta_{\mathfrak{l}}\rangle^2}{|\eta_{\mathfrak{l}}|^2} \right) \eta_{\mathfrak{l}} + \left(|\eta_{\mathfrak{l}}| \exp\left(\frac{\langle u, \eta_{\mathfrak{l}}\rangle}{|\eta_{\mathfrak{l}}|} \right) + 2\langle u, \eta_{\mathfrak{l}}\rangle \right) u \Bigg\rangle = 0,$$

  holds for every $\xi \in \mathfrak{l}$.

- For a homogeneous Finsler space (N, F), there exists at least one homogeneous geodesic, with Matsumoto metric $F = \dfrac{\alpha^2}{\alpha - \beta}$.

- For a homogeneous Finsler space (N, F), there exists at least one homogeneous geodesic, with metric $F = \alpha e^{\frac{\beta}{\alpha}} + \dfrac{\beta^2}{\alpha}$.

- Using above results, we discuss the geodesic vectors for a two-step nilpotent Lie group of dimension five with left-invariant (α, β)-metrics.

In **Chapter 5**, the concept of Ricci curvature in Finsler geometry is discussed. Curvature properties of homogeneous Finsler spaces with (α, β)-metrics are among the most significant topics in Finsler geometry. Here, we have obtained the formulae for Ricci curvature of homogeneous Finsler spaces with special (α, β)-metrics. Based on this formula, we have discussed the condition for vanishing S-curvature for the space $(G/H, F)$. We have obtained the following results:

- A compact homogeneous Finsler space G/H with G-invariant special metric $F = \alpha + \sqrt{\alpha^2 + \beta^2}$. Then the Ricci curvature is given by Eq. (5.3.1).

- Let $(N = G/H, F)$ be a compact connected homogeneous Finsler space with G-invariant special metric $F = \alpha + \sqrt{\alpha^2 + \beta^2}$ on G/H. Suppose that (N, F) has vanishing S-curvature. Then Ricci curvature is given by

$$\begin{aligned} Ric(Z) =& Ric^{\alpha}(Z) - \frac{c^2}{4}(C^n_{q0})^2 K_{19} + \frac{c}{4}\alpha(Z)\Big(2C^n_{m0}C^q_{qm} + C^n_{qm}C^0_{qm}\Big)K_{24} \\ & - \frac{c^2}{4}\alpha^2(Z)(C^n_{ik})^2 K_{26}, \end{aligned}$$

where, $Z(\neq 0) \in \mathfrak{l}$ and $K_{19} = \dfrac{-2(s^2\sqrt{s^2+1} - s^2 - \phi)}{\phi^2\sqrt{s^2+1}}, \quad K_{24} = \dfrac{2s}{\phi}, \quad K_{26} = \dfrac{-s^2}{\phi^2}.$

Chapter 6 focuses on some properties of Finsler space with (α, β)-metrics. In this chapter we discuss the nonholonomic Finsler frames, hypersurface and projective flatness of Finsler space with (α, β)-metrics. Nonholonomic frames have been studied by many geometers and the concept of nonholonomic Finsler frames was introduced by P. R. Holland in 1982, when he studied electromagnetism by considering the charged particles moving in an external electromagnetic field. Many researchers have worked on this concept with different (α, β)-metrics. In 1985, M. Matsumoto studied the theory of Finslerian hypersurfaces, a hyperplane of the first kind, a hyperplane of the second kind, and a hyperplane of the third kind are three different forms of Finslerian hypersurfaces that he investigated. Next we have discussed the projectively flat Finsler spaces with special (α, β)-metric. The condition for a Finsler space to be projectively flat was studies by L. Berwald and this work was completed by M. Matsumoto. We have discussed the following results:

- For the deformed Finsler metric $F = (\alpha+\beta)\left(\dfrac{\alpha^2}{\alpha-\beta}\right)$, we have obtained Finsler invariants $\rho, \rho_0, \rho_{-1}, \rho_{-2}$ which satisfies the condition, $\rho_{-1}\beta + \rho_{-2}\alpha^2 = 0$. Then we have constructed a nonholonomic Finsler frames, which are given by,

  $$V_j^i = X_k^i Y_j^k,$$

  where X_k^i and Y_j^k are given by Eqs. (6.2.13) and (6.2.14) respectively.

- For the deformed Finsler metric $F = (\alpha+\beta)\left(\alpha+\beta+\dfrac{\beta^2}{\alpha}\right)$, we have obtained Finsler invariants $\rho, \rho_0, \rho_{-1}, \rho_{-2}$ which satisfies the condition, $\rho_{-1}\beta + \rho_{-2}\alpha^2 = 0$. Then we have constructed a nonholonomic Finsler frames, which are given by,

  $$V_j^i = X_k^i Y_j^k,$$

  where X_k^i and Y_j^k are given by Eqs. (6.2.16) and (6.2.17) respectively.

- By considering the hypersurface of a Finsler space with generalized Matsumoto metric, we have obtained the following results:

  (a) The induced metric structure of the generalized Matsumoto metric on the hypersurface F^{n-1} and obtained the scalar function $b(x)$ given by $b_i(x(u)) = \sqrt{\dfrac{b^2}{1+(m^2+m)b^2}}N_i$ and $b^i = \sqrt{b^2\left((m^2+m)b^2+1\right)}N^i + \dfrac{mb^2}{\alpha}\eta^i$, where N_i is a unit normal vector.

  (b) For the generalized Matsumoto metric on the Finsler hypersurface F^{n-1} the second fundamental tensor is given by $M_{\alpha\beta} = \dfrac{m}{2\alpha}\sqrt{\dfrac{b^2}{1+(m^2+m)b^2}}h_{\alpha\beta}, \quad M_\alpha = 0$.

  (c) Further, using Matsumoto's results, we have discussed the properties of hypersurface $F^{n-1}(c)$ that it is a hyperplane of a first and second kind but not of third kind.

- A Finsler space F^n with an (α,β)-metric $F = \alpha + \sqrt{\alpha^2+\beta^2}$ provided $b^2 \neq 1$ is projectively flat if and only if the associated Riemannian space (N^n, α) is projectively flat and $b_{i;j} = 0$.