A THESIS ENTITLED

THE STUDY ON CONTACT AND PARACONTACT MANIFOLDS

Submitted to the

Faculty of Science and Technology

KUVEMPU UNIVERSITY

For the Award of the Degree of

Doctor of Philosophy

in

MATHEMATICS

by

BHANUMATHI N

Research Supervisor

Dr. VENKATESHA

Professor

Department of P.G. Studies and Research in Mathematics,

Jnana Sahyadri, Shankaraghatta - 577 451,

Shivamogga, Karnataka, India.

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DECLARATION

I hereby declare that the thesis entitled **A Study on Contact and Paracontact Manifolds**, submitted to the Faculty of Science and Technology, Kuvempu University for the award of the degree of Doctor of Philosophy in Mathematics is the result of research work carried out by me in the Department of Mathematics, Kuvempu University under the guidance of **Dr. Venkatesha**, Professor, Department of P.G. Studies and Research in Mathematics, Kuvempu University, Jnanasahyadri, Shankaraghatta.

I further declare that this thesis or part thereof has not been previously formed the basis of the award of any degree, associateship etc., of any other University or Institution.

Place: Jnanasahyadri

Date: 05-05-2023

Bhanumathi

Bhanumathi N.

CERTIFICATE

This is to certify that the thesis entitled **The Study on Contact and Paracontact Manifolds**, submitted to the Faculty of Science and Technology, Kuvempu University for the award of the degree of Doctor of Philosophy in Mathematics by **Bhanumathi N.** is the result of bonafide research work carried out by her under my guidance in the Department of P. G. Studies and Research in Mathematics, Kuvempu University, Jnanasahyadri, Shankaraghatta.

This thesis or part thereof has not been previously formed the basis of the award of any degree, associateship etc., of any other University or Institution.

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Place: Jnana Sahyadri

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Notations

Notation	**Abbreviation**
$\otimes$	Tensor product
$\odot$	Kulkarni-Nomizu product of symmetric tensors
∇	Levi-Civita connection
J	Almost complex structure
I	Identity endomorphism
R	Riemann curvature tensor
g	Riemannian or pseudo-Riemannian metric
Ric	Ricci curvature tensor
$Ric^\sharp$	Ricci operator
s	Scalar curvature
$[Y, Z]$	Lie bracket of vector fields Y and Z
$\pounds$	Lie derivative
$Hess^f$	Hessian of smooth function f
$grad$	Gradient operator
div	Divergence operator
tr	Trace
Ker	Kernel of operator
Δ	Laplacian operator
$\mathfrak{X}(M)$	Lie algebra of all smooth vector fields on M
$\mathbb{R}^n$	Real Euclidean space of dimension n
$\mathbb{C}^n$	Complex Euclidean space of dimension n
$\mathbb{S}^n$	Sphere of dimension n
$\mathbb{H}^n$	Hyperbolic space of dimension n

d	Exterior derivative
$C^\infty(M)$	Set of smooth functions on M
$\mathscr{W}$	Weyl conformal curvature tensor
$\mathcal{W}^*$	$*$-Weyl conformal curvature tensor
C	Cotton tensor
B	Bach tensor
$X^\flat$	Canonical 1-form associated to the vector field X
$\mathbb{R}$	Set of real numbers
$\mathbb{N}$	Set of natural numbers
$\{u_i\}$	Orthonormal frame
$\mathbb{CP}^n$	Complex projective space
$\mathbb{CH}^n$	Complex hyperbolic space
A	Shape operator
$O(1,2)$	Pseudo-orthogonal group
$\widetilde{SL}(2,\mathbb{R})$	Special linear group
V^b	Metric dual to V

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Preface

On the 10th of June 1854, Riemann gave his famous inaugural lecture at Gottingen and discussed the foundations of geometry, introduced n-dimensional manifolds, formulated the concept of Riemannian manifolds and defined their curvature. Since every manifold admits a Riemannian metric, Riemannian geometry often helps us to solve problems of differential topology. Most remarkably, by applying Riemannian geometry, Perelman solved the famous Poincare's conjecture posed in 1904.

Under the impetus of Einstein's theory of general relativity (1915) a further generalization appeared; the positiveness of the inner product was weakened. Consequently, one has the notion of pseudo-Riemannian manifolds which is a generalization of a Riemannian manifold in which the metric tensor need not be positive-definite, but need only be a non-degenerate bilinear form, which is a weaker condition.

The theory of structures on manifolds is a very interesting and very fruitful fields of Riemannian geometry. In this thesis, we investigate Riemannian and pseudo-Riemannian manifolds admitting different types of structures. In particular, we study contact Riemannian structures, almost Kenmotsu structures, almost coKaehler structures, almost contact pseudo-Riemannian structures and almost paracontact metric structures under several geometric points of view. The entire work in the thesis has been partitioned into five chapters and are summarized as follows:

Chapter 1 gives a brief summary of the main concepts and results about almost contact manifolds and Paracontact manifolds which will be used widely in the rest of chapters.

Chapter 2 we study $\mathbb{H}$-Curvature tensor on almost Kenmotsu manifold with nullity distibution. Also we investigate Generalized Ricci Soliton on Almost Kenmotsu Manifolds.

In the beginning, we proved that if M is a locally ϕ-$\mathbb{H}$ symmetric alomost Kenmotsu manifold with characteristic vector field ξ belonging to the $(\kappa, \mu)'$-nullity distribution and $h \neq 0$, then the manifold M is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4) \times \mathbb{E}^n$. Next we showed that if M is a locally ϕ-$\mathbb{H}$ symmetric alomost Kenmotsu manifold with characteristic vector field ξ belonging to the generalized $(\kappa, \mu)'$-nullity distribution and $h \neq 0$ then M is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4) \times \mathbb{E}^n$. Also we proved that if M is a locally ϕ-$\mathbb{H}$ symmetric almost Kenmotsu manifold with the characteristic vector field ξ belonging to the (κ, μ)-nullity distribution and $h \neq 0$, then the manifold M is an Einstein manifold. And if M is a locally $\phi - \mathbb{H}$ symmetric almost Kenmotsu manifold with the characteristic vector field ξ belonging to the generalized (κ, μ)-nullity distribution and $h \neq 0$, then the manifold M is Einstein. Finally, we study the two classes of almost Kenmotsu manifolds. Firstly, we study a closed generalized Ricci soliton on the Kenmotsu manifold. Secondly, we prove that if a Kenmotsu manifold M admits a generalized Ricci soliton with conformal vector field V, then M is Einstein. Next, we show that a non-Kenmotsu almost Kenmotsu $(\kappa, \mu)'$-manifold admitting a closed generalized Ricci soliton is locally isometric to the Riemannian product $\mathbb{H}^{n+1} \times \mathbb{R}^n$, provided that $\lambda - \frac{\kappa}{\beta}(2n\alpha\beta - 1) = -\frac{2}{\beta}$.

In Chapter 3, we devoted to the study of K-paracontact manifold admitting parallel Cotton tensor, vanishing Cotton tensor and to study Bach flatness on K-paracontact manifold. Also we study vanishing Cotton tensor on (κ, μ)-paracontact manifold for both $\kappa > -1$ and $\kappa < -1$. Further we study Yamabe and Quasi Yamabe soliton on (κ, μ)-paracontact manifold and K-paracontact manifold. First we consider M to be a K-paracontact manifold. Then M has constant scalar curvature if and only if $C(X, \xi)\xi = 0$. Next, we show that if M is a K-paracontact metric manifold, then M has parallel Cotton tensor if and only if M is an η-Einstein manifold and $r = -2n(2n + 1)$. Also we proved that if M is an η-Einstein K-paracontact manifold, and is Bach flat then M is an Einstein manifold. Also, we prove that if M is a (κ, μ)-paracontact manifold for $\kappa \neq 1$, and if M has vanishing Cotton tensor for $\mu \neq \kappa$ then M is an η-Einstein manifold. Next, we study Yamabe and quasi Yamabe soliton on (κ, μ)-paracontact manifold and K-paracontact manifold. Here we prove that, if M is non-para-Sasakian manifold and admits Yamabe soliton for the potential vector field V, then either V is Killing, or M is locally isometric to the product of a flat $(n+1)$-dimensional manifold and n-dimensional manifold of constant negative curvature equal to -4. Next we prove that if a non-para-Sasakian (κ, μ)-paracontact manifold admits a quasi Yamabe gradient soliton then for $\kappa > -1$, M is either $N(\frac{1-n}{n})$-manifold, or M is locally isometric to the product of a flat $(n+1)$-dimensional manifold and n-dimensional manifold of constant negative curvature equal to -4, or the potential function f is constant on M. For $\kappa < -1$ either $\mu \neq \frac{-4}{n+1}$ or the potential function f is constant on M. Lastly, we show that, if a K-paracontact metric g with $Q\varphi = \varphi Q$ represents a quasi Yamabe gradient soliton then either the scalar curvature $r = -2n(2n+1)$, or the potential function f is a constant.

In Chapter 4, we study some geometric properties of extended quasi generalized φ-recurrent para-Kenmotsu manifolds. And a proper example is also provided to demonstrate the existence of an extended quasi-generalized φ-recurrent Kenmotsu manifold. Also we study C-Bochner pseudosymmetric para-Kenmotsu manifold. Firstly we proved that if M is a para-Kenmotsu manifold and if M is an extended quasi φ- recurrent manifold, then M is super generalized Ricci-recurrent manifold. Also we show that if a para-Kenmotsu manifold M is an extended quasi φ- recurrent manifold, then M is an Einstein manifold. Moreover, the associated vector fields χ_1 and χ_2 of 1-forms Π_1 and Π_2 respectively are co-directional. And if a para-Kenmotsu manifold M admitting an extended quasi generalized φ-recurrent, then M is of constant sectional curvature -1. Next we prove that if M is a para-Kenmotsu manifold and if M is an extended quasi φ- recurrent manifold, then the 1-forms Π_1 and Π_2 are related by the equation $dr(W) = [2n(2n+1)+r]\Pi_1(W) - 2(n+1)(2n+1)\Pi_2(W)$. Finally we showed that if a n-dimensional para-Kenmotsu manifold M is C-Bochner Pseudo-symmetric then M_n is locally isometric to a sphere or $L_B = 1$. And we examine if a n-dimensional para-Kenmotsu manifold M satisfies $B(\xi, X) \cdot B = 0$ then M is isometric to a hyperbolic space. Later we showed that an n-dimensional para-Kenmotsu manifold satisfying the condition $B(\xi, X) \cdot R = 0$ is locally isometric to a sphere or $\tau = 2n$. Also we proved that a n-dimensional para-Kenmotsu manifold satisfying $B(\xi, X) \cdot S = 0$ is an Einstein manifold.

in the final Chapter 5 focuses on the study some symmetric properties on $(LCS)_n$-Manifolds. First, we show that a B-pseudosymmetric $(LCS)_n$-manifold is an Einstien manifold if $L_B \neq (\alpha^2 - \rho)$ and we show that if $(LCS)_n$ manifold is Q-pseudosymmetric, then it is an Einstein manifold. Also show that a Q-pseudosymmetric $(LCS)_n$-manifold is Q-semisymmetric if and only if $L_Q = 0$. Finally we show that a Q-Ricci semisymmetric $(LCS)_n$-manifold is an Einstien manifold if $\Psi \neq (\alpha^2 - \rho)(n-1)$. In addition to this we study conditions $Q(\xi, X) \cdot Q(Y, U)Z = 0$ and Q-pseudosymmetric and ϕ-Q-flat on $(LCS)_n$-manifolds. Next, we show that the geometric aspects of a Reeb vector field ξ and an orthogonal vector field V on a Lorentzian para-Sasakian manifold M is a conformal vector field, then it is Killing on M. Next, we prove that if an infinitesimal contact transformation on a Lorentzian para-Sasakian manifold is a holomorphically planar conformal vector field, then it is either collinear with ξ, or strictly infinitesimal contact transformation of M. And finally we showed that if M is a Lorentzian para-Sasakian manifold and V is a orthogonal vector field which is non-zero, then V never be a holomorphically planar conformal vector field on M.

Chapter 1

Introduction

Geometry is an essential part of mathematics concerned with shape, size and corresponding position of figures and with properties of spaces. Many pioneers studied and proved that geometry plays a vital role in describing the beauty of nature in a systematic and effective manner. Geometry was put into a axiomatic form by a great Greek mathematician Euclid. Later in 1854, Bernhard Riemann's ideas concerning geometry of space had a profound effect on the development of modern theoretical physics. The geometry of space was made by Riemann, different to the hyperbolic geometry of Bolyai and Lobachevsky which came to known as elliptic geometry. Riemann developed Riemannian geometry as well as the concept of a manifold, which generalized the ideas of curves and surfaces.

The succesful integration of ideas enabled Riemann to advance when constructing both particular cases of non-Euclidean spaces and a theory of arbitrarily curved spaces. Firstly, Riemann discovered an elliptical geometry which was the opposite to the hyperbolic geometry of Lobachevski. Thus, he was the first to indicate the possiblity of a finite geometrical space. The idea immediately took root and brought about the question as whether our physical space is finite or not. Secondly, he had the courage to build much more general geometries than Euclid's or narrowly the non Euclidean geometry. Nowadays, this geometry is referred to as Riemannian geometry. This geometry is also referred as the second non-Euclidean geometry on a three dimensional hypersphere. The essential property of this three dimensional space is that its volume is finite, so that if a point moves in the same direction it may eventually return to the starting point. Instead of straight lines in Euclidean space, we have in Riemannian spherical geometry of geodesics, or the arcs of great circles.

In the word of Chern, *fundamental objects of study in differential geometry are manifolds*. Roughly, an n-dimensional manifold is a mathematical object that *locally* looks like R^n. The theory of manifolds has a long and complicated history. For centuries, manifolds have been studied as subsets of Euclidean space, given for examples as level sets of equations. The term *manifold* goes back to the 1851 thesis of Benhard Riemann, *Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse* (foundations for a general theory of functions of complex variable) and his 1854 habilitation address *Über die Hypothesen, welche der Geometrie zugrundle liegen* (on the hypotheses underlying geometry). However, in neither refrence Riemann makes an attempt to give a precise definition of the concept. This was done subsequently by many authors, including Riemann himself. Henri Poincaré in his 1895 work *analysis situs*, introduces the idea of a manifold atlas

Albert Einstein's theory of General Relativity from 1916 gave a mojor boost to this new point if view; In his theory, space-time was regarded as a 4-dimensional *curved* manifold with no distinguished coordinates (not even a distinguished separation into space and time) a local observer may want to introduce local (x, y, z, t) coordinates to perform measurements, but all physically meaningful quantities must admit formulations that are coordinate-free. At the same time, it would seem unnatural to try to embed the 4-dimensional curved space-time continuum into some higher dimensional flat space, in the absence of any physical significance for additional dimensions. Some years later, *gauge theory* once again emphasized coordinate-free formulations, and provided physics motivations for more elaborate constructions such as fiber bundles and connections.

In 1930, Schouten and Dantzing initiated to transfer the result of differential geometry of spaces with Riemannian metric, and affine connection to the case of spaces with complex structures. These spaces were also investigated and nurtured by Kähler in 1933 and are now familiarly known as Kähler spaces, which are even dimensional. After 1960, a great deal of work is carried out on Kähler manifolds using the complex structures and differential 1-forms on manifolds. These are known as contact manifolds and are odd dimensional. Contact geometry has been widely used to analyse various physical phenomena, and connected to distinct mathematical structures. The contact structures have wide connection with Riemannian geometry and low dimensional topology,‘ and provide an interesting class of sub elliptic operators. In connection with this, many geometers investigated the different structures Sasakian, K-Contact, Kenmotsu, trans-Sasakian, para-Saskian and others by providing further condition to the contact structures.

In differential geometry, a manifold is a topological space that locally looks like Euclidean space close to each point. In particular, each points of an n-dimensional manifold has a neighbourhood that is homeomorphic to n dimensional Euclidean space. The line and circles are the one-dimensional manifold and surface is the two-dimensional manifold. Plane, Sphere and torus are the three dimensional manifold.

Let M be a Hausdroff topological space. If each point $p \in M$ has a neighbourhood U which is homeomorphic to an open set E in R^n, then M is called an n-dimensional **topological manifold.** A n-dimensional topological manifold with a globally defined differential structure is called **differential manifold** or if n-dimensional topological manifold M has a coordinate neighbouhood system S of class C^r, then M is called n-dimensional **differential manifold**.

A differentiable function $\alpha : I \longrightarrow R^3$ is called **curve** in R^3 that is $\alpha(t) =$ $(\alpha_1(t), \alpha_2(T),$

$\alpha_3(t))$ for all t in open interval I. Straight lines and helices are examples for curves. Differential forms are obtained by adding and multiplying real valued functions with differentials dx_1, dx_2, dx_3 of natural coordinate functions. Associative and distributive laws are commonly holds true for differential forms, but commutative under multiplication is not true for these forms. A **tangent vector** W_P includes two points, one is vector part W and other one is point of application P, and the union of all tangent vectors is called **tangent space** and is denoted by T_pM or $\chi(M)$. A differentiable real valued function f

is said to have **derivative** with respect to W_P if it satisfies $W_P[f] = \frac{d(f(p+tW))}{dt}$ at $t = 0$.

The fundamental ideas of carrying properties along a curve or family of curves in a consistent and similar manner is called **connection**. There are various types of connections in differential geometry, depending on what sort of properties are required to carry. In Differential geometry we are using many connections particularly, affine connection, quarter symmetric connection, quarter symmetric non metric connection, semi symmetric connection, Tanaka Webster Okumar connection (g-TWO), generalized symmetric metric connection, canonical connections and many more. The affine connection is the most important type of connection, gives a means for transporting tangent vectors to a manifold from one point to another along a curve. All connections are typically defined in the form of covariant derivative, which gives the means for taking directional derivatives of vector fields: the infinitestimal transport of a vector field in a given direction.

In the following, we present a breif summary of results about a contact manifolds, almost contact manifolds, almost para-contact metric manifolds, Lorentzian manifolds which will be widely used to study our main concepts.

1.1 Contact manifold

Contact Riemannian manifolds are an odd dimensional analogue of symplectic manifolds and has been used as a proper different context(particularly) those related to physics. It has been used as a proper framework for classical thermodynamics, and as a geometrical approach to magnetic field. Also, it was studied in relation with the Yang-Mills theory, quantum mechanics, gravitational waves etc.

A differentiable manifold M is said to be a contact manifold, if it carried a global differential 1-form η such that [3]

$$\eta \wedge d\eta^n \neq 0, \tag{1.1}$$

everywhere on M. For a given contact form η, it is well known that there exists a unique vector field ξ, called the characterstic vector field of η satisifying

$$\eta(\xi) = 1 \quad and \quad d\eta(X, \xi) = 0, \tag{1.2}$$

for every vector field X on M.

A Riemannian metric g is said to be associated with a contact manifold if there exists a tensor field ϕ of type $(1,1)$, such that

$$\eta(X) = g(X, \xi), \tag{1.3}$$

$$d\eta(X, Y) = g(X, \phi Y), \tag{1.4}$$

$$\phi^2 = -I + \eta \otimes \xi, \tag{1.5}$$

for every vector fields X, Y on M, where I denotes the identity map of the tangent space T_pM and the symbol $\otimes$ is the tensor product. Then the structure (ϕ, ξ, η, g) on M is called a contact metric structure and the manifold M equipped with such a structure is known as contact metric manifold.

Example 1.1.1. The simple contact Riemannian manifold is $R^{2n+1}(x^1, \cdots, x^n, y^1, \cdots, y^n, z)$ with the contact form

$$\eta = \frac{1}{2}\left(dz - \sum_{i=1}^{n} y^i dx^i\right).$$

The Reeb vector field ξ is $2\frac{\partial}{\partial z}$ and the contact subbundle D is spanned by $X - i = \frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial z}, X_{n+i} = \frac{\partial}{\partial y^i}, i = 1, \cdots, n$. The Riemannian metric

$$g = \eta \otimes \eta + \frac{1}{4} \sum_{i=1}^{n} \left((dx^i)^2 + (dy^i)^2\right),$$

gives a contact metric structure on R^{2n+1}. The tensor field ϕ is given by the matrix

$$\begin{bmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & y^j & 0 \end{bmatrix}$$

and the vector fields $X_i = 2\frac{\partial}{\partial y^i}, X_{n=i} = 2\frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial z}$ and ξ forms a $\phi-$basis for the contact metric structure.

Consider the restriction of ϕ to the contact subbundle $\mathcal{D}$ (defined by $\eta = 0$), and denote this by $\mathcal{J}$. Then $\mathcal{J}^2 = -id$ and $G = -(d\eta)(\cdot, \mathcal{J}\cdot)$ defines the almost Hermitian structure on $\mathcal{D}$. Thus $(M, \eta, \mathcal{J})$ a contact strongly pseudo-convex integrable CR-manifold (see [80]). We call $(M, \eta, \mathcal{J})$ a contact strongly pseudo-convex integrable CR-manifold when the complex distribution $\{X - i\mathcal{J}; X \text{ in } \mathcal{D}\}$ is integrable. Tanno [80] give this integrability condition by

$$(\nabla_W \phi)X = g(W + hW, X)\xi - \eta(X)(W + hW), \tag{1.6}$$

where ∇ is the Riemannian connection of g. The following identities are valid for every contact Riemannian manifold;

$$\nabla\xi = -\phi - \phi h, \tag{1.7}$$

$$S(\xi,\xi) = trace_g(\ell) = 2n - ||h||^2, \tag{1.8}$$

$$(\nabla_{\phi W}\phi)\phi X + (\nabla_W \phi)X = 2g(W,X)\xi - \eta(X)W - \eta(W)\eta(X)\xi, \tag{1.9}$$

$$\nabla_\xi h = \phi - \phi h^2 \phi \ell, \tag{1.10}$$

$$Div(\phi h)W = 2n\eta(W) - S(\xi, W), \tag{1.11}$$

where Div is the dovergence operator.

If the characterstic vector field ξ is Killing (equivalently, $h = 0$) with respect to g, then the contact Riemannian manifold M is said to be K-*contact* and the following identities are valid on it:

$$\nabla_W \xi = -\phi W, \tag{1.12}$$

$$R(\xi, W)X = (\nabla_W \phi)X. \tag{1.13}$$

A contact metric structure on M is said to be *Sasakian* if the almost Käehler structure on the metric cone $(M \times \mathbb{R}^+, r^2 g + dr^2)$ over M, is Kähler. Also, a normal contact Riemannian manifold is called Sasakian. Moreover, M is Sasakian if and only if any of the following identities hold on M

$$(\nabla_W \phi)X = g(W,X)\xi - \eta(X)W, \tag{1.14}$$

$$R(W,X)\xi = \eta(X)W - \eta(W)X, \tag{1.15}$$

$$Q\xi = 2n\xi. \tag{1.16}$$

It is easy to observe that every Sasakian manifold is K-contact, but the converse need not be true, except in dimension 3. We know for a fact that in a Sasakian manifold, the Ricci operator Q commutes with ϕ; but this commutativity need not hold for a contact Riemannian manifold. In this setting, we recall the following lemma (see [39]):

Lemma 1.1.1. *On a K-contact manifold M we have*

$$(\nabla_W Q)\xi = (Q\phi - 2n\phi)W, \tag{1.17}$$

$$(\nabla_\xi Q)W = (Q\phi - \phi Q)W. \tag{1.18}$$

Blair et al [5] introduced a contact Riemannian (κ, μ)-manifold which is a contact Riemannian manifold M whose curvature tensor satisfies

$$R(W, X)\xi = \kappa\{\eta(X)W - \eta(W)X\} + \mu\{\eta(X)hW - \eta(W)hX\}, \tag{1.19}$$

for some real numbers (κ, μ). Later on, BoeckX [6] classified these manifolds completely. In particular, if $\mu = 0$, then contact Riemannian (κ, μ)-manifold, introduced by Tanno [79]. On contact Riemannian (κ, μ)-manifolds, the following identities hold true:

$$h^2 = (\kappa - 1)\phi^2, \qquad \kappa < 1, \tag{1.20}$$

$$Q = [2(n-1) - n\mu]id + [2(n-1) + \mu]hid + [2(1-n) + n(2\kappa + \mu)]\eta \otimes \xi, \tag{1.21}$$

$$Q\xi = 2n\kappa\xi. \tag{1.22}$$

Moreover, the scalar curvature τ of such a manifold is given by

$$\tau = 2n(2(n-1) + \kappa - n\mu), \tag{1.23}$$

which is constant. On a contact Riemannian (κ, μ)-manifold we have

$$(\nabla_W Q)\xi = Q(\phi + \phi h)W - 2n\kappa(\phi + \phi h)W, \tag{1.24}$$

$$(\nabla_\xi Q)X = \mu(2(n-1) + \mu)h\phi X, \tag{1.25}$$

for any vector field X on M. This class of manifold contains Sasakian manifolds (for $\kappa = 1$) and the trivial sphere bundle $\mathbb{E}^{n+1} \times \mathbb{S}^n(4)$ (for $\kappa = \mu = 0$). In this connection, we reveal the following result (see [4]):

Lemma 1.1.2. *A contact metric manifold M with $R(W, X)\xi = 0$ is locally isometric to the trivial sphere bundle $\mathbb{E}^{n+1} \times \mathbb{S}^n(4)$.*

Lemma 1.1.3. *If ν is a smooth function on a contact Riemannian manifold M such that $d\nu = (\xi\nu)\eta$ (d denotes the operator of exterior differentiation), then ν is constnat on M.*

1.2 Almost Contact manifold

As a topological point of view, smooth manifolds with almost contact structures were studied by Gary, Boothby and Hatakeyama. Thereafter, many geometers applied this concept to define different manifolds and their geometric properties. In those, one of the predominant manifolds are K-contact, Sasakian, Kenmotsu, cosympletic and trans-Sasakian manifolds.

A differentiable manifold M is said to have an almost contact structure (ϕ, ξ, η), if it carries a tensor field ϕ of type $(1, 1)$ a vector field ξ and a 1-form η on M, such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad \phi\xi = 0. \tag{1.26}$$

Thus, the manifold M equipped with this structure is called an almost contact manifold, and is denoted by (M, ϕ, ξ, η). If g is a Riemannian metric on an almost contact manifold M such that,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X), \tag{1.27}$$

$$g(X, \phi Y) = -g(\phi X, Y), \tag{1.28}$$

$$(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y), \tag{1.29}$$

where X and Y are vector fields defined on M, then M is said to have an almost contact metric structure (ϕ, ξ, η, g) and M with this structure is called an almost contact metric manifold, which is denoted by (M, ϕ, ξ, η, g).

If on (M, ϕ, ξ, η, g), the exterior derivative of 1-form η satisfies

$$d\eta(X, Y) = g(X, \phi Y), \tag{1.30}$$

then (ϕ, ξ, η, g) is said to be a contact metric structure and M equipped with a contact metric structure is called a contact metric manifold.

In a contact manifold, the Riemannian curvature tensor and Ricci curvature are given respectively by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{1.31}$$

$$S(X, Y) = R(e_i, X, Y, e_i). \tag{1.32}$$

An almost contact metric manifold is said to be Sasakian manifold [3, 67] if and only if

$$(\nabla_Z \phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{1.33}$$

where ∇ is the operator of covariant defferentiation with respect to g. In a Sasakian manifold M, the following relations holds:

$$\nabla_X \xi = -\phi X, \tag{1.34}$$

$$g(R(\xi, X)Y, \xi) = g(X, Y) - \eta(X)\eta(Y), \tag{1.35}$$

$$R(\xi, X)\xi = \eta(X)\xi - X, \tag{1.36}$$

$$S(X, \xi) = 2n\eta(X), \tag{1.37}$$

for any vector fields X, Y and Z. In the whole thesis R, S, Q, and r respectively indicates the Riemannian curvature tensor of type $(1, 3)$, Ricci tensor of type $(0, 2)$, Ricci operator and scalar curvature tensor of $(2n + 1)$-dimensional manifolds. An almost contact metric manifold is a Kenmotsu manifold if and only if [49]

$$(\nabla_X \phi)Y = -g(X, \phi Y)\xi - \eta(Y)\phi X, \tag{1.38}$$

$$\nabla_X \xi = X - \eta(X)\xi, \qquad (\nabla_X \eta)Y = g(\nabla_X \xi, Y), \tag{1.39}$$

for all vector fields X and Y. In Kenmotsu manifolds the following relations holds true;

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{1.40}$$

$$S(X, \xi) = -2n\eta(X), \tag{1.41}$$

for any vector fields X, Y, Z.

The Riemannian curvature tensor plays a fundamental role in Riemannian geometry and the curvature tensor R completely defined by sectional curvature of a manifold. For any point $p \in M$ and any plane section $\pi \subseteq T_pM$, the sectional curvature $K(\pi)$ is defined by $K(\pi) = g(R(X,Y)Y,X)$, where X and Y are orthonormal vector fields in π and also indicates $K(\pi)$ by $K(X \wedge Y)$. A Riemannian manifold with constant sectional curvature c is called as real-space-form and its curvature tensor satisfies the condition

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}. \tag{1.42}$$

The above condition become Euclidean space at $c = 0$, the sphere at $c > 0$ and the hyperbolic space for $c < 0$.

A similar manner can be found in the study of complex manifolds from a Riemannian point of view. If (M, J, g) is a Kähler manifold with holomorphic sectional curvature $K(X \wedge JX) = c$, then it is said to be a complex space-form and it is well-known that its curvature tensor is given by

$$R(X,Y)Z = \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(X,JZ)JY - g(Y,JZ)JX + 2g(X,JY)JZ\}. \tag{1.43}$$

The models now are complex projective space (cP^n) if $c > 0$, complex hyperbolic space (cH^n) if $c < 0$ and Euclidean space (c^n) if $c = 0$.

Now, we give some definitions which are widely used in the rest of the chapters:

Definition 1.2.1. On almost contact Riemannian manifold M, if the Ricci operator satisfies

$$Q = \alpha id + \beta\eta \otimes \xi, \tag{1.44}$$

where both α and β are smooth functions on M, then M is said to be an *η-Einstein manifold* and the associated metric g is referred to as *η-Einstein metric.* Naturally, an *η-Einstein manifold* with $\beta = 0$ and α, a constant becomes the more familiar *Einstein manifold.*

Definition 1.2.2. A vector field V on a manifold M is said to be an *infinitesimal contact transformation* (or often called *contact vector field*) if

$$\pounds_V \eta = \sigma\eta, \tag{1.45}$$

where $\sigma \in C^\infty(M)$ and $\pounds$ denotes Lie derivative operator. V is strictly infinitesimal contact transformation when $\sigma = 0$. It is known from Blair (see page 34 in [4]) that a vector field V is contact if and only if there is a function ν on M such that

$$V = -\frac{1}{2}\phi g d\nu + \nu\xi, \tag{1.46}$$

namely one puts $\sigma = (\xi\nu)$.

Definition 1.2.3. A vector field V is called *conformal* if there exists $\rho \in C^\infty(M)$ such that

$$\pounds_V g = 2\rho g, \tag{1.47}$$

where ρ is constant (resp. $\rho = 0$). For a conformal vector field, we call up the following formulas (see Yano [99]):

$$\begin{aligned}(\pounds_V R)(W, X)Y &= g(\nabla_W gd\rho, Y)X - g(\nabla_X gd\rho, Y)W \qquad (1.48)\\ &\quad + g(W, Y)\nabla_X gd\rho - g(X, Y)\nabla_W gd\rho,\\ (\pounds_V S)(W, X) &= -(2n-1)g(\nabla_W gd\rho, X) - (\Delta_\rho)g(W, X), \qquad (1.49)\\ \pounds_V \tau &= -2\rho\tau - 4n\Delta\rho, \qquad (1.50)\end{aligned}$$

for any vector fields W, X, Y on M, where R is the Riemannian curvature tensor, ∇ is the Levi-Civita connection of g, S is the Ricci tensor.

1.3 Almost Kenmotsu manifolds

In literature, Kenmotsu manifolds were firstly introduced and investigated by Kenmotsu [49] in 1972. Such manifolds were generalized to almost Kenmotsu manifolds by Janssens and Vanhecke [47] in 1981. One of the main reasons that people are interested in Kenmotsu geometry lines in the fact that a Kenmotsu manifold of constant sectional curvature is locally isometric to the hyperbolic space $H^{2n+1}(-1), n \geq 1$ (see [49]). This result was generalized to almost Kenmotsu manifolds (see [33]), namely an almost Kenmotsu manifold of constant sectional curvature is locally isometric to $H^{2n+1}(-1), n \geq 1$. This means that in some sense, geomety of Kenmotsu manifolds corresponds to that of the hyperbolic spaces.

According to Janssens and Vanhecke, an almost contact Riemannian manifold with structure (ϕ, η, ξ, g) is said to be an *almost Kenmotsu manifold* if $d\eta = 0$ and $d\omega = 2\eta \wedge \omega$, where ω is the fundamental two-form and is defined by $\omega = g(\cdot, \phi)$. In an almost Kenmotsu manifold we have the following relations [47, 61, 33]:

$$\nabla_X \xi = X - \eta(X)\xi + h'X, \tag{1.51}$$

$$\nabla_\xi h = \phi - 2h - \phi h^2 - \phi l, \tag{1.52}$$

$$l - \phi l \phi = 2(\phi^2 - h^2), \tag{1.53}$$

$$R(W, X)\xi = \eta(W)(X + h'X) - \eta(X)(W + h'W) + (\nabla_W h')X - (\nabla_X h')X, \tag{1.54}$$

$$S(\xi, \xi) = trace_g(l) = g(Q\xi, \xi) = -2n - trace_g(h^2). \tag{1.55}$$

An almost Kenmotsu manifold with $h \neq 0$ is called *proper* (or sometimes called *strict*) almost Kenmotsu manifold. An almost Kenmotsu manifold such that $h = 0$ is locally isometric to a wrapped product $[-\epsilon, \epsilon] \times_f N^{2n}$, N^{2n} being an almost Kähler manifold and $f = cexp(t), c$ a positive constant. In particular, if M is Kenmotsu manifold, that is, M is normal, then N^{2n} is Kähler.

Let M be a proper almost Kenmotsu 3-manifold and U be an open subset of M, then there exists a local orthonormal basis $\{\xi, \nu, \phi\nu\}$. On U, we set $h\nu = \gamma\nu$ and hence $h\phi\nu = -\gamma\phi\nu$, where γ is a positive function on U. Applying (1.51), the following result was revealed by Cho and Kimura (See [21]);

Lemma 1.3.1. *On U we have*

$$\begin{aligned} &\nabla_\xi \xi = 0, \qquad \nabla_\xi \nu = a\phi\nu, \qquad \nabla_\xi \phi\nu = -a\nu, \\ &\nabla_\nu \xi = \nu - \gamma\phi\nu, \qquad \nabla_\nu \nu = -\xi - b\phi\nu, \qquad \nabla_\nu \phi\nu = \gamma\xi + b\nu, \\ &\nabla_{\phi\nu} \xi = -\gamma\nu + \phi\nu, \qquad \nabla_{\phi\nu} \nu = \gamma\xi + c\phi\nu, \qquad \nabla_{\phi\nu} \phi\nu = -\xi - c\nu, \end{aligned} \tag{1.56}$$

where a, b, c are smooth fuctions.

An almost Kenmotsu manifold M is said to be an *almost Kenmotsu $(\kappa, \mu)'$-manifold* if it satisfies

$$R(W, X)\xi = \kappa\{\eta(X)W - \eta(W)X\} + \mu\{\eta(X)h'W - \eta(W)h'X\}, \tag{1.57}$$

for some constants κ and μ. Also, if M is satisfying

$$R(W, X)\xi = \kappa\{\eta(X)W - \eta(W)X\} + \mu\{\eta(X)hW - \eta(W)hX\}, \tag{1.58}$$

then M called *almost Kenmotsu (κ, μ)-manifold.* Suppose $\kappa, \mu \in C^\infty(M)$, then (1.57) and (1.58) respectively are referred to as *generalized almost Kenmotsu $(\kappa, \mu)'$-manifold* and *generalized almost Kenmotsu (κ, μ)-manifold.* The following identities hold on an almost Kenmotsu $(\kappa, \mu)'$-manifold (also (κ, μ)-space):

$$h'^2 = (\kappa + 1)\phi^2, \quad or \quad h^2 = (\kappa + 1)\phi^2, \tag{1.59}$$

$$Q\xi = 2n\kappa\xi. \tag{1.60}$$

Let M be an almost Kenmotsu $(\kappa, \mu)'$-manifold and $h' \neq 0$. Then $\kappa < -1, \mu = -2$ and M is locally isometric to the warped products

$$B^{n+1}(\kappa + 2\vartheta) \times_f R^n, \qquad H^{n+1}(\kappa - 2\vartheta) \times_f R^n,$$

where $\vartheta = \sqrt{-(\kappa+1)}$, $B^{n+1}(\kappa+2\vartheta)$ is a space of constant curvature $\kappa+2\vartheta \leq 0$, $H^{n+1}(\kappa-2\vartheta)$ is the hyberbolic space of constant curvature $\kappa - 2\vartheta < -1$, $f = cexp((1+\vartheta)t)$ and $f' = c'exp((1+\vartheta)t)$, with c, c' positive constants. Furthermore, M is locally isometric to $H^{n+1}(-4) \times R^n$ when $\kappa = -2$ or equivalently $\vartheta = 1$ (see [34]). Now, we recall the following important results which is found in [34]:

Lemma 1.3.2. *In a proper almost Kenmotsu $(\kappa, \mu)'$-manifold, the Ricci operator satisfy*

$$QW = -2nW + 2n(\kappa+1)\eta(W)\xi - 2nh'W. \tag{1.61}$$

Let M be a generalized almost Kenmotsu (κ, μ)-manifold such that $h \neq 0$. Then the Ricci operator can be expressed as

$$QW = -2nW + 2n(\kappa+1)\eta(W)\xi - 2(n-1)h'W + \mu hW. \tag{1.62}$$

In both spaces, the scalar curvature τ is $2n(\kappa - 2n)$.

A normal almost Kenmotsu manifold is called *Kenmotsu manifold* (see [49]), and this normality condition is expressed as

$$(\nabla_W \phi)X = g(\phi W, X)\xi - \eta(X)\phi W, \tag{1.63}$$

for any vector field $W, X \in \Gamma(TM)$. The following equalities hold for any Kenmotsu manifold (see [49]):

$$\nabla_W \xi = W - \eta(W)\xi, \tag{1.64}$$

$$R(W,X)\xi = \eta(W)X - \eta(X)W, \tag{1.65}$$

$$Q\xi = -2n\xi. \tag{1.66}$$

We may verify that a Kenmotsu manifold M is η-Einstein if and only if

$$S = \left(1 + \frac{\tau}{2n}\right) g - \left(2n + 1 + \frac{\tau}{2n}\right) \eta \otimes \eta. \tag{1.67}$$

As a result of (1.63)-(1.66), the authors in [84] revealed the following outcome:

Lemma 1.3.3. *On a Kenmotsu manifold M, the Ricci operator satisfies*

$$(\nabla_W Q)\xi = -QW - 2nW, \tag{1.68}$$

$$(\nabla_\xi Q)W = -2QW - 4nW. \tag{1.69}$$

1.4 Almost Contact Pseudo-Riemannian manifold

Studying almost contact structures with pseudo-Riemannian metrics was started by Takahashi in [76], and he just studied the Sasakian case. Afterwards, a systematic study of almost contact pseudo-Riemannian manifolds was undertaken by Calvaruso and Perrone [9] in 2010, introducing all the technical apparatus which is needed for further investigations.

If an almost contact manifold M equipped a pseudo-Riemannian metric g such that

$$g(\phi W, \phi X) = g(W,X) - \epsilon\eta(W)\eta(X), \tag{1.70}$$

where $\epsilon = \pm 1$, then (M, g) is said to be an *almost contact pseudo-Riemannian manifold.* From the above relation, it can be see that

$$\eta(W) = \epsilon g(\xi, W) \quad along \;\; with \;, \quad g(\phi W, X) = -g(W, \phi X),$$

for any vector field $W, X \in \Gamma(TM)$. In particular, in an almost contact pseudo-metric manifold, it follows that $g(\xi, \xi) = \epsilon$ and so, the charecteristic vector field ξ is a unit vector field, which is either space-like or time-like, but cannot be light-like. The following result can be easily obtained (see [101]):

Lemma 1.4.1. *An almost contact pseudo-Riemannian manifold is normal if and only if*

$$(\nabla_{\phi W}\phi)X - \phi(\nabla_W \phi)X + (\nabla_W \eta)(X)\xi,$$

for any vector fields $W, X \in \Gamma(TM)$.

1.5 Contact pseudo-Riemannian manifolds

An almost pseudo-Riemannian is called a *contact pseudo-Riemannian manifold* if $d\eta = \omega$, where $\omega = g(\cdot, \phi\cdot)$ is a fundamental two-form. We define a self-adjoint $(1,1)$-tensor field $h = \frac{1}{2}(\pounds_\xi \phi)$ and $\ell = R(\cdot, \xi)\xi$ and it satisfies

$$h\xi = 0 = \ell\xi, \qquad h\phi = -\phi h, \qquad trace_g(h) = trace_g(\phi h) = 0. \tag{1.71}$$

We now accumulate some relations which are valid for a contact pseudo-Riemannian manifold:

$$\nabla_W \xi = -\epsilon\phi W - \phi h W, \tag{1.72}$$

$$(\nabla_\xi h)W = \phi W - h^2\phi W + \phi R(\xi, W\xi), \tag{1.73}$$

$$trace_g(\nabla\phi) = 2n\xi, \qquad Div(\xi) = Div(\eta) = 0. \tag{1.74}$$

If Reeb vector filed ξ of contact pseudo-Riemannian manifold M is Killing (equivalently $h = 0$), then M is called *K-contact pseudo-Riemannina manifold.* A *Sasakian pseudo-Riemannian manifold* is a contact pseudo-Riemannian manifold whose almost contact structure (ϕ, ξ, η) is normal, and this normality can be expressed as

$$(\nabla_W \phi)X = g(W, X)\xi - \epsilon\eta(X)W. \tag{1.75}$$

Any Sasakian pseudo-Riemannian manifold is always $K-$contact and the converse holds when $n = 1$, that is, for 3-dimensional spaces. It is worthwhile to mention that, on a Sasakian pseudo-Riemannian manifold we obtain

$$R(W, X)\xi = \eta(W)X - \eta(X)W. \tag{1.76}$$

In contact Riemannian case, the above equation shows that the manifold is Sasakian, but this is not valid in case of contact pseudo-Riemannian [64]. However, we call up the following (see [64]):

Lemma 1.5.1. *A K-contact pseudo-Riemannian manifold M is Sasakian if and only if the curvature tensor R satisfies (1.76).*

In [37], introduced the notion of *contact pseudo-Riemannian (κ, μ)-manifold.* According to them a contact pseudo-Reimannian (κ, μ)-manifold is a contact pseudo-Riemannian manifold whose curvature tensor R satisfies

$$R(W, X)\xi = \epsilon\kappa\{\eta(X)W - \eta(W)X\} + \epsilon\mu\{\eta(X)hW - \eta(W)hX\}, \tag{1.77}$$

for some real numbers κ, μ. For contact pseudo-Riemannian $(\kappa, \mu)-$ manifold we have the following relation (see [37]):

$$h^2 = (\epsilon\kappa - 1)\phi^2, \tag{1.78}$$

$$Q\xi = 2n\kappa\xi. \tag{1.79}$$

Ghaffarzadeh and Faghfouri reveals the following result in [37]:

Lemma 1.5.2. *In any contact pseudo-Riemannian* $(\kappa, \mu)-$*manifold* M*, the Ricci operator* Q *can be expressed as*

$$QW = \epsilon(2(n-1) - n\mu)W + (2(n-1) + \mu)hW + (2(1-n)\epsilon + 2n\kappa + n\mu\epsilon)\eta(W)\xi, \tag{1.80}$$

where $\epsilon\kappa < 1$*. Further, the scalar curvature is* $2n(2(n-1)\epsilon - n\mu\epsilon + \kappa)$*.*

1.6 Kenmotsu pseudo-Riemannian manifolds

In [92], Wang introduced the geometry of almost Kenmotsu pseudo-Riemannian manifolds, emphasizing the analogies and differences with respect to the Riemannian case and providing some technical apparatus needed for further investigations. An almost contact pseudo-Riemannian manifold with structure (ϕ, η, ξ, g) where structure operators satisfy $d\eta = 0$ and $d\omega = 2\eta \wedge \omega$ is referred to as an *almost Kenmotsu pseudo-Riemannian manifold*. A normal almost Kenmotsu pseudo-Riemannian manifold is called a *Kenmotsu pseudo-Riemannian manifold*. Equivalently, from Lemma (1.4.1) the normality of this class can be expressed as (see [92]):

$$(\nabla_W \phi)X = -\eta(X)\phi W - \epsilon g(W, \phi X)\xi, \tag{1.81}$$

for any vector fields $W, X \in \Gamma(TM)$. For any Kenmotsu pseudo-Riemannian manifold the following identities were explored [83]:

$$\nabla\xi = id - \eta \otimes \xi, \tag{1.82}$$

$$R(W, X)\xi = \eta(W)X - \eta(X)W, \tag{1.83}$$

$$Q\xi = -2n\epsilon\xi. \tag{1.84}$$

Now, we recall the following formula which is valid for any Kenmotsu pseudo-Riemannian manifold

$$(\nabla_W Q)\xi = -QW - 2n\epsilon W, \tag{1.85}$$

$$(\nabla_\xi Q)W = -2QW - 4n\epsilon W, \tag{1.86}$$

for any vector field $W \in \Gamma(TM)$. The complete proof of this can be found in [83]:

1.7 Almost Paracontact Metric manifolds

Almost paracontact structures are the natural odd-dimensional analogue of almost para-Hermition structures, just like almost contact metric structures correspond to the almost Hermition ones. The study of almost paracontact metric manifolds started in [48], and for long time focused on the special case of paraSasakian manifolds. In 2009, Zamkovoy [101] undertook a systematic study of almost paracontact metric manifolds.

A pseudo-Riemannian manifold M is said to admit an *almost paracontact structure* if the triple (ϕ, ξ, η), where ϕ is a $(1, 1)$-type tensor field, ξ is a vector field (called *Reeb vector* *field* or *characterstic vector field*) and η is one-form, satisfies the following conditions:

$$\phi^2 = id - \eta \otimes \xi, \qquad \eta(\xi) = 1. \tag{1.87}$$

The tensor field ϕ induces an almost paracomplex structure on each fibre of $D = Ker(\eta)$, i.e. the ± 1-eigendistributions $D^{\pm} = D_{\phi}(\pm)$ of ϕ have equal dimension n. If an almost paracontact manifold M acquires a pseudo-Riemannian metric g such that

$$g(\phi W, \phi X) = -g(W, X) + \eta(W)\eta(X), \tag{1.88}$$

then M is called *almost paracontact metric manifold.* Moreover, we can define symmetric tensor field ω by $\omega = g(\cdot, \phi\cdot)$, usually called *fundamental two-form.* We denote by $\ell = R(\cdot, \xi)\xi$ and $h = \frac{1}{2}\pounds_{\xi}\phi$, which are symmetric and they satisfies

$$h\xi = \ell\xi = 0, \qquad trace_g(h) = trace_g(\phi h) = 0, \qquad h\phi + \phi h = 0. \tag{1.89}$$

1.8 Paracontact Metric manifolds

An almost paracontact metric manifold becomes a *paracontact metric manifold* if $d\eta = \omega$. In this case, it is not difficult to examine that η is a *contact form*, that is $\eta \wedge (d\eta)^n \neq 0$. We also have the following relations on paracontact metric manifolds

$$\begin{aligned}
\nabla_W \xi &= -\phi W + \phi h W, && (1.90)\\
\nabla_{\xi} h &= -\phi + \phi h^2 - \phi l, && (1.91)\\
Ric(\xi, \xi) &= trace_g(\ell) = trace_g(h^2) - 2n, && (1.92)\\
(\nabla_{\phi W}\phi)\phi X - (\nabla_{W\phi})X &= 2g(W, X)\xi - \eta(X)(W - hW + \eta(W)\xi). && (1.93)
\end{aligned}$$

A paracontact metric manifold M is said to be

- $K-paracontact$ if ξ is a Killing $(i.e., \pounds_\xi g = 0)$ or equivalently $h = 0$. For a $K-paracontact$ manifold M, the following equation were found in [101]:

$$\begin{aligned}\nabla_W \xi &= -\phi W, \quad (1.94)\\ R(W,\xi)\xi &= -W + \eta(W)\xi, \quad (1.95)\\ Q\xi &= -2n\xi. \quad (1.96)\end{aligned}$$

- *paraSasakian* if the paracontact structure is normal, i.e., satisfies $[\phi,\phi]+2d\eta\otimes\xi = 0$. This condition is equivalent to

$$(\nabla_W \phi)X = -g(W,X)\xi + \eta(X)W. \quad (1.97)$$

A paraSasakian manifold is in particular $K-$paracontact. The converse holds only in dimension 3 [9]. Every paraSasakian manifold satisfies

$$R(W,X)\xi = \eta(W)X - \eta(X)W, \quad (1.98)$$

$$R(W,\xi)X = g(W,X)\xi - \eta(X)W. \quad (1.99)$$

We note that unlike in the contact Riemannian case, in general the condition $h^2 = 0$ does not imply that the manifold is $K-$ paracontact (see example in [9] describe some cases with $h^2 = 0$ and $h \neq 0$).

Cappelletti-Montano et al [16] defined the notion of *paracontact* (κ,μ)*-manifold*, that is, the curvature tensor of a paracontact metric manifold satisfies

$$R(W,X)\xi = \kappa\{\eta(X)W - \eta(W)X\} + \mu\{\eta(X)hW - \eta(W)hX\}, \quad (1.100)$$

for any $\kappa, \mu \in R$. On paracontact (κ, μ)-manifolds one has (see [16]):

$$h^2 = (\kappa + 1)\phi^2, \tag{1.101}$$

$$Q\xi = 2n\kappa\xi. \tag{1.102}$$

In [102], Zamkovoy and Tzanov reveals the following outcome:

Lemma 1.8.1. *If a paracontact metric manifold M satisfies $R(W, X)\xi = 0$, then it is locally isometric to the product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of constant negative curvature -4.*

1.9 ParaKenmotsu manifolds

Recently, Zamkovoy [103] studied a class of paracontact metric manifolds satisfying some special conditions. These manifolds are analogues to Kenmotsu manifolds and they belong of the class G_6 of the classification given in [102]. He characterize these manifolds by tensor equations and study their properties. An almost paracontact metric manifold M is called *paraKenmotsu manifold* if it satisfies

$$(\nabla_W \phi)X = g(\phi W, X)\xi - \eta(X)\phi W, \tag{1.103}$$

for any vector field $W \in \Gamma(TM)$. From the definition by means of the tensor equations, it can be easily verified that the structure is normal, but quasi-paraSasakian (and hence not paraSasakian). Also, in a paraKenmotsu manifold, we have the following relations (see [103]):

$$\nabla_W \xi = W - \eta(W)\xi, \tag{1.104}$$

$$R(W, X)\xi = \eta(W)X - \eta(X)W, \tag{1.105}$$

$$Q\xi = -2n\xi. \tag{1.106}$$

1.10 Trans-paraSasakian manifolds

The author in [104] considered trans-paraSasakian manifolds as an analogue of the trans-Sasakian manifolds. A trans-paraSasakian structure is a trans-paraSasakian structure of type (α, β), where α and β are smooth functions. According to the author in [104], if an almost paracontact metric manifold M satisfies

$$(\nabla_W \phi)X = \alpha\{-g(W, X)\xi + \eta(X)W\} + \beta\{g(W, \phi X)\xi + \eta(X)\phi W\}, \tag{1.107}$$

then the manifold M said to be a *trans-paraSasakian manifold.* As a result of (1.107), it follows that

$$\nabla_W \xi = -\alpha\phi W - \beta(W - \eta(W)\xi). \tag{1.108}$$

One can easily verify that, this class of manfiold is normal. It is clear that a trans-paraSasakian manfiold of type $(1, 0), (0, 1)$ and $(0, 0)$ are respectively called paraSasakian manifold, paraKenmotsu manifold and paracosympletic manifold.

Let M be a trans-paraSasakian 3-manifold of type (α, β). Then for each point p of M, we may choose a local pseudo-orthonormal frame $\{\xi, \nu, \phi\nu\}$ on certain neighbourhood

U of p. By virtue of (1.107) and (1.108), we can find that the Levi-Civita connection ∇ of M has the following;

$$\begin{aligned} \nabla_\nu \xi = -\alpha\phi\nu - \beta\nu, \quad \nabla_{\phi\nu}\xi = -\alpha\nu - \beta\phi\nu, \quad \nabla_\xi \xi = 0, \\ \nabla_\nu \nu = \gamma\phi\nu + \beta\xi, \quad \nabla_\nu \phi\nu = -\gamma\nu + \alpha\xi, \quad \nabla_\xi \nu = \vartheta\phi\nu, \\ \nabla_{\phi\nu}\phi\nu = \delta\nu + \beta\xi, \quad \nabla_{\phi\nu}\nu = -\delta\phi\nu + \alpha\xi, \quad \nabla_\xi \phi\nu = -\vartheta\nu, \end{aligned} \tag{1.109}$$

where ϑ, δ and γ are smooth functions.

1.11 Lorentzian Manifolds

The modern study of gravitation is primarily grounded in the theory of general relartivity, where space and time are modelled together as points on 4-dimensional manifolds similar to Riemannian manifolds but without the requirment that the metric be positive-definite. The manifolds have a similar structure to their Riemannian counterparts but pick up an additional causal structure and associated physical notions. Since Einstein's development of this geometry of space-time in the early $20th$ century, the field of *Lorentizian Geometry* has flourished and there has been much interplay between the physical study of gravitation and relativity and the mathematical study of differential geometry. General relativity has been one of the most successful physical theories to date.

An n-dimensional Lorentzian manifold M is a smooth connected paracompact Hausdroff manifold with a Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of type $(0, 2)$ such that for each point $p \in M$, the tensor g_p is a non-degenerate inner product of signature $(-, +, \cdots, +)$. A non-zero vector $\Upsilon \in \Gamma(TM)$ is said to be time-like (resp., non-spacelike, null, spacelike) if it satisfies $g_p(\Upsilon, \Upsilon) < 0$ (resp., $\leq 0, = 0, > 0$).

1.12 Lorentzian concircular structure (LCS) Manifolds

The extended version of LP-Sasakian manifold is the Lorentzian concircular structure manifolds (shortly (LCS)-manifold). An $(2n+1)-$dimensional Lorentzian manifold M is a smooth connected para compact Hausdorff manifold with a Lorentzian metric g i.e., M admits a smooth tensor field g of type $(0, 2)$ such that for each point $p \in M$, the tensor $T_pM \times T_pM \longrightarrow R$ is a non degenerate inner product of signature $(-, +, \cdots -, +)$, where T_pM denotes the tangent space of M at p and R is the real number space. It is to be pointed that the most important case is $(LCS)-$manifold remains invariant under a $D-$homothetic transformation, which does not hold for an $LP-$Sasakian manifold [70].

In a Lorentzian manifold (M, g), the vector field P defined by

$$g(W, P) = A(W), \tag{1.110}$$

for any vector field $W \in (T_pM)$ is said to be concircular vector field [100], if

$$(\nabla_W A)(X) = \alpha\{g(W, X) + \omega(W)\omega(X)\}, \tag{1.111}$$

where α is a non zero scalar function, A is a $1-$ form.

In a Lorentzian manifold M of dimension $(2n+1)$ admitting a unit timelike concircular vector field ξ, called the characteristic vector field of the manifold. Then we have

$$g(\xi, \xi) = -1, \qquad g(W, \xi) = \eta(W),$$

for any vector $W \in \Gamma(TM)$. The equation of the following form holds:

$$(\nabla_W \eta)X = \alpha\{g(W, X) + \eta(W)\eta(X)\}, \qquad \alpha \neq 0, \tag{1.112}$$

that is,

$$\nabla_W = \alpha\{W + \eta(W)\xi\}, \tag{1.113}$$

where α is a nonzero scalar function satisfying $\nabla_W \alpha = W(\alpha) = d\alpha(W) = \rho\eta(W), \rho$ being a certain scalar function given by $\rho = -\xi(\alpha)$. If we put

$$\nabla_W \xi = \frac{1}{\alpha}\phi W, \tag{1.114}$$

then from (1.112) and (1.114), we can find

$$\phi W = W + \eta(W)\xi,$$

from which it follows that ϕ is a symmetric $(1, 1)$-tensor. Thus the Lorentzian manifold M together with the unit timelike concircular vector field ξ, its associated one-form η, and a $(1, 1)$-type tensor field ϕ is said to be a Lorentzian concircular structure manifold (breifly, (LCS)-manifold), which was introduced by Shaikh [70] along with their existance and applications to the general theory of relativity and cosmology. Particularly, if we take $\alpha = 1$, then we can obtain the $LP-$Sasakian structure of Matsumoto [52].

An $(2n+1)$-dimensional differentiable manifold M is called an (LCS) manifold if it admits a $(1, 1)$ tensor field ϕ, a contravatiant vector filed ξ, a 1-form η, and a Lorentzian metric

g such that [71]

$$\eta(\xi) = -1, \qquad \phi\xi = 0, \qquad \eta \circ \phi = 0, \tag{1.115}$$

$$\phi^2 W = W + \eta(W)\xi, \tag{1.116}$$

$$g(\phi W, \phi X) = g(W, X) + \eta(W)\eta(X), \tag{1.117}$$

$$(\nabla_W \phi)X = \alpha\{g(W, X)\xi + 2\eta(W)\eta(X)\xi + \eta(X)W\}, \tag{1.118}$$

where ∇ is the covariant differentiation operator of Lorentzian metric g. It is easy to see that, the following relations hold in an (LCS)-manifold [71]:

$$R(W, X)Y = (\alpha^2 - \rho)\{g(X, Y)W - g(W, Y)X\}, \tag{1.119}$$

$$R(\xi, X)Y = (\alpha^2 - \rho)\{g(X, Y)\xi - \eta(Y)X\}, \tag{1.120}$$

$$R(W, X)\xi = (\alpha^2 - \rho)\{\eta(X)W - \eta(W)X\}, \tag{1.121}$$

$$\eta(R(W, X)Y) = (\alpha^2 - \rho)\{g(X, Y)\eta(W) - g(W, Y)\eta(X)\}, \tag{1.122}$$

$$S(W, \xi) = 2n(\alpha^2 - \rho)\eta(W), \tag{1.123}$$

for all vector fields $W, X, Y \in \Gamma(TM)$ and $\alpha^2 - \rho \neq 0$.

Chapter 2

Almost Kenmotsu Manifolds

2.1 Introduction

Recently, Nurowski and Randall [58] introduced the concept of generalized Ricci soliton. Let (M, g) be a pseudo-Riemannian manifold and we say the metric g is a generalized Ricci soliton, if there is a smooth vector field V, and $\alpha, \beta, \lambda \in R$ such that

$$\pounds_V g + 2\alpha Ric + 2\beta V^b \otimes V^b = 2\lambda g, \tag{2.1}$$

where $\pounds$ represents the Lie-derivative, V^b is the metric-dual to V and Ric is the Ricci operator. In particular, if V^b is closed, then we say the generalized Ricci soliton is closed. According to Nurowski and Randall [58], the Eq. (2.1) unifies several important equations such as

- Homothetic vector field: $\pounds_V g = 2\lambda g$;
- Killing vector field: $\pounds_V g = 0$;
- Ricci soliton: $\pounds_V g + 2Ric = 2\lambda g$; [22]

- metric projective structure equation in the projective class for which the Ricci tensor is skew-symmetric:$\pounds_V g - Ric + 2V^b \otimes V^b = 0$; [65]

- the equation of vacuum near horizon geometry:$\pounds_V g + \frac{2}{n-1} Ric + 2V^b \otimes V^b = 2 \wedge g$, where $\wedge$ is the cosmological constant; [20]

- the equation of Einstein-Weyl: $\pounds_V g + \frac{2}{n-1} Ric + 2V^b \otimes V^b = 2\lambda g$. [8]

If V is the gradient of a smooth function f (i.e., $V = Df$), then (2.1) yields to

$$Hessf + \alpha Ric + \beta df \otimes df = \lambda g. \tag{2.2}$$

For $\alpha = 1$ and $\beta = 0$, the equation (2.2) is the equation of gradient Ricci soliton. Moreover, we have gradient almost Ricci soliton when $\lambda \in C^\infty(M)$. For this reason, we call Eq. (2.2) as gradient generalized almost Ricci soliton, when $\lambda \in C^\infty(M)$.

It is well known that the conformal curvature tensor is invariant under conformal transformation, and the projective curvature tensor is invariant under projective transformation. The conformal curvature tensor C and the projective curvature tensor P in a Riemannian or Lorentzian manifold of dimension $2n+1$ are defined as follows:

$$\begin{aligned} C(X,Y)Z =& R(X,Y)Z - \frac{1}{2n-1}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\} \\ & - \frac{r}{2n(2n-1)}\{g(Y,X)X - g(X,Z)Y,\} \end{aligned} \tag{2.3}$$

and

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}\{S(Y,Z)X - S(X,Z)Y,\} \tag{2.4}$$

where R is the Riemannian curvature tensor of type $(1,3)$, the Ricci operator Q is defined by $g(QU,V) = S(U,V)$, and r denotes the scalar curvature. Conformal curvature tensor and projective curvature tensor play an important role in differential geometry as well as in the theory of relativity. In 2020 U. C. De [28] and others were introduce a new tensor named $\mathbb{H}$-tensor of type $(1,3)$ which is a linear combination of conformal and projective curvature tensors and is defined by

$$\mathbb{H}(X,Y)Z = aC(X,Y)Z + (a + (2n-1)b)P(X,Y)Z, \tag{2.5}$$

where a and b are real numbers (not simultaneously zero). If $a = 1$ and $b = -\frac{1}{2n-1}$, then $\mathbb{H} \equiv C$, also if $a = 0$ and $b = \frac{1}{2n-1}$, then $\mathbb{H} \equiv P$. Since the conformal curvature tensor vanishes for $n = 3$, we consider the dimension of the manifold $n > 3$.

In the study of Riemannian manifolds (M,g), Gray [44] and Tanno [81] introduced the notion of κ-nullity distribution, which is defined for any $p \in M$ and $\kappa \in R$ as follows:

$$N_p(\kappa) = \{Z \in T_pM \colon R(X,Y)Z = \kappa[g(Y,Z)X - g(X,Z)Y]\} \tag{2.6}$$

for any $X, Y \in T_pM$, where T_pM denotes the tangent vector space of M at any point $p \in M$ and R denotes the Riemannian curvature tensor of type $(1,3)$. Moreover, if κ is a smooth function then the distribution is called generalized κ-nullity distribution.

Recently, Blair, Koufogiorgos and Papantoniou [5] introduced a generalized notion of the κ-nullity distribution on a contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, namely (κ, μ)-nullity distribution which is defined for any $p \in M^{2n+1}$ and $\kappa, \mu \in R$ as follows:

$$N_p(\kappa, \mu) = \{Z \in T_pM^{2n+1} \colon R(X,Y)Z = \kappa[g(Y,Z)X - g(X,Z)Y] + \mu[(g(Y,Z)hX - g(X,Z)hY]\}, \tag{2.7}$$

where $h = \frac{1}{2}\pounds_\xi \phi$ and $\pounds$ denotes the Lie differentiation. Next, Dileo and Pastore [33] introduced another generalized notion of the κ- nullity distribution which is as named the $(\kappa, \mu)'$-nullity distribution on an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ and is defined for any $p \in M^{2n+1}$, $\kappa, \mu \in R$ as follows:

$$N_p(\kappa, \mu)' = \{Z \in T_pM^{2n+1} \colon R(X,Y)Z = \kappa[g(Y,Z)X - g(X,Z)Y] + \mu[g(Y,Z)h'X - g(X,Z)h'Y]\}, \tag{2.8}$$

where $h' = h \circ \phi$.

Definition 2.1.1. A vector field V on a Riemannian manifold is said to be conformal if there exists a smooth function ν such that

$$\pounds_V g = 2\nu g. \tag{2.9}$$

If ν vanishes, then we say that V is Killing.

Definition 2.1.2. On an almost contact metric manifold M, a vector field V is said to be infinitesimal contact transformation if $\pounds_V \eta = \sigma\eta$, for some function σ. In particular, we call V as a strict infinitesimal contact transformation if $\pounds_V \eta = 0$.

We also have the following formulas given in ([33] - [35])

$$\begin{aligned} \nabla_X\xi &= X-\eta(X)\xi-\phi hX, && (2.10)\\ R(X,Y)\xi &= \eta(X)(Y-\phi hY)-\eta(Y)(X-\phi hX) && (2.11)\\ &+ (\nabla_Y\phi h)X-(\nabla_X\phi h)Y, \\ (\nabla_X\phi)Y-(\nabla_{\phi X}\phi)\phi Y &= -\eta(Y)\phi X-2g(X,\phi Y)\xi-\eta(Y)hX, && (2.12) \end{aligned}$$

for any X, Y on M^{2n+1}. The $(1,1)$-type symmetric tensor field $h' = h\circ\phi$ is anticommuting with ϕ and $h'\xi = 0$.

2.2 Generalized Ricci soliton on Kenmotsu manifold

In this section, we study generalized Ricci soliton on Kenmotsu manifold. In [33], Dileo and Pastore was proved that an almost Kenmotsu manifold is normal if and only if the foliations of the distribution D (where D is the distribution orthogonal to ξ, that is, $D = ker\eta$) are Kählerian and tensor field h vanishes. On any Kenmostu manifold, the following formulae hold:

$$\nabla_X\xi = X-\eta(X)\xi, \tag{2.13}$$

$$R(X,Y)\xi = \eta(X)Y-\eta(Y)X, \tag{2.14}$$

$$Q\xi = -2n\xi, \tag{2.15}$$

where R is the curvature tensor. From the (2.13) and (2.15) one can prove (for details see [84]) that

$$(\nabla_X Q)\xi = -QX - 2nX, \tag{2.16}$$

$$(\nabla_\xi Q)X = -2QX - 4nX. \tag{2.17}$$

Before going to the main results we state the following ;

Lemma 2.2.1. *For a closed m-quasi Einstein metric the following formula holds:*

$$\begin{aligned} R(X,Y)V =&\alpha((\nabla_Y Q)X - (\nabla_X Q)Y) + \alpha\beta(V^b(Y)QX - V^b(X)QY) \\ &+ \beta\lambda(V^b(X)Y - V^b(Y)X) \end{aligned} \tag{2.18}$$

Proof. Because of V^b is closed. Equation (2.1) can be written as

$$\nabla_X V = -\alpha QX - \beta V^b(X)V + \lambda X. \tag{2.19}$$

We known that the expression of curvature tensor is

$$R(X,Y)V = \nabla_Y \nabla_X V - \nabla_X \nabla_Y V - \nabla_{[X,Y]} V. \tag{2.20}$$

Now applying (2.19) in the above relation (2.20) one can directly get the relation (2.18). This completes the proof. □

Now we are entering to prove the following results

Theorem 2.2.2. *If a Kenmotsu manifold M admits a closed generalizd Ricci tensor. Then one of the following conditions occurs*

1. *V is pointwise collinear with ξ and in such a case M is η-Einstein.*
2. *V is strictly infinitesimal contact transformation.*
3. *M is Einstein.*

Proof. Replace Y by ξ in (2.18) and employ (2.16) and (2.17) to get

$$\begin{aligned} R(X,\xi)V =& -\alpha(QX+2nX)+(\lambda+2n\alpha)\beta V^b(X)\xi \\ &+\eta(V)(\alpha\beta QX-\beta\lambda X). \end{aligned} \tag{2.21}$$

From (2.14) we can write

$$g(R(X,\xi)Y,V)=g(X,Y)\eta(V)-\eta(Y)g(X,V). \tag{2.22}$$

The previous equation together with (2.21) we obtain

$$\begin{aligned} -(\beta\lambda+2n\alpha\beta+1)V^b(X)\xi+\eta(V)((\lambda\beta+1)-\alpha\beta QX) \\ +\alpha QX+2n\alpha X=0. \end{aligned} \tag{2.23}$$

Now taking the scalar product of (2.23) with ξ and applying equation (2.15) in that gives

$$(\beta\lambda+2n\alpha\beta+1)(g(X,V)-\eta(X)\eta(V))=0. \tag{2.24}$$

Since α,β and λ are constants, the above equation (2.24) involves two cases, that either $V=\eta(V)\xi$ or $\lambda=-\left(\frac{2n\alpha\beta+1}{\beta}\right)$.

Case 1. Consider $V=\eta(V)\xi$, differentiating this along X and using (2.13) gives

$$\begin{aligned} \nabla_X V =&(\nabla_X\eta)(V)\xi+g(\nabla_X V,\xi)\xi+\eta(V)\nabla_X\xi, \\ =&g(\nabla_X V,\xi)\xi+\eta(V)(X-\eta(X)\xi). \end{aligned} \tag{2.25}$$

On combining the above relation (2.25) with (2.19) one can get

$$\begin{aligned} -\alpha QX-\beta V^b(X)+\lambda X =&2n\alpha\eta(X)\xi-\beta V^b(X)\eta(V)\xi \\ &+\eta(V)(X-\eta(X)\xi). \end{aligned} \tag{2.26}$$

Applying ϕ on both sides of the above relation implies

$$\alpha\phi QX=(\lambda-\eta(V))\phi X, \tag{2.27}$$

where we applied $\phi V=0$ and put X by ϕX in the previous relation and recall that Ricci operator Q and ϕ commutes on M(see Lemma 4.1 [42]) to obtain

$$\alpha QX=(\lambda-\eta(V))X-(\lambda-\eta(V)+2n\alpha)\eta(X)\xi. \tag{2.28}$$

Contraction of the above relation over X gives

$$\alpha\left(\frac{r}{2n}+1\right)=(\lambda-\eta(V)). \tag{2.29}$$

Using (2.29) in (2.28), we reach at

$$QX=\left(\frac{r}{2n}+1\right)-\left(\frac{r}{2n}+2n+1\right)\eta(X)\xi, \tag{2.30}$$

Showing that M is η-Einstein manifold. Hence this completes the proof of (1).

Case 2. Consider $\lambda=-\left(\frac{2n\alpha\beta+1}{\beta}\right)$ making use of this by (2.23), we get

$$(\eta(V)\alpha\beta+\alpha)(QX+2nX)=0. \tag{2.31}$$

If we consider $\eta(V)=-\frac{1}{\beta}$, then from (2.15) and (2.19) we have

$$\nabla_\xi V=\eta(V)\xi+V. \tag{2.32}$$

As the result of (2.16), one can show that $\pounds_V\xi=2(V+\eta(V)\xi)$. Then from (2.3) implies $(\pounds_V g)(X,\xi)=2(\eta(V)\eta(X)+g(X,V))$ which is turns, show that $\pounds_V\eta=0$, which means that V is a strictly infinitesimal contact transformation. Let us suppose that $\eta(V)\neq\frac{1}{\beta}$, then (2.31) shows that M is Einstein with constant scalar curvature $-2n(2n+1)$. □

Theorem 2.2.3. *Let M be a Kenmotsu manifold with dimension $2n+1$. If the metric of M is generalized Ricci soliton and V is a conformal vector field, then M is Einstein with constant scalar curvature $-2n(2n+1)$.*

Proof. If V is conformal vector field then (2.3) becomes

$$\alpha Ric(X,Y)=(\lambda-\nu)g(X,Y)-\beta V^b(X)V^b(Y). \tag{2.33}$$

Differentiating the above relation (2.33) gives

$$\alpha(\nabla_Z Ric)(X,Y)=-(Z\nu)g(X,Y)-\beta((\nabla_Z V^b)(X)V^b(Z)+V^b(Y)(\nabla_Z V^b)(Y)).$$

Taking the cyclic sum of the above relation over (X,Y,Z) and remembering V is conformal, we aim at obtianing

$$\begin{aligned}\alpha(\nabla_X Ric)(Y,Z)+\alpha(\nabla_Y Ric)(Z,X)+\alpha(\nabla_Z Ric)(X,Y)\\ +\{(X\nu)+2\nu\beta V^b(X)\}g(Y,Z)+\{(Y\nu)+2\nu\beta V^b(Y)\}g(Z,X)\\ +\{(Z\nu)+2\nu\beta B^b(Z)\}g(X,Y)=0.\end{aligned} \tag{2.34}$$

Contraction of the above relation over Y, Z implies

$$\frac{2\alpha}{2n+3}(Xr) + (X\nu) + 2\nu\beta V^b(X) = 0. \tag{2.35}$$

Using this result in (2.34) gives

$$\begin{aligned} &\alpha(\nabla_X Ric)(Y,Z) + \alpha(\nabla_Y Ric)(Z,X) + \alpha(\nabla_Z Ric)(X,Y) \\ -&\frac{2\alpha}{2n+3}\{(Xr)g(Y,Z) + (Yr)g(Z,X) + (Zr)g(X,Y)\} = 0. \end{aligned} \tag{2.36}$$

Now plugging $Y = Z = \xi$ in (2.36) and using (2.16) and (2.17) we have

$$(Xr) + 2(\xi r)\eta(X) = 0. \tag{2.37}$$

The trace of (2.17) gives

$$\pounds_\xi r = -2(r + 2n(2n+1)). \tag{2.38}$$

Applying d to this equation and since $\pounds_\xi$ commutes with d we have $\pounds dr = -2dr$. Gradient operator D write in terms of D then the last relation can be written as $\pounds_\xi Dr = -2Dr$. This relation together with $\nabla_X\xi = X - \eta(X)\xi$ implies that

$$\nabla_\xi Dr = -Dr - (\xi r)\xi. \tag{2.39}$$

We know that, any Kenmotsu manifold satisfies $Dr = (\xi r)\xi$. Using this in the above relation gives $(Xr) = 0$, which implies r is constant. Therefore

$$(\nabla_X Ric)(Y,Z) + (\nabla_Y Ric)(Z,X) + (\nabla_Z Ric)(X,Y) = 0. \tag{2.40}$$

Now substituting $Y = \xi$ in the above equation (2.40) and using (2.16) and (2.17) we get $QX = -2nX$, this shows that M is Einstein with negative scalar curvature. $\square$

2.3 Generalized Ricci tensor on almost Kenmotsu $(\kappa, \mu)'$ manifold.

Amost Kenmotsu $(\kappa, \mu)'$-manifold have been studied by many geometers, for instance see [29, 33, 85, 87, 88, 93, 95, 96] and reference therein. Suppose the curvature tensor R of an almost Kenmotsu manifolds M satisfies

$$R(X,Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)h'X - \eta(X)h'Y\} \tag{2.41}$$

for any vector field X, Y and κ, μ are constants, then we say that M is almost Kenmotsu $(\kappa,\mu)'$ manifold. From the results of [34], any almost Kenmotsu $(\kappa,\mu)'$-manifold satisifies $\mu = -2$ and $h'^2 = (\kappa+1)\phi^2$. From this, we have $\kappa \leq -1$ and the equality holds only if $h = 0$. We begin this section by recalling the following Lemmas for our later use.

Lemma 2.3.1. *(Lemma 3 in [97]).The expression of Ricci operator Q on an almost Kenmotsu $(\kappa,\mu)'$-manifold M is of the form*

$$QX = -2nX + 2n(\kappa+1)\eta(X)\xi - 2nh'X, \tag{2.42}$$

where $\kappa \leq -1$. Moreover, the scalar curvature of M is $2n(\kappa - 2n)$.

Lemma 2.3.2. *(Lemma 4.1 in [34]). On an almost Kenmotsu $(\kappa,\mu)'$-manifold with $\kappa \leq -1$, we have*

$$\begin{aligned}(\nabla_X h')Y =& g((\kappa+1)X - h'X, Y)\xi + \eta(Y)((\kappa+1)X - h'X) \\ &- 2(\kappa+1)\eta(X)\eta(Y)\xi.\end{aligned} \tag{2.43}$$

In this section we study closed generalized Ricci soliton on an almost Kenmotsu $(\kappa,\mu)'$-manifold M.

Theorem 2.3.3. *Let M be a non-Kenmotsu almost Kenmotsu $(\kappa,\mu)'$-manifold of dimension $2n+1$. If metric of M is closed generalized Ricci soliton, then M is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold, provided that $\lambda - \frac{\kappa}{\beta}(2n\alpha\beta - 1) = -\frac{2}{\beta}$.*

Proof. By using the Lemma 2. we can find that

$$\begin{aligned}(\nabla_Y Q)X - (\nabla_X Q)Y =& -2n\{(\nabla_Y h')X - (\nabla_X h')Y\} \\ &- 2n(\kappa+1)\{\eta(Y)(X + h'X) - \eta(X)(Y + h'Y)\}.\end{aligned} \tag{2.44}$$

As a result of (2.21) and Lemma 3, we obtain

$$g(R(X,Y)V,\xi) = (2n\kappa\alpha\beta - \beta\lambda)\{V^b(Y)\eta(X) - V^b(X)\eta(Y)\}, \tag{2.45}$$

where we used $Q\xi = 2n\kappa\xi$. From (2.41) the above relation becomes

$$\begin{aligned}(2n\kappa\alpha\beta - \beta\lambda - \kappa)\{V^b(Y)\eta(X) - V^b(X)\eta(Y)\} \\ -2\{V^b(h'X)\eta(Y) - V^b(h'Y)\eta(X)\} = 0.\end{aligned} \tag{2.46}$$

Substituting $X = \xi$ in the previous eqution gives

$$2V^b(h'Y) = (2n\kappa\alpha\beta - \beta\lambda - \kappa)\{\eta(V)\eta(Y) - V^b(Y)\}. \tag{2.47}$$

Since $\lambda - \frac{\kappa}{\beta}(2n\alpha\beta - 1) = -\frac{2}{\beta}$ and h' is a symmetric operator, the equation (2.47) becomes

$$h'V = \eta(V)\xi - V. \tag{2.48}$$

Applying h' on both sides of the above relation and using the equation $h'^2 = (\kappa+1)\phi^2$ implies

$$(\kappa+1)(V - \eta(V)\xi) = h'V. \tag{2.49}$$

On comparing (2.48) and (2.49) we get

$$(\kappa+2)(V - \eta(V)\xi) = 0. \tag{2.50}$$

From (2.50) we obtain two cases either $\kappa = -2$ or $V = \eta(V)\xi = f\xi$. Let us suppose $V = f\xi$ where $f = \eta(V)$ is a smooth function. The co-variant derivative of this along X provides

$$\begin{aligned}\nabla_X V &= (Xf)\xi + f(\nabla_X\xi) \\ &= (Xf)\xi + f(X - \eta(X)\xi + h'X).\end{aligned} \tag{2.51}$$

Since V is closed, we have

$$\nabla_X V = -\alpha QX - \beta V^b(X)V + \lambda X. \tag{2.52}$$

By compairing (2.51) and (2.52) one can get

$$-\alpha QX - \beta V^b(X)V + \lambda X = (Xf)\xi + f(X - \eta(X)\xi + h'X). \tag{2.53}$$

Applying QX value from (2.42) in the above relation gives

$$(\lambda + 2n\alpha - f)X + (2n\alpha - f)h'X + +\{f\eta(X) - (Xf) - 2n\alpha(\kappa+1)\eta(X)\}\xi$$
$$-\beta V^b(X)V = 0. \qquad (2.54)$$

On applying h' on both side of the above relation and by $h'\xi = 0$ implies

$$(\lambda + 2n\alpha - f)h'X + (2n\alpha - f)h'^2X = 0, \qquad (2.55)$$

where we have applied $h' = 0$. Now recalling $h'^2 = (\kappa+1)\phi^2$ we have

$$\{\lambda + 2(2n\alpha - f) + \kappa(2n\alpha - f)\}h'X + (2n\alpha - f)(\kappa+1)(X - \eta(X)\xi) = 0.$$

Contracting the above equation yields

$$2n(\kappa+1)(2n\alpha - g) = 0, \qquad (2.56)$$

and this implies $f = 2n\alpha$ because of $\kappa < -1$ and substituting this value in (2.54) implies

$$\lambda X - 2n(\alpha\kappa + 2n\beta)\eta(X)\xi = 0. \qquad (2.57)$$

Now taking X orthogonal to ξ we have $\lambda = 0$. Thus the above relation becomes $(\alpha\kappa + 2n\beta)\eta(X)\xi = 0$. Which implies $\kappa = -\frac{2n\beta}{\alpha} > -1$, which contradicts our assumption. So the only posibility is $\kappa = -2$. According to Corollary 4.2 and Proposition 4.1 [33] we claim that M is locally isometric to the Riemannian product $H^{n+1} \times R^n$.

$\square$

2.4 Basic Lemma's of the nullity conditions

In this section we present some basic Lemma's related to the properties of the nullity conditions

Lemma 2.4.1. *(Prop. 3.1 and Prop. 5.1 of [60]). Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold satisfying either the generalized (κ, μ)-nullity condition or the generalized $(\kappa, \mu)'$-nullity condition (the term generalized means κ, μ both are smooth functions),* *with $h \neq 0$. Then, one has*

$$h'^2 = (\kappa+1)\phi^2 \Leftrightarrow h^2 = (\kappa+1)\phi^2, \tag{2.58}$$

$$S(X,\xi) = 2nk\eta(X), \tag{2.59}$$

for any vector field X on M^{2n+1}. Furthermore, in the case of generalized (κ,μ)-nullity condition, one has

$$R(\xi,X)Y = \kappa[g(X,Y)\xi - \eta(Y)X] + \mu[g(hX,Y)\xi - \eta(Y)hX], \tag{2.60}$$

and in the case of generalized $(\kappa,\mu)'$-nullity condition, one has

$$R(\xi,X)Y = \kappa[g(X,Y)\xi - \eta(Y)X] + \mu[g(h'X,Y)\xi - \eta(Y)h'X], \tag{2.61}$$

for any $X, Y \in T_PM$. In addition if $n > 1$, then one has

$$(\nabla_X h')Y = -g(h'X + h'^2X, Y)\xi - \eta(Y)(h'X + h'^2X) - (\mu+2)\eta(X)h'Y, \tag{2.62}$$

for any $X, Y \in T_PM$.

Let $X \in D$ be the eigenvector of h' corresponding to the eigenvalue λ. It follows from (3.2) that $\lambda^2 = -(\kappa+1)$, a constant. Therefore $\kappa \leq -1$ and $\lambda = \pm\sqrt{-\kappa-1}$. We denote by $[\lambda]'$ and $[-\lambda]'$ the corresponding eigenspaces associated with h' corresponding to the non-zero eigenvalues λ and $-\lambda$ respectively.

Lemma 2.4.2. *(Prop. 4.1 and Prop. 4.3 of [33]). Let $(M^{2n+1},\phi,\xi,\eta,g)$ be an almost Kenmotsu manifold such that ξ belongs to the $(\kappa,\mu)'$-nullity distribution and $h' \neq 0$. Then $\kappa < -1$, $\mu = -2$ and Spec $(h') = 0, \lambda, -\lambda$, with 0 as simple eigen value and $\lambda = \sqrt{-\kappa-1}$. The distributions $[\xi] \oplus [\lambda]'$ and $[\xi] \oplus [-\lambda]'$ are integrable with totally geodesic leaves. The distributions $[\lambda]'$ and $[-\lambda]'$ are integrable with totally umbilical leaves. Furthermore, the* *sectional curvature are given as following:*

$$
\begin{aligned}
(a) \qquad & K(X,\xi) = \kappa - 2\lambda \quad if \quad X \in [\lambda]' \quad and \\
& K(X,\xi) = \kappa + 2\lambda \quad if \quad X \in [-\lambda]', \\
(b) \qquad & K(X,Y) = \kappa - 2\lambda \quad if \quad X, Y \in [\lambda]', \\
& K(X,Y) = \kappa + 2\lambda \quad if \quad X, Y \in [-\lambda]', \\
& K(X,Y) = -(\kappa + 2) \quad if \quad X \in [\lambda]',\ Y \in [-\lambda]', \\
(c) \qquad & M^{2n+1} \text{ has constant negative scalar curvature } r = 2n(\kappa - 2n).
\end{aligned}
$$

Lemma 2.4.3. *(Lemma 3 of [94]) Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with ξ belonging to the generalized $(\kappa, \mu)'$ -nullity distribution and $h' \neq 0$. If $n > 1$, then the Ricci operator Q of M^{2n+1} is given by*

$$Q = -2nid + 2n(\kappa + 1)\eta \otimes \xi + [\mu - 2(n-1)]h'. \tag{2.63}$$

Moreover, if both κ and μ are constant, then we have

$$Q = -2nid + 2n(\kappa + 1)\eta \otimes \xi - 2nh'. \tag{2.64}$$

In both cases, the scalar curvature of M^{2n+1} is $2n(\kappa - 2n)$.

Lemma 2.4.4. *(Theorem 5.1 and Proposition 5.2 of [60]). Let $(M^{2n+1}, \phi, \xi, \eta, g)$, $(n > 1)$ be an almost Kenmotsu manifold such that $h' \neq 0$ and ξ belongs to the generalized $(\kappa, \mu)'$-nullity distribution. Then for any $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$, the Riemannian curvature tensor satisfies:*

$$
\begin{aligned}
& R(X_\lambda, Y_\lambda)Z_{-\lambda} = 0, \\
& R(X_{-\lambda}, Y_{-\lambda})Z_\lambda = 0, \\
& R(X_\lambda, Y_{-\lambda})Z_\lambda = (-\kappa + 2)g(X_\lambda, Z_\lambda)Y_{-\lambda}, \\
& R(X_\lambda, Y_{-\lambda})Z_{-\lambda} = (\kappa + 2)g(Y_{-\lambda}, Z_{-\lambda})X_\lambda, \\
& R(X_\lambda, Y_\lambda)Z_\lambda = (\kappa - 2n)[g(y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda], \\
& R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = (\kappa + 2n)[g(y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}].
\end{aligned}
$$

Further, for the sectional curvature we have:

$$(a) \qquad K(X,\xi) = \kappa + \lambda\mu \quad if \quad X \in [\lambda]' \quad and$$
$$K(X,\xi) = \kappa - \lambda\mu \quad if \quad X \in [-\lambda]',$$
$$(b) \qquad K(X,Y) = \kappa - 2\lambda \quad if \quad X, Y \in [\lambda]',$$
$$K(X,Y) = \kappa + 2\lambda \quad if \quad X, Y \in [-\lambda]'$$
$$K(X,Y) = -(\kappa + 2) \quad if \quad X \in [\lambda]',\ Y \in [-\lambda]'$$
$$(c) \qquad M^{2n+1} \text{ has constant negative scalar curvature } r = 2n(\kappa - 2n).$$

Lemma 2.4.5. *(Proposition 4.2 of [33]). Let $(M^{2n+1}, \phi, \xi, \eta, g)$, be an almost Kenmotsu manifold such that $h' \neq 0$ and ξ belongs to the generalized $(\kappa, -2)'$-nullity distribution. Then for any $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$, the Riemannian curvature tensor satisfies:*

$$R(X_\lambda, Y_\lambda)Z_{-\lambda} = 0,$$
$$R(X_{-\lambda}, Y_{-\lambda})Z_\lambda = 0,$$
$$R(X_\lambda, Y_{-\lambda})Z_\lambda = (-\kappa + 2)g(X_\lambda, Z_\lambda)Y_{-\lambda},$$
$$R(X_\lambda, Y_{-\lambda})Z_{-\lambda} = (\kappa + 2)g(Y_{-\lambda}, Z_{-\lambda})X_\lambda,$$
$$R(X_\lambda, Y_\lambda)Z_\lambda = (\kappa - 2n)[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda],$$
$$R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = (\kappa + 2n)[g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}].$$

Lemma 2.4.6. *(Lemma 4.1 of [33]). Let $(M^{2n+1}, \phi, \xi, \eta, g)$, be an almost Kenmotsu manifold such that $h' \neq 0$ and ξ belongs to the $(\kappa, -2)'$-nullity distribution. Then for any $X, Y \in T_pM$,*

$$(\nabla_X h')Y = -g(h'X + h'^2X, Y)\xi - \eta(Y)(h'X + h'^2X). \tag{2.65}$$

Lemma 2.4.7. *(Theorem 4.1 of [33]). Let M be an almost Kenmotsu manifold of dimension $2n+1$. Suppose that the characteristic vector field ξ belonging to the (κ, μ)-nullity distribution. Then $\kappa = -1, h = 0$ and M is locally a warped product of an open interval and an almost Kähler manifold.*

2.5 Locally $\phi - \mathbb{H}$-conformally symmetric almost Kenmotsu manifolds

We begin this section with the following;

Definition 2.5.1. An almost Kenmotsu manifold is said to be ϕ-symmetric if it satisfies

$$\phi^2((\nabla_W R)(X,Y)Z) = 0, \tag{2.66}$$

for any vector fields $W, X, Y, Z \in T_pM$. In addition, if the vector fields W, X, Y, Z are orthogonal to ξ, then the manifold is called locally ϕ-symmetric.

Definition 2.5.2. An almost Kenmotsu manifold is said to be ϕ-conformally symmetric if it satisfies

$$\phi^2((\nabla_W C)(X,Y)Z) = 0, \tag{2.67}$$

for any vector fields $W, X, Y, Z \in T_pM$ In addition, if the vector fields W, X, Y, Z are orthogonal to ξ, then the manifold is called locally ϕ-conformally symmetric.

Theorem 2.5.1. *Let M^{2n+1} be a locally $\phi - \mathbb{H}$-conformally symmetric alomost Kenmotsu manifold with characteristic vectro field ξ belonging to the $(\kappa,\mu)'$-nullity distribution and $h \neq 0$. Then the manifold M^{2n+1} is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold.*

Proof. Consider an almost Kenmotsu manifold M^{2n+1} and suppose the manifold M^{2n+1} is a locallly $\phi - \mathbb{H}$-conformally symmetric with the characteristic vector field ξ belonging to the $(\kappa,\mu)'$-nullity distribution. This implies

$$\phi^2((\nabla_Z \mathbb{H})(X,Y)W) = 0, \tag{2.68}$$

for any vector fields X, Y, Z, W orthogonal to ξ. Substituting $W = \xi$ in the above equation we get,

$$\phi^2((\nabla_Z \mathbb{H})(X,Y)\xi) = 0. \tag{2.69}$$

Using (2.64) and (2.8) in (2.5) implies

$$\begin{aligned} H(X,Y)\xi = & \left(a\mu + \frac{2na}{2n-1} + (a+(2n-1)b)\mu\right)(\eta(Y)h'X - \eta(X)h'Y), \\ = & \left((2a+(2n-1)b)\mu + \frac{2na}{2n-1}\right)(\eta(Y)h'X - \eta(X)h'Y), \end{aligned} \tag{2.70}$$

for any $X, Y \in T_PM$. Taking covariant derivative of (2.70) along $Z \in T_PM$ then we get

$$\begin{aligned} (\nabla_Z \mathbb{H})(X,Y)\xi = \left((2a+(2n-1)b)\mu + \frac{2na}{2n-1}\right)&((\nabla_Z\eta)Y(h'X) + \eta(Y)(\nabla_Z h')X \\ &-(\nabla_Z\eta)X(h'Y) - \eta(X)(\nabla_Z h')Y). \end{aligned} \tag{2.71}$$

Now applying ϕ^2 on both sides of the above equation then we obtain

$$\begin{aligned} \phi^2((\nabla_Z \mathbb{H})(X,Y)\xi) = \left((2a+(2n-1)b)\mu + \frac{2na}{2n-1}\right)&[(\nabla_Z\eta)Y(-h'X) + \eta(Y)\phi^2((\nabla_Z h')X) \\ &-(\nabla_Z\eta)X(-h'Y) - \eta(X)\phi^2((\nabla_Z h')Y)]. \end{aligned} \tag{2.72}$$

Adopting (2.10) in the above relation we have

$$\begin{aligned} \phi^2((\nabla_Z \mathbb{H})(X,Y)\xi) = \left((2a+(2n-1)b)\mu + \frac{2na}{2n-1}\right)&[-h'X(g(Y,Z) - \eta(Y)\eta(Z) + g(h'Z,Y)) \\ &+h'Y(g(X,Z) - \eta(X)\eta(Z) + g(h'Z,X)) \\ &+\eta(Y)\phi^2((\nabla_Z h')X) - \eta(X)\phi^2((\nabla_Z h')Y)]. \end{aligned} \tag{2.73}$$

In view of (2.73) and (2.68) and X, Y, Z are orthogonal to ξ gives

$$\left((2a+(2n-1)b)\mu + \frac{2na}{2n-1}\right)\{-h'X[g(Y,Z) + g(h'Z,Y)] + h'Y[g(X,Z) + g(h'Z,X)]\} = 0. \tag{2.74}$$

In [33], Dileo and Pastore proved that if ξ belongs to the $(\kappa,\mu)'$-nullity distribution then $\mu = -2$, where a and b are real numbers (not simultaneously zero), using this result and by the assumption $n > 1$, it follows from (2.74) that

$$\{-h'X[g(Y,Z) + g(h'Z,Y)] + h'Y[g(X,Z) + g(h'Z,X)]\} = 0. \tag{2.75}$$

Letting $X, Y, Z \in [-\lambda]'$ in (2.75) implies that

$$\lambda(1-\lambda)[g(Y,X)X - g(X,Z)Y] = 0. \tag{2.76}$$

Suppose $\lambda = 0$, then $\lambda^2 = -(\kappa + 1)$ we get $\kappa = -1$ and hence $h' = 0$ from (2.58), which contradicts our assumption $h' \neq 0$. Therefore $\lambda \neq 0$, then it follows from (2.76) that $\lambda = 1$ and hence $\kappa = -2$. Then we can write from Lemma [2.4.5] that

$$R(X_\lambda, Y_\lambda)Z_\lambda = -4[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda], \tag{2.77}$$

$$R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = 0 \tag{2.78}$$

for any $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$. And from Lemma [2.4.2] that $K(X, \xi) = -4$ for any $X \in [\lambda]'$ and $K(X, \xi) = 0$ for any $X \in [-\lambda]'$. And also from the same Lemma [2.4.2] we have $K(X, Y) = -4$ for any $X, Y \in [\lambda]'$, $K(X, Y) = 0$ for any $X, Y \in [-\lambda]'$ and $K(X, Y) = 0$ for any $X \in [\lambda]'$, $Y \in [-\lambda]'$. From [33] we see that the distribution $[\xi] \oplus [\lambda]'$ is integrable with totally geodesic leaves and the distribution $[-\lambda]'$ is integrable with totally umbilical leaves by $T = -(1 - \lambda)\xi$, where T is the mean curvature vector field for the leaves of $[-\lambda]'$ immersed in M^{2n+1}. Here $\lambda = 1$, then two orthogonal distributions $[\xi] \oplus [1]'$ and $[-1]'$ are both integrable with totally geodesic leaves immersed in M^{2n+1}. Then we can say that M^{2n+1} is locally isometric to $T^{n+1}(-4) \times R^n$. This completes the proof. □

Theorem 2.5.2. *If M^{2n+1} is a locally $\phi - \mathbb{H}$-conformally symmetric alomost Kenmotsu manifold with characteristic vector field ξ belonging to the generalized $(\kappa, \mu)'$-nullity distribution and $h \neq 0$. Then the manifold M^{2n+1} is locally isometric to the Riemannian product of an $(n + 1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold.*

Proof. Consider an almost Kenmotsu manifold M^{2n+1} and suppose the manifold M^{2n+1} is a locallly $\phi - \mathbb{H}$-conformally symmetric with the characteristic vector field ξ belonging to the generalized $(\kappa, \mu)'$-nullity distribution. This implies

$$\phi^2((\nabla_Z \mathbb{H})(X, Y)W) = 0, \tag{2.79}$$

for any vector fields X, Y, Z, W orthogonal to ξ. Substituting $W = \xi$ in the above equation gives

$$\phi^2((\nabla_Z \mathbb{H})(X, Y)\xi) = 0. \tag{2.80}$$

Using (2.61) and (2.63) in (2.5) we obtain

$$\mathbb{H}(X,Y)\xi = \left\{\frac{(2a(n-1)+(2n-1)(a+(2n-1)b)\mu+2a(n-1)}{(2n-1)}\right\}\{\eta(Y)h'X-\eta(X)h'Y\}, \tag{2.81}$$

for any $X, Y \in T_PM$. Taking covariant derivative of (2.81) along $Z \in T_PM$ then we get

$$\begin{aligned}(\nabla_Z\mathbb{H})(X,Y)\xi = & \left\{\tfrac{(2a(n-1)+(2n-1)(a+(2n-1)b))}{(2n-1)}\right\}\{\eta(Y)h'X-\eta(X)h'Y\}Z(\mu)\\ & +\left\{\tfrac{(2a(n-1)+(2n-1)(a+(2n-1)b)\mu+2a(n-1)}{(2n-1)}\right\}\{(\nabla_Z\eta)Yh'X\\ & +\eta(Y)\nabla_Z(h'X)-(\nabla_Z\eta)(X)h'Y-\eta(X)\nabla_Z(h'Y)\}.\end{aligned} \tag{2.82}$$

Now applying ϕ^2 on both sides of the above equation gives

$$\begin{aligned}\phi^2((\nabla_Z\mathbb{H})(X,Y)\xi) = & \left\{\tfrac{(2a(n-1)+(2n-1)(a+(2n-1)b))}{(2n-1)}\right\}\{\eta(Y)(-h'X)-\eta(X)(-h'Y)\}Z(\mu)\\ & +\left\{\tfrac{(2a(n-1)+(2n-1)(a+(2n-1)b)\mu+2a(n-1)}{(2n-1)}\right\}\{\eta(Y)\phi^2((\nabla_Zh)X)\\ & -(\nabla_Z\eta)Y(h'X)-(\nabla_Z\eta)X(-h'Y)-\eta(X)\phi^2((\nabla_Zh'Y)\}.\end{aligned} \tag{2.83}$$

Here we adopting (2.10) in the above relation we have

$$\begin{aligned}\phi^2((\nabla_Z\mathbb{H})(X,Y)\xi) = & \left\{\tfrac{(2a(n-1)+(2n-1)(a+(2n-1)b))}{(2n-1)}\right\}\{\eta(Y)(-h'X)-\eta(X)(-h'Y)\}Z(\mu)\\ & +\left\{\tfrac{(2a(n-1)+(2n-1)(a+(2n-1)b)\mu+2a(n-1)}{(2n-1)}\right\}\{h'Y[g(X,Z)-\eta(X)\eta(Z)\\ & +g(h'Z,Z)]-h'X[g(Y,Z)-\eta(Y)\eta(Z)+g(h'Z,Y)]\\ & +\eta(Y)\phi^2((\nabla_Zh')X)-\eta(X)\phi^2((\nabla_Zh')Y)\},\end{aligned} \tag{2.84}$$

for any vector field $X, Y \in T_PM$ and from (2.84) noticing X, Y, Z are orthogonal to ξ and using (2.80) we get

$$\begin{aligned}\left\{\frac{(2a(n-1)+(2n-1)(a+(2n-1)b)\mu+2a(n-1)}{(2n-1)}\right\}&\{h'Y[g(X,Z)+g(h'Z,X)]\\ &-h'X[g(X,Z)+g(h'Z,Y)]\}=0.\end{aligned} \tag{2.85}$$

Case 1. In this case $\left\{\frac{(2a(n-1)+(2n-1)(a+(2n-1)b)\mu+2a(n-1)}{(2n-1)}\right\}=0$, so that we acquire

$$\mu = \left\{\frac{-2a(n-1)}{2a(n-1)+(2n-1)(a+(2n-1)b)}\right\}. \tag{2.86}$$

As a result of Pastore and Saltarelli [60], an almost Kenmotsu generalized $(\kappa,\mu)'$-manifold with $h' \neq 0$ satisfy

$$\nabla_\xi h' = -(\mu+2)h', \tag{2.87}$$

and for any $X, Y, Z \in D$ one can find

$$g((\nabla_X h')Y, Z) = 0. \tag{2.88}$$

Using (2.88) in (2.87) gives

$$0 = -(\mu+2)g(h'Y, Z) \tag{2.89}$$

On utilising (2.86) in (2.89) gives

$$0 = -\left\{\frac{-2a(n-1)}{2a(n-1)+(2n-1)(a+(2n-1)b)} + 2\right\} g(h'Y, Z). \tag{2.90}$$

From the above relation we get $b = \frac{-(3n-2)a}{(2n-1)^2}$ and substuting this b in (2.86) then we obtain $\mu = -2$.

Case 2. If $\lambda = 0$, then from $\lambda^2 = -(\kappa+1)$ we get $\kappa = -1$ and hence $h' = 0$ from (2.58), which again contradicts our assumption that $h' \neq 0$. Therefore $\lambda \neq 0$.

Case 3. If $\lambda = 1$ and hence $\kappa = -2$. Then from the Lemma (2.4.4) we can write that

$$R(X_\lambda, Y_\lambda)Z_\lambda = -4[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda], \tag{2.91}$$

$$R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = 0, \tag{2.92}$$

for $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$. From the Lemma (2.4.4) that $K(X,Y) = -4$ for any $X, Y \in [\lambda]'$ and $K(X,Y) = 0$ for any $X, Y \in [-\lambda]'$ and $K(X,Y) = 0$ for any $X \in [\lambda]', Y \in [-\lambda]'$. As From [33] that the distribution $[\xi] \oplus [\lambda]'$ is integrable with totally geodesic leaves and the distribution $[-\lambda]'$ is integrable with totally umbilical leaves by $T = -(1-\lambda)\xi$, where T is the mean curvature vector field for the leaves of $[-\lambda]'$ immersed in M^{2n+1}. Here $\lambda = 1$, then two orthogonal distributions $[\xi] \oplus [1]'$ and $[-1]'$ are both integrable with totally geodesic leaves immersed in M^{2n+1}. Then we can say that M^{2n+1} is locally isometric to $T^{n+1}(-4) \times R^n$. This completes the proof. $\square$

In the next results we study locally $\phi - \mathbb{H}$-conformally symmetric almost Kenmotsu manifolds with ξ belonging to the (κ, μ)-nullity and generalized (κ, μ)-nullity distributions respectively.

Theorem 2.5.3. *If M^{2n+1} is a locally $\phi - \mathbb{H}$- conformally symmetric almost Kenmotsu manifold with the characteristic vector field ξ belonging to the (κ, μ)-nullity distribution and $h \neq 0$, then the manifold M^{2n+1} is an Einstein manifold.*

Proof. Consider an almost Kenmotsu manifold M^{2n+1} and suppose the manifold M^{2n+1} is a locallly $\phi - \mathbb{H}$-conformally symmetric with the characteristic vector field ξ belonging to the (κ, μ)-nullity distribution. This implies

$$\phi^2((\nabla_Z \mathbb{H})(X, Y)W) = 0, \tag{2.93}$$

for any vector fields X, Y, Z, W orthogonal to ξ. Substituting $X = \xi$ in the above equation we get,

$$\phi^2((\nabla_Z \mathbb{H})(\xi, Y)W) = 0. \tag{2.94}$$

On considering equation (2.7) and Lemma [2.4.7] in (2.5) we obtain

$$\begin{aligned} \mathbb{H}(\xi, Y)W = a\left\{\frac{-1}{2n-1}[S(Y, W)\xi - \eta(W)QY] - \frac{2n}{2n-1}[g(Y, W)\xi - \eta(W)Y]\right\} \\ - \left\{\frac{a + (2n-1)b}{2n}\right\}(S(Y, W)\xi + 2n\eta(W)Y) \end{aligned} \tag{2.95}$$

for any $Y, W \in T_P M$. Taking the covariant differentiation along any vector field $Z \in T_P M$ of (2.95) we get

$$\begin{aligned} (\nabla_Z \mathbb{H})(\xi, Y)W = &\left(\frac{-a}{2n-1}\right)\{(\nabla_Z S)(Y, W)\xi + S(Y, W)\nabla_Z \xi - QY(\nabla_Z \eta)(W) - \eta(W)(\nabla_Z Q)Y\} \\ &- \left(\frac{2na}{2n-1}\right)\{(S(Y, W)(\nabla_Z \xi) - (\nabla_Z \eta)(W)Y) \\ &- \left(\frac{a + (2n-1)b}{2n}\right)\{(\nabla_Z S)(Y, W)\xi + S(Y, W)\nabla_Z \xi + 2n(\nabla_Z \eta)(W)Y\}, \end{aligned} \tag{2.96}$$

for any $Y, W \in T_P M$. Using (2.10) in (2.96) gives

$$\begin{aligned}(\nabla_Z \mathbb{H})(\xi, Y)W = &\left(\frac{-a}{2n-1}\right)[g(Y,W)(Z-\eta(Z)\xi) - (g(Z,W)-\eta(Z)\eta(W)Y)]\\ &-\left(\frac{2na}{2n-1}\right)[S(Y,W)(Z-\eta(Z)\xi) - (g(Z,W)-\eta(Z)\eta(W))QY\\ &-\eta(W)(\nabla_Z Q)Y + g((\nabla_Z Q)Y, W)\xi] - \left(\frac{a+(2n-1)b}{2n}\right)[g((\nabla_Z Q)Y, W)\xi\\ &+S(Y,W)(Z-\eta(Z)\xi) + 2n(g(Z,W)-\eta(Z)\eta(W)Y)]. \end{aligned} \tag{2.97}$$

Imposing ϕ^2 on the above equation (2.97) we obtain

$$\begin{aligned}\phi^2((\nabla_Z \mathbb{H})(\xi, Y)W) = &\left(\frac{-a}{2n-1}\right)[g(Y,W)(-Z+\eta(Z)\xi) - (g(Z,W)-\eta(Z)\eta(W))(Y-\eta(Y)\xi)]\\ &-\left(\frac{2na}{2n-1}\right)[S(Y,W)(-Z+\eta(Z)\xi) - (g(Z,W)-\eta(Z)\eta(W))(-QY+\eta(QY)\xi\\ &-\eta(W)\phi^2((\nabla_Z Q)Y)](\nabla_Z Q)Y, W)\xi] - \left(\frac{a+(2n-1)b}{2n}\right)[S(Y,W)(-Z+\eta(Z)\xi)\\ &-2n(g(Z,W)-\eta(Z)\eta(W))(Y-\eta(Y)\xi)], \end{aligned} \tag{2.98}$$

for any $Z, Y, W \in T_P M$. From (2.98) and (2.94) and using the fact Y, Z, W are orthogonal to ξ, we have

$$\begin{aligned}&\left(\frac{a}{2n-1}\right)\{2n(g(Y,W)Z - g(Z,W)Y) + S(Y,W)Z - g(Z,W)QY\}\\ &\quad+\left(\frac{a+(2n-1)b}{2n}\right)(S(Y,W)Z - 2ng(Z,W)Y) = 0,\end{aligned}$$

for any $Z, Y, W \in T_P M$. Taking inner product of the above equation with arbitrary vector field U yields

$$\begin{aligned}&\left(\frac{a}{2n-1}\right)\{2n(g(Y,W)g(Z,U) - g(Z,W)g(Y,U)) + S(Y,W)g(Z,U) - g(Z,W)S(Y,U)\}\\ &\quad+\left(\frac{a+(2n-1)b}{2n}\right)(S(Y,W)g(Z,U) - 2ng(Z,W)g(Y,U)) = 0.\end{aligned}$$

Let $e_i : i = 1, 2, ..., 2n+1$ be a local orthonormal basis of tangent space at each point of the manifold M^{2n+1}. Plugging $Y = U = e_i$ in the above equation and taking summation over $i : 1 \le i \le 2n+1$, we get

$$S(W,Z) = -2n\left(\frac{2na - (2n-1)(2n+1)(a+(2n-1)b)}{2na + (2n-1)(a+(2n-1)b)}\right)g(W,Z). \tag{2.99}$$

for any $Z, W \in T_PM$. In [33], Dileo and Pastore prove that in an almost Kenmotsu manifold with ξ belonging to the (κ, μ)-nullity distribution the sectional curvature $K(X, \xi) = -1$. From this we get in an almost Kenmotsu manifold with ξ belonging to the (κ, μ)-nullity distribution the scalar curvature $r = -2n(2n+1)$. Where a and b are real numbers (not simultaneously zero). If $a = 1$ and $b = -\frac{1}{2n-1}$, then (2.99) becomes $S(Z, W) = -2ng(Z, W)$, also if $a = 0$ and $b = \frac{1}{2n-1}$, then $S(Z, W) = 2n(2n+1)g(Z, W)$. Therefore the manifold M^{2n+1} is an Einstein one. This completes the proof. $\square$

Theorem 2.5.4. *Let M^{2n+1} be a locally $\phi - \mathbb{H}$- conformally symmetric almost Kenmotsu manifold with the characteristic vector field ξ belonging to the generalized (κ, μ)-nullity distribution and $h \neq 0$. Then the manifold M^{2n+1} is an Einstein almost Kenmotsu manifold with ξ belonging to the generalized κ-nullity distribution.*

Proof. Consider an almost Kenmotsu manifold M^{2n+1} and suppose the manifold M^{2n+1} is a locallly $\phi - \mathbb{H}$-conformally symmetric with the characteristic vector field ξ belonging to the generalized (κ, μ)-nullity distribution. This implies

$$\phi^2((\nabla_Z \mathbb{H})(X, Y)W) = 0, \tag{2.100}$$

for any vector fields X, Y, Z, W orthogonal to ξ. Substituting $X = \xi$ in the above equation we get,

$$\phi^2((\nabla_Z \mathbb{H})(\xi, Y)W) = 0. \tag{2.101}$$

Using the result from Pastore and Saltarelli [60], an almost Kenmotsu manifold M^{2n+1}, $n > 1$ satisfying the generalized (κ, μ)-nullity distribution with $h \neq 0$, the scalar curvature is given by $r = 2n(\kappa - 2n)$ and (2.60) we have from (2.5)

$$\begin{aligned} H(\xi, Y)Z =& a\left\{\left(\frac{-2n}{2n-1}\right)(g(Y, Z)\xi - \eta(Z)Y) + \mu(g(hY, Z)\xi - \eta(Z)hY)\right\} \\ &- \left(\frac{a}{2n-1}\right)[S(Y, Z)\xi - \eta(Z)QY] + (a + (2n-1)b)\{\mu(g(hY, Z)\xi \\ &- \eta(Z)hY) - \frac{1}{2n}S(Y, Z)\xi\}, \end{aligned} \tag{2.102}$$

for any $Y, Z \in T_PM$. Taking the covariant differentiation along arbitrary vector field

$W \in T_P M$ of (2.102) we get

$$
\begin{aligned}
(\nabla_W \mathbb{H})(\xi, Y)Z =& a\left\{\left(\frac{-2n}{2n-1}\right)(g(Y,Z)(\nabla_W \xi) - (\nabla_W \eta)(Z)Y)\right\} + aW(\mu)\{g(hY,Z)\xi \\
& - \eta(Z)hY\} + a\mu\{g(hY,Z)(\nabla_W \xi) - (\nabla_W \eta)(Z)hY - \eta(Z)(\nabla_W h)\} \\
& - \left(\frac{a}{2n-1}\right)\{S(Y,Z)(\nabla_W \xi) - (\nabla_W \eta)(Z)QY - \eta(Z(\nabla_W Q)Y \\
& + (\nabla_W S)(Y,Z)\xi)\} + (a + (2n-1)b)\{W(\mu)[g(hY,Z)\xi - \eta(Z)hY] \\
& + \mu[g(hY,Z)(\nabla_W \xi) - (\nabla_W \eta)(Z)hY - \eta(Z)(\nabla_W h)] \\
& - \frac{1}{2n}[(\nabla_W S)(Y,Z)\xi + S(Y,Z)\nabla_W \xi]\}. \qquad (2.103)
\end{aligned}
$$

Using (2.10) and applying ϕ^2 on both sides of the above equation (2.103) and Y, W, Z are orthogonal to ξ then we obtain

$$
\begin{aligned}
\phi^2((\nabla_W \mathbb{H})(\xi, Y)Z) =& \left(\frac{-2na}{2n-1}\right)\{g(Y,Z)[-W + \phi hW] + [g(Z,W) + g(h\phi W, Z)Y]\} \\
& + a\mu\{g(hY,Z)[-W + \phi hW] + g(W,Z)hY + g(h\phi W, Z)hY\} \\
& - \left(\frac{a}{2n-1}\right)\{S(Y,Z)[-W + \phi hW] + [g(W,Z) + g(h\phi W, Z)QY]\} \\
& + (a + (2n-1)b)\{\mu[g(hY,Z)[-W + \phi hW] + g(W,Z)hY \\
& + g(h\phi W, Z)hY] - \frac{1}{2n}S(Y,Z)[-W + \phi hW]\}. \qquad (2.104)
\end{aligned}
$$

From (2.101) and taking inner product of the above relation with any vector field U we obtain

$$
\begin{aligned}
& \left(\frac{-2na}{2n-1}\right)\{g(Y,Z)[-g(W,U) + g(\phi hW, U)] + [g(Z,W) + g(h\phi W, Z)]g(Y,U)\} \\
& + a\mu\{g(hY,Z)[-g(W,U) + g(\phi hW, U)] + [g(W,Z) + g(h\phi W, Z)]g(hY,U)\} \\
& - \left(\frac{a}{2n-1}\right)\{S(Y,Z)[-g(W,U) + g(\phi hW, U)] + [g(W,Z) + g(h\phi W, Z)]g(QY,U)\} \\
& + (a + (2n-1)b)\{\mu\{g(hY,Z)[-g(W,U) + g(\phi hW, U)] + [g(W,Z) + g(h\phi W, Z)]g(hY,U)\} \\
& - \frac{1}{2n}[S(Y,Z)(-g(W,U) + g(\phi hW, U))]\} = 0.
\end{aligned}
$$

Let us consider $e_i : i = 1, 2, ..., 2n+1$ be a local orthonormal basis of tangent space at each point of the manifold M^{2n+1}. Setting $Y = U = e_i$ in the previous equation and taking summation over $i : 1 \leq i \leq 2n+1$, implies

$$\begin{aligned}
&\left(\frac{-2na}{2n-1}\right)\{2ng(W,Z) - 2ng(\phi hW,Z)\} \\
&+ (2a+(2n-1)b)\mu\{-g(hZ,W) + g(hZ,\phi hW)\} \\
&- \left(\frac{a}{2n-1} + \frac{a+(2n-1)b}{2n}\right)\{-S(W,Z) + S(Z,\phi hW)\} \\
&- \left(\frac{a}{2n-1}\right)\{r[g(W,Z) + g(\phi hW)]\} = 0.
\end{aligned} \tag{2.105}$$

Now substituting $W = \phi hW$ and using $h^2 = (\kappa+1)\phi^2$ we have

$$\begin{aligned}
&\left(\frac{-2na}{2n-1}\right)\{2ng(\phi hW,Z) + 2n(\kappa+1)g(W,Z)\} \\
&+ (2a+(2n-1)b)\mu\{-g(hZ,\phi hW) - (\kappa+1)g(hZ,W)\} \\
&- \left(\frac{a}{2n-1} + \frac{a+(2n-1)b}{2n}\right)\{-S(\phi hW,Z) - (\kappa+1)S(Z,W)\} \\
&- \left(\frac{a}{2n-1}\right)\{2n(\kappa-2n)[g(\phi hW,Z) - (\kappa+1)g(W,Z)]\} = 0.
\end{aligned} \tag{2.106}$$

Adding (2.105) and (2.106) and we obtain

$$\begin{aligned}
&(\kappa+2)\left\{\left(\frac{a}{2n-1} + \frac{a+(2n-1)b}{2n}\right)S(Z,W)\right\} \\
-(\kappa+2)&\left\{\left(\frac{2na\kappa}{2n-1}\right)g(W,Z) - (2a+(2n-1)b)\mu g(hZ,W)\right\} = 0.
\end{aligned} \tag{2.107}$$

From (2.107) we see that either $\kappa = -2$ or

$$\left(\frac{a}{2n-1} + \frac{a+(2n-1)b}{2n}\right)S(Z,W) = \left(\frac{2na\kappa}{2n-1}\right)g(W,Z) + (2a+(2n-1)b)\mu g(hZ,W). \tag{2.108}$$

Suppose $\kappa = -2$, a constant, then $\xi(\kappa) = 0$. Now we present a result due to Pastore and Saltarelli [60]: In an almost Kenmotsu manifold with generalized (κ,μ)-nullity distribution and $h \neq 0$, the relation $\xi(\kappa) = -4(\kappa+1)$ holds. Therefore substituting $\kappa = -2$ in this result we get $\xi(\kappa) = 4$. Thus, we have $\xi(\kappa) = 0$ and $\xi(\kappa) = 4$, which is not possible. Hence, it follows from (2.107) that

$$\left(\frac{a}{2n-1} + \frac{a+(2n-1)b}{2n}\right)S(Z,W) = \left(\frac{2na\kappa}{2n-1}\right)g(W,Z) + (2a+(2n-1)b)\mu g(hZ,W). \tag{2.109}$$

But now replacing Z by hZ and using $h^2 = (\kappa+1)\phi^2$ we have

$$\left(\frac{a}{2n-1} + \frac{a+(2n-1)b}{2n}\right) S(hZ,W) = \left(\frac{2na\kappa}{2n-1}\right) g(W,hZ) - (2a+(2n-1)b)(\kappa+1)\mu g(Z,W). \tag{2.110}$$

Plugging $Y = Z = \xi$ in the previous eqution we get

$$(2a+(2n-1)b)\mu(\kappa+1) = 0. \tag{2.111}$$

In the above relation if $a = 1$ and $b = -\frac{1}{2n-1}$ or if $a = 0$ and $b = \frac{1}{2n-1}$, for both values of a and b we have $(2a+(2n-1)b) \neq 0$, then only possiblity is either $\mu = 0$ or $\kappa = -1$. Suppose, $\kappa = -1$. Then from, $h^2 = (\kappa+1)\phi^2$ we have $h = 0$, which contradicts our assumption $h \neq 0$. Hence it follows from (2.111) that

$$\mu = 0 \tag{2.112}$$

Then ξ belongs to the generalized k-nullity distribution. Also substituting the above relation into (2.109) we get $S(Z,W) = 2n\kappa\left(\frac{2na}{4na-a+(2n-1)^2 b}\right) g(Z,W)$ for any $Z, W \in T_P M$. Therefore the manifold M^{2n+1} is an Einstein one. Thus the manifold M^{2n+1} reduces to an Einstein almost Kenmotsu manifold with ξ belonging to the generalized κ-nullity distribution. This completes the proof of our theorem. □

2.6 Conclusion

In this chapter, $\mathbb{H}$-Curvature tensor on almost Kenmotsu manifold with nullity distibution and Generalized Ricci Soliton on Almost Kenmotsu Manifolds are studied. Main conclusions that are drawn following:

- If a Kenmotsu manifold M admits a closed generalizd Ricci tensor, then we have one of the following conditions occurs

    1. V is pointwise collinear with ξ and in such a case M is η-Einstein.

2. V is strictly infinitesimal contact transformation.

3. M is Einstein.

- Let M be a Kenmotsu manifold with the dimension $2n+1$. If the metric of M is generalized Ricci soliton and V is a conformal vector field, then M is Einstein with constant scalar curvature $-2n(2n+1)$.

- Let M be a non-Kenmotsu almost Kenmotsu $(\kappa,\mu)'$-manifold of dimension $2n+1$. If a metric of M is closed generalized Ricci soliton, then M is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold, provided that $\lambda - \frac{\kappa}{\beta}(2n\alpha\beta - 1) = -\frac{2}{\beta}$.

- Let M^{2n+1} be a locally $\phi - \mathbb{H}$-conformally symmetric alomost Kenmotsu manifold with characteristic vectro field ξ belonging to the $(\kappa,\mu)'$-nullity distribution and $h \neq 0$. Then the manifold M^{2n+1} is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold.

- If M^{2n+1} is a locally $\phi - \mathbb{H}$-conformally symmetric alomost Kenmotsu manifold with characteristic vector field ξ belonging to the generalized $(\kappa,\mu)'$-nullity distribution and $h \neq 0$, then the manifold M^{2n+1} is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold.

- If M^{2n+1} is a locally $\phi - \mathbb{H}$- conformally symmetric almost Kenmotsu manifold with the characteristic vector field ξ belonging to the (κ, μ)-nullity distribution and $h \neq 0$, then the manifold M^{2n+1} is an Einstein manifold.

Chapter 3

K-paracontact and (κ, μ)-paracontact Manifolds

It is well known that, the notion of Yamabe flow was first introduced by Richard Hamilton at the same time of a Ricci flow [45]. A Yamabe flow is defined as a tool for constructing metrics of constant as the evolution of the metric g_0 in time t to $g = g(t)$ through the equation

$$\frac{\partial}{\partial t} g(t) = -rg, \qquad g(0) = g_0, \tag{3.1}$$

where r is the scalar curvature of the metric $g(t)$. If a Riemannian manifold M holds the relation

$$\pounds_V g = 2(r - \lambda) g, \tag{3.2}$$

for a smooth vector field V on M and a constant λ, then M is said to have Yamabe soliton. And the soliton is said to be shrinking, steady or expanding if it admits a soliton field for which $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$ respectively. In the recent years, many authors have studied Yamabe soliton on various types of manifolds ([38], [19], [24], [82]). In [46], Guangyue Huang and Haizhang Li defines a generalized form of a Yamabe gradient soliton and which is called as quasi Yamabe gradient soliton.

In 1921, the notion of Bach tensor was introduced by R. Bach [2] to study conformal relativity. This is a symmetric traceless $(0,2)$-type tensor B on an n-dimensional Riemannian manifold (M,g), defined by

$$B(X,Y) = \frac{1}{n-1}\sum_{i,j=1}^{n}\big((\nabla_{e_i}\nabla_{e_j}W)(X,e_i,e_j,Y)\big) + \frac{1}{n-2}\sum_{i,j=1}^{n} Ric(e_i,e_j)W(X,e_i,e_j,Y) \tag{3.3}$$

where $(e_i), i = 1,...,n$, is a local orthonormal frame on $(M;g)$, Ric is the Ricci tensor of type $(0,2)$ and C is the $(0,3)$-type Cotton tensor defined by[40]

$$\begin{aligned} C(X,Y)Z &= (\nabla_X Ric)(Y,Z) - (\nabla_Y Ric)(X,Z) \\ &- \frac{1}{2(n-1)}[g(Y,Z)(X_r) - g(X,Z)(Y_r)], \end{aligned} \tag{3.4}$$

and W denotes the Weyl tensor of type $(0,3)$ defined by[40]

$$\begin{aligned} W(X,Y)Z &= R(X,Y)Z - g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X \\ &- g(QX,Z)Y - \frac{r}{2}(g(Y,Z)X - g(X,Z)Y). \end{aligned} \tag{3.5}$$

After Bach [2], many people worked on Bach tensor; In 1993 Pedersen and Swann[62] studied Einstein-Weyl geometry, the Bach tensor and conformal scalar curvature. In 2013-14 H.D. Cao and others ([14] and [15]) studied Bach tensor on gradient shrinking and steady Ricci soliton. In 2017 Ghosh and Sharma [41] studied Sasakian manifolds with purely transversal Bach tensor. In that article they show that a $(2n+1)$-dimensional Sasakian manifold (M,g) with a purely transversal Bach tensor has constant scalar curvature $\geq 2n(2n+1)$, equality holding if and only if (M,g) is Einstein. For dimension 3, M is locally isometric to the unit sphere S^3. For dimension 5, if in addition (M, g) is complete, then it has positive Ricci curvature and is compact with finite fundamental group $\pi_1(M)$. Recently in 2019 Ghosh and Sharma [40] studied classification of (κ, μ)-contact manifold with divergence free Cotton tensor and vanishing Bach tensor.

Definition 3.0.1. A Riemannian manifold is called an η-Einstein manifold, if it has *Ricci* tensor Q such that

$$QY = aY + b\eta(Y)\xi \tag{3.6}$$

where $a, b \in C^\infty(M^{2n+1})$ and if the function $b = 0$ then it is called Einstein.

Definition 3.0.2. On a Riemannian manifold (M, g) if there exist a smooth function f and two constants m and λ (where m is non-zero) such that

$$\nabla\nabla f = \frac{1}{m} df \otimes df + (r - \lambda)g, \tag{3.7}$$

then M is said to have quasi Yamabe gradient soliton.

Definition 3.0.3. In an almost paracontact metric manifold if $g(X, \varphi Y) = d\eta(X, Y)$ (where $d\eta(X, Y) = \frac{1}{2}\{X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y]\}$ then η is a paracontact form and the almost paracontact metric manifold $(M, \varphi, \eta, \xi, g)$ is said to be a paracontact metric manifold.

Proposition 3.0.1. *On a K-paracontact manifold M, we have (from [59])*

$$(i) \quad (\nabla_X Q)\xi = Q\varphi X + 2n\varphi X, \tag{3.8}$$

$$(ii) \quad (\nabla_\xi Q)X = Q\varphi X - \varphi QX, \tag{3.9}$$

for any vector field X on M.

Definition 3.0.4. (See [16]) A (κ, μ)-paracontact metric manifold M is a paracontact metric manifold for which the curvature tensor field satisfies

$$R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \tag{3.10}$$

for all vector fields $X, Y \in T(M)$ and for some real constants κ and μ.

Further, a paracontact metric manifold M satisfies the following properties

$$
\begin{aligned}
h^2X &= (1+\kappa)\varphi^2 X, && (3.11)\\
Q\xi &= 2n\kappa\xi, && (3.12)\\
(\nabla_X\varphi)Y &= -g(X-hX,Y)\xi && (3.13)\\
&+ \eta(Y)(X-hX), \quad for \quad \kappa=-1, &&\\
(\nabla_X h)Y-(\nabla_Y h)X &= -(1+\kappa)2g(X,\varphi Y)\xi+\eta(X)\varphi Y-\eta(Y)\varphi X && (3.14)\\
&+ (1-\mu)\eta(X)\varphi hY-\eta(Y)\varphi hX, &&\\
QX &= (2(1-n)+n\mu)X+(2(n-1)+\mu)hX && (3.15)\\
&+ (2(n-1)+n(2\kappa-\mu))\eta(X)\xi, \quad for\ \ \kappa>-1, &&\\
QX &= (2(1-n)+n\mu)X+(2(n+1)+\mu)hX && (3.16)\\
&+ (2(n-1)+n(2\kappa-\mu))\eta(X)\xi, \quad for\ \ \kappa<-1, &&\\
r &= 2n\{2(1-n)+n\mu+\kappa\} \quad for \quad \kappa\neq-1, && (3.17)\\
(\nabla_X h)Y &= -((1+\kappa)g(X,\varphi Y)+g(X,\varphi hY))\xi && (3.18)\\
&+ \eta(Y)\varphi h(hX-X)-\mu\eta(X)\varphi hY \quad for \quad \kappa\neq 0. &&
\end{aligned}
$$

Here Q denotes the Ricci operator. From the above relation its clear that r is constant on M.

And from the above conditions we can find

$$
\begin{aligned}
(\nabla_X Q)Y &= (2(n-1)+\mu)(\nabla_X h)Y\\
&+ (2(n-1)+n(2\kappa-\mu))\{(\nabla_X\eta)(Y)\xi+\eta(Y)\nabla_X\xi\}, \quad for\ \ \kappa>-1 \quad (3.19)
\end{aligned}
$$

and

$$\begin{aligned}(\nabla_X Q)Y &= (2(n+1)+\mu)(\nabla_X h)Y \\ &+ (2(n-1)+n(2\kappa-\mu))\{(\nabla_X\eta)(Y)\xi+\eta(Y)\nabla_X\xi\}, \quad for \ \ \kappa < -1. \end{aligned} \tag{3.20}$$

From K. Yano [99], we have

$$2g((\pounds_V\nabla)(X,Y),Z) = (\nabla_X\pounds_V g)(Y,Z) + (\nabla_Y\pounds_V g)(X,Z) - (\nabla_Z\pounds_V g)(X,Y), \tag{3.21}$$

and

$$(\pounds_V R)(X,Y)Z = (\nabla_X\pounds_V\nabla)(Y,Z) + (\nabla_Y\pounds_V\nabla)(X,Z). \tag{3.22}$$

Proposition 3.0.2. *On a paracontact manifold M, if $f \in M$, $Xf = 0$ for all $X \perp \xi$, then f is constant on M.*

Proof. If $Xf = 0$ for all $X \perp \xi$, then operating φ^2 on both sides we have

$$g(X-\eta(X)\xi, \nabla f) = 0. \tag{3.23}$$

Taking covariant derivative of (3.23) along Y we obtain

$$YXf - (\xi f)\nabla_Y\eta(X) - \eta(X)(Y\xi f) = 0. \tag{3.24}$$

Interchange X and Y in the above equation gives

$$XYf - (\xi f)\nabla_X\eta(Y) - \eta(Y)(X\xi f) = 0. \tag{3.25}$$

Replacing X by $[X,Y]$ in (3.23) implies

$$[X,Y]f - (\xi f)\eta[X,Y] = 0. \tag{3.26}$$

Subtracting (3.26) and (3.25) from (3.24) we get

$$(\xi f)\{(\nabla_X\eta)Y - (\nabla_Y\eta)X\} = 0, \tag{3.27}$$

which implies

$$\{g(Y,\nabla_X\xi) - g(X,\nabla_Y\xi)\}(\xi f) = 0. \tag{3.28}$$

From $\nabla_X \xi = -\varphi X + \varphi h X$, which reduces to

$$2g(X, \varphi Y)(\xi f) = 0. \tag{3.29}$$

This shows that $\xi f = 0$. Therefore $Xf = 0$ for all $X \in T(M)$. So that f is constant on M. □

Before start to study quasi Yamabe soliton, here we prove a result regarding the Yamabe soliton on non-para Sasakian (κ, μ)-paracontact manifold.

Let us consider a non-para-Sasakian (κ, μ)-paracontact manifold M. For $dim M > 3$ if M admits a Yamabe soliton for the general vector field V the scalar curvature r of M is constant. Therefore by the use of (3.2) in the formula (3.21), we obtain

$$(\pounds_V \nabla)(Y, Z) = 0. \tag{3.30}$$

Moreover taking the co-variant derivative of above relation in the direction of X results

$$(\nabla_X \pounds_V)(Y, Z) = 0. \tag{3.31}$$

With help of the foregoing equation in the commutation formula (3.22), yeilds

$$(\pounds_V R)(X, Y)Z = 0. \tag{3.32}$$

Contracting the above condition over X with respect to an orthonormal basis provides

$$(\pounds_V S)(Y, Z) = 0. \tag{3.33}$$

Putting $Y = Z = \xi$ in (3.33) and by the use of (3.12) and (3.2) on (ξ, ξ) gives

$$\kappa(r - \lambda) = 0. \tag{3.34}$$

From the above condition we get two cases. **case (i)**: if $(r-\lambda)=0$ and $\kappa\neq 0$, then from equation (3.2) we can say that V is Killing. **case(ii)**: if $\kappa=0$ and $(r-\lambda)\neq 0$. Since V is Killing if and only if $(r-\lambda)=0$, so $\kappa=0$ in (3.10) gives

$$R(X,Y)\xi=\mu\{\eta(Y)hX-\eta(X)hY\}. \tag{3.35}$$

Taking the Lie-derivative of above condition along V gives

$$\begin{aligned}(\pounds_V R)(X,Y)\xi &= \mu\{(\pounds_V\eta)YhX+\eta(Y)(\pounds_V h)X-(\pounds_V\eta)XhY-\eta(X)(\pounds_V h)Y\}\\ &- R(X,Y)\pounds_V\xi. \end{aligned}\tag{3.36}$$

Contracting the above equation over X with respect to an orthonormal basis gives

$$(\pounds_V S)(Y,\xi)=\mu\left\{\eta(Y)\sum_{i=1}^{2n+1}g((\pounds_V h)e_i,e_i)+\sum_{i=1}^{2n+1}g(\pounds_V e_i,\xi)g(hY,e_i)-S(Y,\pounds_V\xi)\right\}.$$

Putting $Y=\xi$ in the above equation and from (3.33) we obtain

$$\mu\sum_{i=1}^{2n+1}\langle(\pounds_V h)e_i,e_i\rangle=0. \tag{3.37}$$

In the above relaton if $\mu=0$ then M becomes $(0,0)$-space. If $\mu\neq 0$, then $\sum_{i=1}^{2n+1}g((\pounds_V h)e_i,e_i)=0$. Since $\kappa=0$ calculating $(\pounds_V S)(Y,Z)$ by the help of (3.15) finds

$$\begin{aligned}(\pounds_V S)(Y,Z)= &\quad (2(1-n)+n\mu)(\pounds_V g)(Y,Z)+(2(n-1)+n(2\kappa-\mu))\eta(Y)(\pounds_V\eta)Z\\ &+(2(n-1)+n(2\kappa-\mu))\eta(Z)(\pounds_V\eta)Y+(2(n-1)+\mu)(\pounds_V g)(hY,Z)\\ &+(2(n-1)+\mu)g((\pounds_V h)Y,Z). \end{aligned}\tag{3.38}$$

Contracting the above equation over Y, Z and by the use of (3.2), (3.33) and $r-\lambda\neq 0$ provides

$$(2(1-n)+n\mu)(2n+1)+(2(n-1)+n(2\kappa-\mu))=0. \tag{3.39}$$

Further substituting $Y = Z = \xi$ in (3.38) gives

$$(2(1-n)+n\mu)+(2(n-1)+n(2\kappa-\mu))=0. \tag{3.40}$$

On solving (3.39) and (3.40) we get

$$2(1-n)+n\mu=0, \tag{3.41}$$

$$2(n-1)+\mu=0, \tag{3.42}$$

which gives $\mu = 0$, whivh contradicts to our assumption. So $\kappa = 0$ and so μ must be zero.

Theorem 3.0.3. *Let M be a non-para-Sasakian (κ,μ)-paracontact manifold and admits a Yamabe soliton then either V is Killing or M is locally isometric to the product of a flat $(n+1)$-dimensional manifold and n-dimensional manifold of constant negative curvature equal to -4.*

3.1 Quasi Yamabe gradient soliton on non-para-Sasakian (κ,μ)-paracontact manifold

Lemma 3.1.1. *Let M be a non-para-Sasakian (κ,μ)-paracontact manifold. If g is a Quasi Yamabe gradient soliton, then either $r-\lambda=0$, or $r=\frac{2n(2n+1)(r-\lambda)}{m}$, or the soliton is steady.*

Proof. Here we study quasi Yamabe gradient soliton on M by exhibiting the relation (3.7) as

$$\nabla_X\nabla f=\frac{1}{m}g(X,\nabla f)\nabla f+(r-\lambda)X, \tag{3.43}$$

and we compute

$$R(X,Y)\nabla f=\frac{r-\lambda}{m}\left\{g(Y,\nabla f)X-g(X,\nabla f)Y\right\}. \tag{3.44}$$

Consdering g-trace of the condition (3.44) leads to get

$$Q\nabla f=\frac{2n(r-\lambda)}{m}\nabla f. \tag{3.45}$$

Taking covariant derivative of (3.45) along X gives

$$\nabla_X Q\nabla f = \frac{2n(r-\lambda)}{m}\nabla_X\nabla f. \tag{3.46}$$

Operating Q on (3.43) gives

$$Q\nabla_X\nabla f = \frac{1}{m}g(X,\nabla f)Q\nabla f + (r-\lambda)QX. \tag{3.47}$$

In view of (3.46) and (3.47) we obtain

$$(\nabla_X Q)\nabla f = \frac{2n(r-\lambda)}{m}\nabla_X\nabla f - \frac{1}{m}g(X,\nabla f)Q\nabla f - (r-\lambda)QX. \tag{3.48}$$

By virtue of (3.43) and (3.45) the preceeding relation transfer to

$$(\nabla_X Q)\nabla f = \frac{2n(r-\lambda)^2}{m}X - (r-\lambda)QX. \tag{3.49}$$

As r is constant on M, tracing this over X leads to

$$(r-\lambda)\left\{r - \frac{2n(2n+1)(r-\lambda)}{m}\right\} = 0. \tag{3.50}$$

This completes the proof. □

Theorem 3.1.2. *If a non-para-Sasakian (κ,μ)-paracontact manifold M admits a quasi Yamabe gradient soliton, then manifold M for $\kappa > -1$, either*

1. *M is a $N(\frac{1-n}{n})$-manifold,*

   or

2. *M is locally isometric to the product of a flat $(n+1)$-dimensional manifold and n-dimensional manifold of constant negative curvature equal to -4,*

   or

3. *f is constant on M.*

Next, for $\kappa < -1$ either

1. *$\mu = \frac{-4}{n+1}$*

   or

2. *f is constant on M.*

Proof. In view of the above lemma, we can come to the conclusion that, if a non-para Sasakian (κ, μ)-para contact metric g is a quasi Yamabe gradient soliton, then either $r - \lambda = 0$, or $r = \frac{2n(2n+1)(r-\lambda)}{m}$ on M.

In (3.45) taking scalar product with ξ and by (3.12) we find

$$\left(\kappa - \frac{r-\lambda}{m}\right)(\xi f) = 0. \tag{3.51}$$

Case 1, if we suppose $r - \lambda \neq 0$ then $r = \frac{2n(2n+1)(r-\lambda)}{m}$, so in the above condition if $\xi f = 0$, then which in (3.43) for $X = \xi$ and inner product with ξ results $(r - \lambda) = 0$. But this is not possible as concern to our assumption. Therefore ξf must be non-zero and $\kappa = \frac{r-\lambda}{m}$, which finds $r = 2n(2n+1)\kappa$. Comparing this with the expression of r in (3.17) leads to get κ as $\kappa = \frac{1-n}{n} + \mu$. Further by taking the scalar product of (3.44) with ξ we obtain

$$g(R(X,Y)\nabla f, \xi) = \frac{r-\lambda}{m}\left\{g(Y, \nabla f)\eta(X) - g(X, \nabla f)\eta(Y)\right\}. \tag{3.52}$$

Similarly considering an inner product of (3.10) with ∇f gives

$$\begin{aligned} g(R(X,Y)\xi, \nabla f) &= \kappa\left\{\eta(Y)g(X, \nabla f) - \eta(X)g(Y, \nabla f)\right\} \\ &+ \mu\left\{\eta(Y)g(hX, \nabla f) - \eta(X)g(hY, \nabla f)\right\}. \end{aligned}$$

Comparing these two equations provides

$$\mu\left\{\eta(Y)g(hX, \nabla f) - \eta(X)g(hY, \nabla f)\right\} = 0, \tag{3.53}$$

for $X = \xi$ and $Y \in D$ the expression reducess to $\mu(Yf) = 0$. Since $\xi f \neq 0$ by Lemma [3.4.2] we can say that f is non constant and this finds $\mu = 0$ for $\kappa > -1$ case. Hence M is $\left(\frac{1-n}{n}, 0\right)$-manifold. But for $\kappa < -1$, $\mu(Yf) = 0$ leads to a contradiction. Hence for $\kappa < -1$, $r - \lambda$ must be zero.

Case 2, If $r - \lambda = 0$ and $r \neq \frac{2n(2n+1)(r-\lambda)}{m}$ then equations (3.43), (3.44) and (3.45) turns to

$$R(X,Y)\nabla f = 0, \tag{3.54}$$

$$Q\nabla f = 0, \tag{3.55}$$

$$\nabla_X \nabla f = \frac{1}{m} g(X, \nabla f)\nabla f \tag{3.56}$$

and also

$$(\nabla_X Q)\nabla f = 0. \tag{3.57}$$

Considering an inner product of (3.54) with ξ, $Y = \xi$ and in view of (3.10) shows

$$\kappa(\eta(X)(\xi f) - Xf) - \mu(hXf) = 0, \tag{3.58}$$

and the expression (3.51) gives

$$\kappa(\xi f) = 0. \tag{3.59}$$

Because of (3.59) there arises three cases

First case: $\kappa = 0$ and $\xi f \neq 0$, then $\kappa > -1$

Since $\xi f \neq 0$ by the help of Lemma [3.4.2] we can say that f is never be a constant. Therefore by (3.58) μ must be zero. So in this case we arrived at condition 2.

Next, for Second case: $\kappa \neq 0$ and $\xi f = 0$.

The condition (3.58) provides

$$\kappa(Xf) + \mu(hXf) = 0. \tag{3.60}$$

Applying h on (3.60) and from (3.11) we deduce that

$$\kappa(hXf) + \mu(\kappa + 1)(Xf) = 0. \tag{3.61}$$

Using (3.60) in (3.61) provides

$$(\kappa^2 - \mu^2(\kappa + 1))(hXf) = 0. \tag{3.62}$$

For $\kappa > -1$

In the foregoing equation, we proceed with the assumption that f is non constant. This implies $\kappa^2 - \mu^2(\kappa+1) = 0$. Clearly, $\kappa = 0$ if and only if $\mu = 0$ so we find that μ must not equal to zero.

Further in (3.19) (for $\kappa > -1$) putting $Y = \nabla f$ and $X = \xi$ and from the use of (3.57) and (3.18) we have that

$$(2n - 2 + \mu)\mu\nabla f = 0, \tag{3.63}$$

which shows

$$2n - 2 + \mu = 0. \tag{3.64}$$

Use of the above relation in calculation of (3.15) for $Y = \nabla f$ and from (3.55) we obtain

$$2 - 2n + n\mu = 0. \tag{3.65}$$

Solving the expressions (3.64) and (3.65) yeilds $\mu = 0$. This contradicts our data condition $\kappa \neq 0$. Hence the soliton function f must be a constant on M.

For $\kappa < -1$.

Similarlly, in (3.108) (for $\kappa < -1$) putting $Y = \nabla f$ and $X = \xi$ and from use of (3.57) and (3.18) we have that

$$(2(n+1) + \mu)\mu\nabla f = 0, \tag{3.66}$$

this gives

$$(2(n+1) + \mu) = 0. \tag{3.67}$$

Use of the above relation in calculation of (3.16) for $Y = \nabla f$ and from (3.55) we obtain

$$2(1-n) + n\mu = 0. \tag{3.68}$$

Solving the above two relations yields $\mu = \frac{-4}{n+1}$ for the condition $\kappa \neq 0$. Hence for $\mu \neq \frac{-4}{n+1}$ the soliton function f is not a constant on M.

For third case: $\kappa = 0$ and $\xi f = 0$, then $\kappa > -1$, which in (3.58) determines

$$\mu(Xf) = 0, \tag{3.69}$$

for all $X \in D$. This shows either f is constant or $\mu = 0$ i.e., M is $(0,0)$-space. Hence this completes the proof. □

3.2 Quasi Yamabe gradient soliton on K-paracontact manifolds.

Theorem 3.2.1. *Let M be a K-paracontact manifold with $Q\varphi = \varphi Q$ and if M holds a Yamabe soliton, then either $r = -2n(2n+1)$ or f is a constant.*

Suppose M is K-para contact manifold and satisifies (3.7), then form relation (3.43), we find $R(X,Y)\nabla f$ as

$$R(X,Y)\nabla f = \frac{r-\lambda}{m}\{g(Y,\nabla f)X - g(X,\nabla f)Y\} + (Xr)Y - (Yr)X. \tag{3.70}$$

Contracting the above condition over X and Y leads to get

$$Q\nabla f = \frac{2n(r-\lambda)}{m} + 2n\nabla r. \tag{3.71}$$

On K-para contact manifold, as $Q\varphi = \varphi Q$ from the relation (3.9) we can obtain

$$\begin{aligned} (\nabla_\xi Q)X &= 0, && (3.72) \\ \xi r &= 0. && (3.73) \end{aligned}$$

So by taking inner product of (3.71) with ξ yeilds

$$\left\{1 + \frac{(r-\lambda)}{m}\right\}\xi f = 0. \tag{3.74}$$

Differentiating equation (3.71) along X and from (3.43) gives

$$\nabla_X Q\nabla f = \frac{2n(r-\lambda)}{m^2}g(X,\nabla f)\nabla f + \frac{2n(r-\lambda)^2}{m}X + 2n(Xr)\nabla f. \tag{3.75}$$

Operating Q on both side of (3.43) we get

$$Q\nabla_X\nabla f = \frac{2n(r-\lambda)}{m^2}g(X,\nabla f)\nabla f + \frac{2n}{m}g(X,\nabla f)\nabla r + (r-\lambda)QX. \tag{3.76}$$

For $X = \xi$ in (3.75) and (3.76) and from (3.72) we obtain

$$(r-\lambda)\left\{\frac{r-\lambda}{m} + 1\right\} = 0. \tag{3.77}$$

Since m is non zero constant, so either $r = \lambda$ or $r - \lambda = -m$. If $r = \lambda$, then in (3.43) for $X = \xi$ and scalar product with ξ gives $\xi f = 0$. Again, in equation (3.43) taking an inner product with ξ provides $Xf = 0$. Hence f is constant. Next, if we suppose $r - \lambda \neq 0$ and $r - \lambda = -m$, which implies $\xi f \neq 0$ and moreover the relations (3.75) and (3.76) provides

$$(\nabla_X Q)\nabla f = -(r-\lambda)(2nX + QX), \tag{3.78}$$

taking g-trace results $r = -2n(2n+1)$. Hence the proof completed.

3.3 Vanishing Cotton tensor on K-paracontact manifold.

Proposition 3.3.1. *Let M^{2n+1} be a K-paracontact manifold. Then M^{2n+1} has constant scalar curvature if and only if $C(X,\xi)\xi = 0$.*

Proof. Setting $Z = \xi$ in (3.4) we get.

$$C(X,Y)\xi = g((\nabla_X Q)\xi, Y) - g((\nabla_Y Q)\xi, X) - \frac{1}{4n}[(Xr)\eta(Y) - (Yr)\eta(X)]. \quad (3.79)$$

Using equation (3.8) from Proposition [3.0.1] in the above equation, we get

$$\begin{aligned} C(X,Y)\xi = {} & -4ng(\varphi X, Y) + g(Q\varphi X, Y) - g(Q\varphi Y, X) \\ & -\frac{1}{4n}[(Xr)\eta(Y) - (Yr)\eta(X)]. \end{aligned} \quad (3.80)$$

Replacing X by φX and Y by φY in (3.80) we obtain,

$$C(\varphi X, \varphi Y)\xi = 4ng(\varphi X, Y) + g(Q\varphi^2 X, \varphi Y) - g(Q\varphi^2 Y, \varphi X) = 0, \quad (3.81)$$

which gives

$$-4ng(\varphi X, Y) - g(X, Q\varphi Y) + g(Q\varphi X, Y) = 0. \quad (3.82)$$

Admitting (3.82) in (3.80), we get,

$$(Xr)\eta(Y) - (Yr)\eta(X) = 0. \quad (3.83)$$

Putting $Y = \xi$ and taking X orthogonal to ξ in the above equation gives

$$Xr = 0. \quad (3.84)$$

As M is paracontact manifold and $X \in ker\eta$ which implies $Xr = 0$, $\forall X \in T_P M$. So r is constant

Conversly, if r is constant then substituting $Y = \xi$ in the equation (3.80) gives $C(X,\xi)\xi = 0$.

Hence the proof. □

3.4 Parallel Cotton tensor on K-paracontact manifold M^{2n+1}

Definition 3.4.1. In a Riemannian manifold M^{2n+1}, if there is a Cotton tensor C such that its covariant differenciation i.e., $(\nabla_W C) = 0$ then the manifold is said to have parallel Cotton tensor.

Theorem 3.4.1. *Let M^{2n+1} be a K-paracontact metric manifold. Then M has parallel Cotton tensor if and only if M^{2n+1} is an η-Einstein manifold and $r = -2n(2n+1)$.*

Proof. For a K-paracontact manifold M^{2n+1}, the equation (3.4) for $Y = \xi$ and $Z = Y$ is gives

$$C(X,\xi)Y = 2ng(\varphi X, Y) + g(Q\varphi X, Y) - \frac{1}{4n}\{(Xr)\eta(Y)\}. \tag{3.85}$$

Taking $Y = \xi$ in the above equation, we get

$$C(X,\xi)\xi = -\frac{1}{4n}\{(Xr)\}. \tag{3.86}$$

Using (3.86) in (3.79) we calculate the following relations

$$\begin{aligned}
\nabla_W C(X,\xi)\xi = &\quad -\frac{1}{4n}\{g(\nabla_W X, Dr) + g(X, \nabla_W Dr)\}, &(3.87)\\
C(\nabla_W X,\xi)\xi = &\quad -\frac{1}{4n}\{g(\nabla_W X, Dr)\}, &(3.88)\\
C(X,\varphi W)\xi = &\quad 4ng(\varphi X,\varphi W) + g(Q\varphi X,\varphi W) - g(Q\varphi^2 W, X) &(3.89)\\
&\quad -\frac{1}{4n}\{-(\varphi W r)\eta(X)\},\\
C(X,\xi)\varphi W = &\quad 2ng(\varphi X,\varphi W) + g(Q\varphi X,\varphi W). &(3.90)
\end{aligned}$$

Making use of above group of equations we obtain

$$\begin{aligned}
(\nabla_W C)(X,\xi)\xi = &\quad -\frac{1}{4n}\{g(X,\nabla_W Dr)\} + 4ng(\varphi X,\varphi W) + g(Q\varphi X,\varphi W) &(3.91)\\
&\quad -g(Q\varphi^2 W, X) - \frac{1}{4n}\{(\varphi W r)\eta(X)\} + 2ng(\varphi X,\varphi W) + g(\varphi Q X,\varphi W).
\end{aligned}$$

Putting $W = \xi$ in the above equation, the parallel Cotton tensor becomes

$$(\nabla_\xi C)(X,\xi)\xi = -\frac{1}{4n}\{g(X,\nabla_\xi Dr)\} = 0. \tag{3.92}$$

As $\pounds_\xi r = 0, \nabla_\xi Dr = \nabla_{Dr}\xi = -\varphi Dr$, which implies $g(X, \varphi Dr) = 0$, which gives $Dr = 0$ and so r is constant. Then the relation (3.91) becomes

$$6ng(\varphi X, \varphi W) + g(Q\varphi X, \varphi W) - g(X, QW) - 2n\eta(X)\eta(W)$$
$$-g(X, QW) - 2n\eta(X)\eta(W) = 0. \tag{3.93}$$

Replacing X by φX and W by φW in (3.93) and simplifying we get

$$g(Q\varphi X, \varphi W) = -3ng(\varphi X, \varphi W) + \frac{1}{2}g(QX, W) + n\eta(X)\eta(W). \tag{3.94}$$

Feeding (3.94) in (3.93) we obtain

$$6ng(X, W) + 6n\eta(X)\eta(W) - 3ng(X, W) + 3n\eta(X)\eta(W) + \frac{1}{2}g(X, \varphi W)$$
$$+n\eta(X)\eta(W) - 4n\eta(X)\eta(W) - 2g(X, QW) = 0. \tag{3.95}$$

Contracting the equation (3.95) over X and W we have $r = -2n(2n+1)$ and M is an η-Einstein manifold.

Conversly, Suppose M is an η-Einstein manifold and $r = -2n(2n+1)$, which implies $QY = -2nY$. And so this gives $C(X, Y)Z = 0$.

Hence the proof. □

Lemma 3.4.2. *Let $M^{2n+1}(n > 1)$ be a K-paracontact manifold. If M^{2n+1} satisifies (3.6), then a and b are constant functions*

Proof. From the condition (3.6) we have,

$$(\nabla_X Q)Y = (Xa)Y + (Xb)\eta(Y)\xi + b\left\{g(X, \varphi Y)\xi + \eta(Y)\nabla_X\xi\right\}. \tag{3.96}$$

From η-Einstein condition, $-2n = a + b$, so $(Xa) = -(Xb)$.

Therefore

$$(\nabla_X Q)Y = (Xa)Y - (Xa)\eta(Y)\xi + \{-2n - a\}\left\{g(X, \varphi Y)\xi - \eta(Y)\varphi X\right\}. \tag{3.97}$$

Contracting the above equation over X with respect to the orthonormal frame field we get

$$\sum_{i=1}^{2n+1} \epsilon_i \langle(\nabla_{e_i}Q)Y, e_i\rangle = \sum_{i=1}^{2n+1} \epsilon_i(e_i a)g(Y, e_i) + (\xi a) \tag{3.98}$$

where $\xi = g(e_i, e_i)$, as $\xi r = 0$ gives $\xi a = 0$. But we know that $\sum_{i=1}^{2n+1} \langle (\nabla_{e_i} Q)Y, e_i\rangle = \frac{1}{2}(Yr)$ which gives

$$\frac{1}{2}(Yr) \quad = \quad g(Y, Da) \tag{3.99}$$

as $Yr = 2$, so $(n-1)Ya = 0$ for $n > 1$ becomes $Ya = 0$, therefore a is constant.
This completes the proof. □

3.5 Bach tensor on η-Einstein K-paracontact manifolds for $(n > 1)$

Bach tensor for $2n+1$-dimensional manifold is

$$B(X,Y) = \frac{1}{2n-1}\left\{\sum_{i=1}^{2n+1} \epsilon_i (\nabla_{e_i} C)(e_i, X, Y) + \sum_{i,j=1}^{2n+1} \epsilon_i g(Qe_i, e_j) W(X, e_i, e_j, Y)\right\}. \tag{3.100}$$

By lemma (3.4.2) we know that a and b are constants, and so the equation (3.96) becomes

$$(\nabla_X Q)Y = b\left\{g(X, \varphi Y)\xi - \eta(Y)\varphi X\right\}. \tag{3.101}$$

simplifying the cotton tensor using (3.101)

$$C(X,Y)Z \quad = \quad bg(X, \varphi Y)\eta(Z) - b\eta(Y)g(\varphi X, Z) - bg(Y, \varphi X)\eta(Z) + bg(\varphi Y, Z)\eta(X).$$

Applying ∇_W on both sides of the above equation gives

$$\begin{aligned}(\nabla_W C)(X,Y)Z \quad &= \quad b\nabla_W \left\{2g(X, \varphi Y)\eta(Z) + \eta(X)g(\varphi Y, Z) + \eta(Y)g(X, \varphi Z)\right\} \qquad (3.102)\\ &= \quad b2g(X, (\nabla_W \varphi)Y)\eta(Z) + bg(X, \varphi Y)g(W, \varphi Z) + bg((\nabla_W \varphi)Y, Z)\eta(X)\\ &+ \quad bg(\varphi Y, Z)g(W, \varphi X) + bg(X, (\nabla_W \varphi)Z)\eta(Y) + bg(X, \varphi Z)g(W, \varphi Y).\end{aligned}$$

On contracting above equation over X and W gives

$$\begin{aligned}\sum_{i=1}^{2n+1} \epsilon_i(\nabla_{e_i}C)(e_i, Y)Z &= b\left\{\sum_{i=1}^{2n+1} \epsilon_i g(e_i, (\nabla_{e_i}\varphi)Y)\eta(Z + g(e_i, (\nabla_{e_i}\varphi)Z)\eta(Y)\right\} + 2bg(\varphi Y, \varphi Z)\\ &= b\left\{\sum_{i=1}^{2n+1} \epsilon_i \langle R(\xi, e_i)Y, e_i\rangle \eta(Z) + g(R(\xi, e_i)Z, e_i)\eta(Y)\right\} + 2bg(\varphi Y, \varphi Z)\\ &= b\{-S(Y, \xi)\eta(Z) - S(Z, \xi)\eta(Y)\} + 2bg(\varphi Y, \varphi Z)\\ &= b\{4n\eta(Y)\eta(Z) + 2g(\varphi Y, \varphi Z)\}.\end{aligned}$$

Now we caluculate the right hand side of the Bach tensor that is

$$\sum_{i,j=1}^{2n+1} \epsilon_i g(Qe_i, e_j)g(W(X, e_i)e_j, Y) = -\sum_{i,j=1}^{2n+1} \epsilon_i g(Qe_i, W(X, e_i)Y).$$

By η-Einstein condition $Qe_i = ae_i + b\eta(e_i)\xi$, which gives

$$\begin{aligned}\sum_{i,j=1}^{2n+1} \epsilon_i g(Qe_i, e_j)g(W(X, e_i)e_j, Y) &= -\sum_{i,j=1}^{2n+1} \epsilon_i g(e_i + b\eta(e_i)\xi, W(X, e_i)Y) \qquad (3.103)\\ &= \sum_{i=1}^{2n+1} \epsilon_i g(W(X, e_i)e_i, Y) + bg(W(X, \xi)\xi, Y).\end{aligned}$$

From the expression of Weyl tensor W we deduce the following relation

$$\begin{aligned}\sum_{i=1}^{2n+1} \epsilon_i \langle W(X, e_i)e_i, Y\rangle = & \sum_{i=1}^{2n+1} \epsilon_i(\langle R(X, e_i)e_i, Y\rangle - \frac{1}{2n-1}[g(Qe_i, e_i)g(X, Y)\\ & -g(QX, e_i)g(e_i, Y) + g(e_i, e_i)g(QX, Y) - g(X, e_i)g(Qe_i, Y)]\\ & +\frac{r}{2n(2n-1)}[g(e_i, e_i)g(X, Y) - g(X, e_i)g(e_i, Y)]) \qquad (3.104)\\ = & S(X, Y) - \frac{1}{2n-1}[rg(X, Y) - S(X, Y) + (2n+1)S(X, Y) - S(X, Y)]\\ & +\frac{r}{2n(2n-1)}[(2n+1)g(X, Y) - g(X, Y)]\\ = & 0.\end{aligned}$$

Taking inner product of $W(X,\xi)\xi$ with Y we get,

$$\begin{aligned}\langle W(X,\xi)\xi,Y\rangle = \quad & \langle R(X,\xi)\xi,Y\rangle - \frac{1}{2n-1}(-2n\langle X,Y\rangle + 2n\eta(X)\eta(Y) + \langle QX,Y\rangle \\ & +2n\eta(X)\eta(Y)) + \frac{r}{2n(2n-1)}(\langle X,Y\rangle - \eta(X)\eta(Y)) \\ = \quad & \langle \varphi\nabla_X\xi,Y\rangle + \frac{2n}{2n-1}(X,Y) - \frac{4n}{2n-1}\eta(X)\eta(Y) + \frac{r}{2n(2n-1)}(X,Y) \\ & -\frac{r}{2n(2n-1)}\eta(X)\eta(Y) - \frac{1}{2n-1}S(X,Y)\end{aligned}$$

But $\langle\varphi X,\varphi Y\rangle = -\langle X,Y\rangle + \eta(X)\eta(Y)$, so we get

$$\begin{aligned}\langle W(X,\xi)\xi,Y\rangle = \quad & \frac{1}{2n-1}\left\{\left(1+\frac{r}{2n}\right)(X,Y) - \left(1+2n+\frac{r}{2n}\right)\eta(X)\eta(Y)\right\} \\ & -\frac{1}{2n-1}S(X,Y)\end{aligned} \tag{3.105}$$

Using the value of $S(X,Y) = \left(1+\frac{r}{2n}\right)g(X,Y) - \left(1+2n+\frac{r}{2n}\right)\eta(X)\eta(Y)$ in (3.105) gives

$$\langle W(X,\xi)\xi,Y\rangle = 0. \tag{3.106}$$

Therefore if g is Bach flat,

$$B(Y,Z) = 0 = \frac{b}{2n-1}\{4n\eta(Y)\eta(Z) + 2g(\varphi Y,\varphi Z)\}. \tag{3.107}$$

For $Y = Z = \xi$, we obtain $b = 0$.

Hence we can state this result:

Theorem 3.5.1. *Let M^{2n+1} be an η-Einstein K-paracontact manifold. If it has Bach flat then M^{2n+1} is an Einstein manifold.*

3.6 (κ, μ)-paracontact manifold, for $\kappa \neq -1$.

In this section we deal with paracontact (κ, μ)-manifolds such that $\kappa > -1$ and $\kappa < -1$.

First for $\kappa > -1$, using (3.15) we calculate,

$$\begin{aligned}(\nabla_X Q)Y = \quad & g(2(n-1)+\mu)(\nabla_X h)Y \\ & +(2(n-1)+n(2\kappa-\mu))\{(\nabla_X \eta)Y\xi + \eta(Y)\nabla_X \xi\}\end{aligned} \tag{3.108}$$

Now considering the Cotton tensor on (κ, μ)-paracontact manifold as from (3.15), r is constant, which implies

$$C(X,Y)Z = g((\nabla_X Q)Y, Z) + g((\nabla_Y Q)X, Z). \tag{3.109}$$

Using equation (3.108) we obtain

$$\begin{aligned}C(X,Y)Z = \quad & (2(n-1)+\mu)\{-(1+\kappa)(2g(X,\varphi Y)\eta(Z) + \eta(X)g(\varphi Y, X) - \eta(Y)g(\varphi X, Z)) \\ & +(1+\mu)(\eta(X)g(\varphi hY, Z) - \eta(Y)g(\varphi hX, Z))\} \\ & +2(2(n-1)+n(2\kappa-\mu))g(X,\varphi Y)\eta(Z) \\ & +(2(n-1)+n(2\kappa-\mu))\{\eta(Y)g(-\varphi X + \varphi hX, Z) - \eta(X)g(-\varphi Y + \varphi hY, Z)\}.\end{aligned} \tag{3.110}$$

Replacing X, Y, Z by $\varphi X, \varphi Y, \varphi Z$ respectively in the above equation then we get

$C(\varphi X, \varphi Y)\varphi Z = 0.$

Similarly for $\kappa < -1$ we have from (3.16)

$$(\nabla_X Q)Y \;\; = \;\; g(2(n-1)+\mu)(\nabla_X h)Y + (2(n+1)+n(2\kappa-\mu))\{(\nabla_X \eta)Y\xi + \eta(Y)\nabla_X \xi\}.$$

Now consider the Cotton tensor with r is constant and substitute $(\nabla_X Q)Y$ and $(\nabla_Y Q)X$ values in Cotton tensor then we get

$$C(X,Y)Z = g((\nabla_X Q)Y,Z) + g((\nabla_Y Q)X,Z) \tag{3.111}$$

$$\begin{aligned} &= (2(n+1)+\mu)\{-(1+\kappa)(2g(X,\varphi Y)\eta(Z)+\eta(X)g(\varphi Y,X)-\eta(Y)g(\varphi X,Z)) \\ &\quad +(1+\mu)(\eta(X)g(\varphi hY,Z)-\eta(Y)g(\varphi hX,Z))\} \\ &\quad +2(2(n+1)+n(2\kappa-\mu))g(X,\varphi Y)\eta(Z) \\ &\quad +(2(n+1)+n(2\kappa-\mu))\{\eta(Y)g(-\varphi X+\varphi hX,Z)-\eta(X)g(-\varphi Y+\varphi hY,Z)\}. \end{aligned} \tag{3.112}$$

Replacing X,Y and Z by $\varphi X,\varphi Y$ and φZ respectively in the above equation, then $C(\varphi X,\varphi Y)\varphi Z=0$.

Form the above two cases, when $\kappa\neq -1$ we obtain the following result;

Proposition 3.6.1. *On a (κ,μ)-paracontact metric manifold for $\kappa\neq -1$ the projection of the image of Cotton tensor $C/_{\varphi T_P(M^{2n+1})X\varphi T_P(M^{2n+1})}$ in $\varphi T_p(M^{2n+1})$ is zero, i.e., $C(\varphi X,\varphi Y)\varphi Z=0$, $\forall X,Y,Z\in T_P(M^{2n+1})$.*

3.7 Vanishing Cotton tensor on (κ,μ)-paracontact manifold, for $\kappa\neq -1$

In this section we deal with paracontact (κ,μ)-manifolds such that $\kappa<-1$ and $\kappa>-1$ then we have the Cotton tensor $C(X,Y)Z=0$.

For $\kappa > -1$, replacing Z by ξ in equation (3.112) then we get

$$\begin{aligned} C(X,Y)\xi = 0 = \quad & (2(n-1)+\mu)\{-(1+\kappa)(2g(X,\varphi Y))\} + 2(2(n-1) \\ & +n(n(2\kappa-\mu))g(X,\varphi Y) \\ \implies \quad & (2(n-1)+\mu)(1+\kappa) + (2(n-1)+n(2n-\mu)) = 0 \end{aligned} \tag{3.113}$$

Similarly, admitting ξ in the place of X in equation (3.112) gives,

$$\begin{aligned} C(\xi,Y)Z = 0 = \quad & (2(n-1)+\mu)\{-(1+\kappa)g(\varphi Y,Z) + (1+\mu)g(\varphi hY,Z)\} \\ & +(2(n-1)n(2\kappa-\mu))\{g(\varphi Y,Z) - g(\varphi hY,Z)\}. \end{aligned} \tag{3.114}$$

Symmetrizing the above equation and replacing Y by hY we obtain

$$(1+\kappa)\{(2(n-1)+\mu)(1+\mu) - (2(n-1)+n(2\kappa-\mu)\} = 0.$$

From equation (3.113) it gives,

$$(1+\kappa)\{(2(n-1)+\mu)(1+\mu) - (2(n-1)+\mu)(1+\kappa)\} = 0.$$

$$\implies (1+\kappa)(\mu-\kappa)(2(n-1)+\mu) = 0.$$

The above calculations leads this result:

Case(i) If $\mu \neq \kappa$ then $(2(n-1)+\mu) = 0$. Therefore M^{2n+1} is η-Einstein.

Case(ii) If $\mu = \kappa$ then from equation (3.113) $\mu = \kappa = 0$ or $\mu = \kappa = 0$. Therefore we have the following result.

Lemma 3.7.1. *Let M^{2n+1} be a (κ,μ)-paracontact manifold, admitting vanishing Cotton tensor for $\kappa > -1$ then we have*

i). If $\mu \neq \kappa$ then M^{2n+1} is an η-Einstein manifold, *ii). If $(2(n-1)+\mu) \neq 0$ then $\mu = \kappa = 0$.*

Next for $\kappa < -1$, the Cotton tensor is

$$\begin{aligned}
C(X,Y)Z = \quad & (2(n+1)+\mu)\{(\nabla_X\eta)Y - (\nabla_Y\eta)X\} + (2(n+1)+n(\kappa-\mu))\{(\nabla_X\eta)Y\eta(Z) \\
& -(\nabla_X\eta)X\eta(Z)\} + (2(n-1)+n(2\kappa-\mu))\{\eta(Y)\nabla_X\xi - \eta(X)\nabla_Y\xi\} \\
= \quad & (2(n+1)+\mu)\{-(1+\kappa)2g(X,\varphi Y)\eta(Z) + \eta(X)g(\varphi Y,Z) - \eta(Y)g(\varphi X,Z)\} \\
& +(1+\mu)(\eta(X)g(\varphi hX,Z) - \eta(Y)g(\varphi hX,Z)) \\
& +2(2(n-1)+n(2\kappa-\mu))g(X,\varphi Y)\eta(Z) \qquad (3.115) \\
& +(2(n-1)+n(2\kappa-\mu)\{\eta(Y)g(-\varphi X+\varphi hX,Z) - \eta(X)g(-\varphi Y+\varphi hY,Z)\}.
\end{aligned}$$

Substitute Z by ξ in the above equation become

$$C(X,Y)\xi = 0 = \{(2(n+1)+\mu)(1+\kappa) - (2(n-1)+n(2\kappa-\mu)\} \qquad (3.116)$$

Replace X by ξ in the equation (3.115) gives

$$\begin{aligned}
C(\xi,Y)Z = 0 = \quad & (-2(n-1)+\mu)(1+\kappa)g(\varphi Y,Z) \qquad (3.117) \\
& +(2(n-1)+\mu)(1+\mu)g(\varphi hY,Z) \\
& +(2(n-1)+n(2\kappa+\mu))\{g(\varphi Y,Z) - g(\varphi hY,Z)\}.
\end{aligned}$$

On symmetrizing the above equation we have

$$(1+\kappa)(2(n+1)+\mu)(\mu-\kappa) = 0. \qquad (3.118)$$

Therefore we can state the following lemma

Lemma 3.7.2. *Let M^{2n+1} be a (κ, μ) paracontact metric manifold for $\kappa < -1$, if M^{2n+1} has vanishing Cotton tensor for $\mu \neq \kappa$ then M^{2n+1} is an η- Einstein manifold.*

From case (i) of lemma (3.7.1) and lemma (3.7.2) we get the following result.

Theorem 3.7.3. *Let M^{2n+1} be a (κ, μ)-paracontact manifold for $\kappa \neq -1$. If M^{2n+1} has vanishing Cotton tensor for $\mu \neq \kappa$, then M^{2n+1} is an η-Einstein manifold.*

3.8 Conclusion

In this chapter, we studied the Yamabe and Quasi Yamabe soliton on (κ, μ)-paracontact manifold and K-paracontact manifold and K-paracontact manifold admitting parallel Cotton tensor, vanishing Cotton tensor and the Bach flatness on K-paracontact manifold. Also we studied vanishing Cotton tensor on (κ, μ)-paracontact manifold for both $\kappa > -1$ and $\kappa < -1$. Main conclusions that can be drawn are:

- Let M be a non-para-Sasakian (κ, μ)-paracontact manifold and admits a Yamabe soliton then either V is Killing or M is locally isometric to the product of a flat $(n+1)$-dimensional manifold and n-dimensional manifold of constant negative curvature equal to -4.

- If a non-para-Sasakian (κ, μ)-paracontact manifold M admits a quasi Yamabe gradient soliton, then for $\kappa > -1$, either

  a. M is a $N(\frac{1-n}{n})$-manifold,

  or

  b. M is locally isometric to the product of a flat $(n+1)$-dimensional manifold and

n-dimensional manifold of constant negative curvature equal to -4,

or

c. f is constant on M.

Next, for $\kappa < -1$ either

a. $\mu = \frac{-4}{n+1}$

or

b. f is constant on M.

- Let M be a K-paracontact manifold with $Q\varphi = \varphi Q$ and if M holds a Yamabe soliton, then either $r = -2n(2n+1)$ or f is a constant.

- Let M be a K-paracontact manifold. Then M has constant scalar curvature if and only if $C(X, \xi)\xi = 0$.

- Let M be a K-paracontact metric manifold. Then M has parallel Cotton tensor if and only if M is an η-Einstein manifold and $r = -2n(2n+1)$.

- Let M be an η-Einstein K-paracontact manifold. If it is Bach flat then M is an Einstein manifold.

- On a (κ, μ)-paracontact metric manifold for $\kappa \neq -1$ the projection of the image of Cotton tensor $C/_{\varphi T_P(M) X \varphi T_P(M)}$ in $\varphi T_p(M)$ is zero, i.e., $C(\varphi X, \varphi Y)\varphi Z = 0$, $\forall X, Y, Z \in T_P(M)$.

- Let M be a (κ, μ)-paracontact manifold for $\kappa \neq -1$. If M has vanishing Cotton tensor for $\mu \neq \kappa$, then M is an η-Einstein manifold.

Chapter 4

Para- Kenmotsu Manifolds

4.1 Introduction

Para-Kenmotsu manifold known not only as a special case of almost para-contact structures but also as an analogue of para-Sasakian manifold and closely related to almost product structure.

The notion of local symmetry of a Riemannian manifold has been weakend by many authors in several ways to a different extent. As a weaker version of local symmetry, Takahashi [77] introduced the notion of locally φ-symmetry on a Sasakian manifold. Generalizing the notion of φ-symmetry, De [30] introduced the notion of φ-recurrent Sasakian manifold. In the context of contact geometry the notion of φ-symmetry is introduced and studied by Boeckx, Buecken and Vanhecke [7] with several examples. Recently Chaubey and Prasad worked on generalized φ-recurrent Kenmotsu manifolds. Ventatesha et al [86] introduced and studied quasi generalized φ- recurrent Sasakian manifold. In this chapter, we have extended the notation of quasi φ-recurrent structure called extended quasi φ-recurrent structure on para-Kenmotsu manifolds. Also in this chapter we study the para-Kenmotsu manifolds with C-Bochner curvature tensor.

A Riemannian manifold M is said to be pseudosymmetric in the sense of Deszcz [32] if

$$R(X,Y)\cdot R(U,V)Z = L_R((X\wedge Y)\cdot R(U,V)Z), \tag{4.1}$$

holds on $U_R = \{X \in M | R - \frac{r}{n(n-1)}G \neq 0 \quad at \quad x\}$, where G is the $(0,4)$ tensor defined by $G(X_1, X_2, X_3, X_4) = g((X_1 \wedge X_2)X_3, X_4)$, L_R is some smooth function on U_R and $(X\wedge Y)$ is an endomorphism defined by

$$(X\wedge Y)Z = g(Y,Z)X - g(X,Z)Y. \tag{4.2}$$

A Riemannian manifold M is said to be C-Bochner pseudosymmetric [26] if

$$R(X,Y)\cdot B(U,V)Z = L_B((X\wedge Y)\cdot B(U,V)Z), \tag{4.3}$$

holds on the set $U_B = \{x \in M : B \neq 0 \;\; at \quad x\}$, where L_B is some function on U_B and B is the C-Bochner curvature tensor.

C-Bochner curvature tensor on an almost contact metric manifold was defined by

Matsumoto and Chuman [54] and is given by

$$\begin{aligned}
B(X,Y)Z &= R(X,Y)Z + \frac{1}{2(n+2)}\{S(X,Z)Y - S(Y,Z)X + g(X,Z)QY \\
&- g(Y,Z)QX + S(\phi X,Z)\phi Y - S(\phi Y,Z)\phi X + g(\phi X,Z)Q\phi Y \\
&- g(\phi Y,Z)Q\phi X + 2S(\phi X,Y)\phi Z + 2g(\phi X,Y)Q\phi Z - S(X,Z)\eta(Y)\xi \\
&+ S(Y,Z)\eta(X)\xi - \eta(X)\eta(Z)QY + \eta(Y)\eta(Z)QX\} \\
&- \frac{\tau+2n}{2(n+2)}\{g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X + 2g(\phi X,Y)\phi Z\} \\
&- \frac{\tau-4}{2(n+2)}\{g(X,Z)Y - g(Y,Z)X\} + \frac{\tau}{2(n+2)}\{g(X,Z)\eta(Y)\xi \\
&- g(Y,Z)\eta(X)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\}, \qquad (4.4)
\end{aligned}$$

where $\tau = \frac{r+2n}{2(n+2)}$, Q is the Ricci operator i.e. $g(QX,Y) = S(X,Y)$ and r is the scalar curvature of the manifold.

In a para-Kenmotsu manifold, we have the following formulas (see [103]):

$$S(X,\xi) = -(n-1)\eta(X) \qquad (4.5)$$

$$Q\xi = -(n-1)\xi \qquad (4.6)$$

$$g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X) \qquad (4.7)$$

$$R(\xi,X)Y = \eta(Y)X - g(X,Y)\xi \qquad (4.8)$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X; \ \ \textit{when} \ \ X \ \ \textit{is} \ \ \textit{orthogonal} \ \ \textit{to} \ \ \xi \qquad (4.9)$$

$$R(X,\xi)Y = g(X,Y)\xi - \eta(Y)X, \qquad (4.10)$$

$$Ric(X,\xi) = -2n\eta(X) \qquad (4.11)$$

$$(\nabla_W R)(X,Y)\xi = g(W,X)Y - g(W,Y)X - R(X,Y)W. \qquad (4.12)$$

where S is the Ricci tensor and R is the Riemannian curvature tensor.

Using (4.5)-(4.9), one can get

$$B(X,\xi)Z = H\{\eta(Z)X - g(X,Z)\xi\}, \tag{4.13}$$

$$B(X,Y)\xi = H\{\eta(X)Y - \eta(Y)X\}, \tag{4.14}$$

$$B(\xi,Y)Z = H\{g(Y,Z)\xi - \eta(Z)Y\}, \tag{4.15}$$

$$\eta(B(X,Y)Z) = H\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}, \tag{4.16}$$

where H is a constant i.e., $H = \{1 + \frac{n-1}{n+2} + \frac{\tau-4}{2(n+2)} - \frac{\tau}{2(n+2)}\}$.

Definition 4.1.1. A $(2n+1)$-dimensional Sasakian manifold M is said to be η-Einstein if its Ricci tensor Ric is of the form

$$Ric(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) \tag{4.17}$$

for any vector fields X and Y where a and b are smooth functions. If $b = 0$, then the manifold M is an Einstein manifold.

4.2 Extended Quasi Generalized φ-recurrent Para-Kenmotsu Manifolds

In this section we study extended quasi generalized φ-recurrent Para-Kenmotsu manifolds

Definition 4.2.1. A Kenmotsu manifold M is said to be extended quasi generalized φ-recurrent manifold if its curvature tensor R satisfies the condition

$$\varphi^2((\nabla_W R)(X,Y)Z) = \Pi_1(W)\varphi^2 R(X,Y)Z + \Pi_2(W)\varphi^2 F(X,Y)Z, \tag{4.18}$$

for all $X, Y, Z \in TM$, where Π_1 and Π_2 are two non-vanishing 1-forms such that $\Pi_1(X) = g(X,\chi_1)$, $\Pi_2(X) = g(X,\chi_2)$ and the tensor F is defined by

$$\begin{aligned} F(X,Y)Z =& g(Y,Z)X - g(X,Z)Y + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \\ &+ g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi, \end{aligned} \tag{4.19}$$

for all $X, Y, Z \in TM$. Here χ_1 and χ_2 are vector fields associated with 1-forms Π_1 and Π_2 respectively. Especially, if the 1-form Π_2 vanishes, then (4.18) turns into the notion of φ-recurrent manifold.

Now we start this section with the following:

Theorem 4.2.1. *Let M be a para-Kenmotsu manifold. If M is an extended quasi φ-recurrent manifold, then M is super generalized Ricci-recurrent manifold.*

Proof. Let us consider an extended quasi generalized φ-recurrent Kenmotsu manifold. Then by the use of (1.87), we have from (4.18) that

$$\begin{aligned}(\nabla_W R)(X,Y)Z - \eta((\nabla_W R)(X,Y)Z)\xi =& \Pi_1(W)\left\{R(X,Y)Z - \eta(R(X,Y)Z)\xi\right\}\\ &+ \Pi_2(W)\left\{F(X,Y)Z - \eta(F(X,Y)Z)\xi\right\}.\end{aligned} \tag{4.20}$$

The above equation can also be written as

$$\begin{aligned}g((\nabla_W R)(X,Y)Z,U) - \eta((\nabla_W R)(X,Y)Z)\eta(U) =& \Pi_1(W)\left\{g(R(X,Y)Z,U) - \eta(R(X,Y)Z)\eta(U)\right\}\\ &+\Pi_2(W)\left\{g(F(X,Y)Z,U) - \eta(F(X,Y)Z)\eta(U)\right\}.\end{aligned} \tag{4.21}$$

Applying $X = U = e_i$ in (4.21) and taking $\sum_{i=1}^{2n+1}$, and then using (4.19), we obtain

$$\begin{aligned}(\nabla_W Ric)(Y,Z) - \sum_{i=1}^{2n+1}\eta((\nabla_W R)(e_i,Y)Z)\eta(e_i) =& \Pi_1(W)\left\{Ric(Y,Z) - \eta(R(\xi,Y)Z)\right\}\\ &+ \Pi_2(W)\left\{(2n-1)g(Y,Z) + (2n+1)\eta(Y)\eta(Z)\right\}.\end{aligned} \tag{4.22}$$

The second term of left hand side in above relation becomes

$$\sum_{i=1}^{2n+1}\eta((\nabla_W R)(X,Y)Z)\eta(e_i) = g((\nabla_W R)(\xi,Y)Z,\xi). \tag{4.23}$$

As a result of (4.10), (4.12) and $g((\nabla_W R)(X,Y)Z,U) = -g((\nabla_W R)(X,Y)U,Z)$ we get

$$g((\nabla_W R)(\xi,Y)Z,\xi) = 0. \tag{4.24}$$

Considering (4.23) and (4.24), it follows from (4.22) that

$$\begin{aligned}(\nabla_W Ric)(Y,Z) =&\Pi_1(W)S(Ric,Z) + \{\Pi_1(W) + (2n-1)\Pi_2(W)\}\, g(Y,Z)\\ &+ \{\Pi_2(W)(2n+1) - \Pi_1(W)\}\, \eta(Y)\eta(Z),\end{aligned} \tag{4.25}$$

showing that M is super generalized Ricci-recurrent manifold. Taking $Z = \xi$ in (4.25) and then using (4.11), we obtain

$$(\nabla_W Ric)(Y,\xi) = 2n\{2\Pi_2(W) - \Pi_1(W)\}\, \eta(Y). \tag{4.26}$$

Now differentiating (4.11) and we obtain

$$(\nabla_W Ric)(Y,\xi) = -2ng(Y,W) - Ric(Y,W). \tag{4.27}$$

On comparing (4.26) and (4.27) we get,

$$Ric(Y,W) = -2n[2\Pi_2(W) - \Pi_1(W)]\eta(Y) - 2ng(Y,W). \tag{4.28}$$

Plugging $Y = \xi$ in the above equation and then using (4.12) we have,

$$2\Pi_2(W) - \Pi_1(W) = 0. \tag{4.29}$$

On using (4.29) in (4.28) we obtain,

$$Ric(Y,W) = -2ng(Y,W). \tag{4.30}$$

Now from (4.28) and (4.30) we can state the following; □

Theorem 4.2.2. *If a para-Kenmotsu manifold M is an extended quasi φ- recurrent manifold, then M is an Einstein manifold. Moreover, the associated vector fields χ_1 and χ_2 of the 1-forms Π_1 and Π_2 respectively are co-directional.*

If we take $\Pi_2 = 0$ in (4.18), then by the above theorem we conclude the following;

Corollary 4.2.3. *Every generalized φ-recurrent para-Kenmotsu manifold is an Einstein manifold.*

Corollary 4.2.4. *Every locally φ-symmetric para-Kenmotsu manifold is an Einstein manifold.*

Let us consider (4.20) and changing W, X, Y cyclically and adding them, we get by the view of Bianchi identity and (4.29) that

$$\Pi_2(W)\left\{2g(R(X,Y)Z,U-2\eta(R(X,Y)Z)\eta(U)+g(F(X,Y)Z,U)-\eta(F(X,Y)Z)\eta(U)\right\}$$
$$+\Pi_2(X)\left\{2g(R(Y,W)Z,U)-2\eta(R(Y,W)Z)\eta(U)+g(F(Y,W)Z,U)-\eta(F(Y,W)Z)\eta(U)\right\}$$
$$\Pi_2(Y)\left\{2g(R(W,X)Z,U)-\eta(R(W,X)Z)\eta(U)+g(F(W,X)Z,U)-\eta(F(W,X)Z)\eta(U)\right\}=0.$$

Contracting the above relation over X and U and using (4.19), we get

$$\Pi_2(W)\left\{2Ric(Y,Z)+(2n+1)g(Y,Z)+(2n-1)\eta(Y)\eta(Z)\right\}+2\Pi_2(R(Y,W)Z)+g(W,Z)B(Y)$$
$$g(Y,Z\Pi_2B(W)+\eta(W)\eta(Z)B(Y)-\eta(Y)\eta(Z)\Pi_2(W)+\left\{g(W,Z)\eta(Y)-g(Y,Z)\eta(W)\right\}\Pi_2(\xi)$$
$$+\Pi_2(Y)\left\{-2Ric(Y,Z)-(2n+1)g(W,Z)-(2n-1)\eta(W)\eta(z)\right\}=0. \tag{4.31}$$

Again contracting above equation (4.31) over Y and Z we have

$$Ric(W,\chi_2)=\left\{\frac{2r+2n(2n-1)}{r}\right\}g(W,\chi_2)-\frac{(2n+1)}{2}\eta(W)\eta(\chi_2). \tag{4.32}$$

From this we can conclude the following ;

Theorem 4.2.5. *In an extended quasi generalized φ-recurrent para-Kenmotsu manifold, the Ricci tensor Ric and vector field χ_2 are related by the equation (4.32).*

$$(\nabla_W R)(X,Y)\xi=g(W,X)Y-g(W,Y)X-R(X,Y)W. \tag{4.33}$$

Taking inner product of (4.12) with ξ and using the relation $\eta(R(X,Y)Z)=-\eta(X)g(Y,Z)+\eta(Y)g(X,Z)$, we obtain

$$\eta((\nabla_W R)(X,Y)\xi)=0. \tag{4.34}$$

Considering (4.34) and (4.10) in (4.20), we reach at

$$(\nabla_W R)(X,Y)\xi = \{A(W) - 2B(W)\}\{\eta(X)Y - \eta(Y)X\}. \tag{4.35}$$

As a result of (4.33) and (4.35), we obtain

$$R(X,Y)W = g(W,X)Y - g(W,Y)X - (A(W) - 2B(W))\{\eta(X)Y - \eta(Y)X\}. \tag{4.36}$$

Making use of (4.29) in the above equation, we obtain

$$R(X,Y)W = g(W,X)Y - g(W,Y)X, \tag{4.37}$$

for all $X, Y, W \in TM$.

Theorem 4.2.6. *An extended quasi generalized concirrcular φ-recurrent para-Kenmotsu manifold M is of constant sectional curvature -1.*

Now we consider an extended quasi generalized concircular φ reccurent para-Kenmotsu manifold M. Then from (4.18), we have

$$\varphi^2((\nabla_W D)(X,Y)Z) = A(W)\varphi^2 D(X,Y)Z + B(W)\varphi^2 F(X,Y)Z, \tag{4.38}$$

where A, B and F are defined as in (4.18) and (4.19), and D is a concircular curvature tensor and is defined by

$$D(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)}\{g(Y,Z)X - g(X,Z)Y\}. \tag{4.39}$$

Then by the virtue of (1.87) we get

$$\begin{aligned}(\nabla_W D)(X,Y)Z - \eta((\nabla_W D)(X,Y)Z)\xi =& \Pi_1(W)\{D(X,Y)Z - \eta(D(X,Y)Z)\xi\}\\ &+ \Pi_2(W)\{F(X,Y)Z - \eta(F(X,Y)X)\xi\}.\end{aligned} \tag{4.40}$$

This can also we write it as

$$g((\nabla_W D)(X,Y)Z,U) - \eta((\nabla_W D)(X,Y)Z)\eta(U) = \Pi_1(W)\{g(D(X,Y)Z,U) - \eta(D(X,Y)Z)\eta(U)\}$$
$$+\Pi_2(W)\{g(F(X,Y)Z,U) - \eta(F(X,Y)Z)\eta(U)\}. \tag{4.41}$$

Contracting the above equation over X and U and using (4.19), (4.24) and (4.39), we have

$$\begin{aligned}(\nabla_W Ric)(Y,Z) =& \Pi_1(W)Ric(Y,Z)\\ &+(2n-1)\left[\frac{dr(W)}{2n(2n+1)} - \left(\frac{r}{2n(2n+1)} - \frac{1}{2n-1}\right)\Pi_1(W) + \Pi_2(W)\right]g(Y,Z)\\ &-\left[\left(1+\frac{r}{2n(2n+1)}\right)\Pi_1(W) - \frac{dr(W)}{2n(2n+1)} - (2n+1)\Pi_2(W)\right]\eta(Y)\eta(X).\end{aligned} \tag{4.42}$$

The above equation can also be written as

$$\nabla S = \Pi_1 \otimes Ric + \psi \otimes g + \beta \otimes \eta \otimes \eta, \tag{4.43}$$

where

$$\psi(W) = (2n-1)\left[\frac{dr(W)}{2n(2n+1)} - \left(\frac{r}{2n(2n+1)} - \frac{1}{2n-1}\right)\Pi_1(W) + \Pi_2(W)\right] \tag{4.44}$$
$$\beta(W) = -\left[\left(1+\frac{r}{2n(2n+1)}\right)\Pi_1(W) - \frac{dr(W)}{2n(2n+1)} - (2n+1)\Pi_2(W)\right] \tag{4.45}$$

From the above we can state the following.

Theorem 4.2.7. *Let M be a para-Kenmotsu manifold. If M is an extended quasi φ-recurrent manifold, then M is super generalized Ricci-recurrent manifold.*

Plugging $Y = Z = \xi$ in (4.42) and using (4.5), we have

$$dr(W) = [2n(2n+1) + r]\Pi_1(W) - 2(n+1)(2n+1)\Pi_2(W). \tag{4.46}$$

From this we can state the following theorem

Theorem 4.2.8. *Let M be a para-Kenmotsu manifold and if M is an extended quasi φ-recurrent manifold, then the 1-forms Π_1 and Π_2 are related by the equation (4.46).*

Corollary 4.2.9. *In an extended quasi generalized concircularly φ-recurrent Para-Kenmotsu manifold with non-zero constant scalar curvature, the associated 1-forms Π_1 and Π_2 are related by*

$$[2n(2n+1)+r]\Pi_1 - 2(n+1)(2n+1)\Pi_2 = 0. \tag{4.47}$$

4.3 Example :-

In the present section we give an example of a extended quasi generalized φ-recurrent para-Kenmotsu manifold. We consider three-dimensional manifold $M = \{(x, y, z) \in R^3, z \neq 0\}$ with the cartesian coordinates (x, y, z) and the vector fields:

$$\partial_1 = \varphi\partial_2, \quad \partial_2 = \varphi\partial_1, \quad \varphi\partial_3 = 0, \tag{4.48}$$

where

$$\partial_1 = \frac{\partial}{\partial x}, \quad \partial_2 = \frac{\partial}{\partial y}, \quad \partial_3 = -\frac{\partial}{\partial z}. \tag{4.49}$$

The 1-form $\eta = dz$ defines an almost paracontact structure on M with characteristic vector field $\xi = \partial_3 = -\frac{\partial}{\partial z}$. Let g be a pseudo-Riemannian metric defined by

$$g(\partial_1, \partial_1) = 1, \quad g(\partial_2, \partial_2) = -1, \quad g(\partial_3, \partial_3) = 1 \tag{4.50}$$

$$g(\partial_1, \partial_2) = 0, \quad g(\partial_1, \partial_3) = 0, \quad g(\partial_2, \partial_3) = 0.$$

Then using the linearity of η and g we have

$$\eta(\partial_3) = 1, \quad \varphi^2 W = W - \eta(W)\partial_3, \tag{4.51}$$

$$g(\varphi W, \varphi U) = -g(W, U) + \eta(W)\eta(U), \tag{4.52}$$

for any $U, W \in TM$. Let ∇ be the Levi-Civita connection with respect to the metric g. Using Koszul formula, we have:

$$\nabla_{\partial_1}\partial_1 = -\partial_3, \quad \nabla_{\partial_1}\partial_2 = 0, \quad \nabla_{\partial_1}\partial_3 = \partial_1,$$

$$\nabla_{\partial_2}\partial_1 = 0 \quad \nabla_{\partial_2}\partial_2 = \partial_3, \quad \nabla_{\partial_2}\partial_3 = \partial_2,$$

$$\nabla_{\partial_3}\partial_1 = \partial_1, \quad \nabla_{\partial_3}\partial_2 = \partial_2, \quad \nabla_{\partial_3}\partial_3 = 0.$$

From the above, the manifold under consideration is a paraKenmotsu manifold. With the help of the above results, we can calculate the components of the curvature tensor R as follows;

$$R(\partial_2, \partial_1)\partial_1 = -\partial_2, \quad R(\partial_1, \partial_2)\partial_2 = \partial_1, \quad R(\partial_1, \partial_2)\partial_3 = 0,$$

$$R(\partial_3, \partial_1)\partial_1 = \partial_3, \quad R(\partial_1, \partial_3)\partial_2 = 0, \quad R(\partial_1, \partial_3)\partial_3 = -\partial_1$$

$$R(\partial_2, \partial_3)\partial_1 = 0, \quad R(\partial_3, \partial_2)\partial_2 = -\partial_3, \quad R(\partial_2, \partial_3)\partial_3 = -\partial_2.$$

Since $\partial_1, \partial_2, \partial_3$ forms a basis of the three-dimensional para-Kenmotsu manifold, vector fields $X, Y, Z \in TM$ can be written as

$$X = a_1 + \partial_1 + b_1\partial_2 + c_1\partial_3$$

$$Y = a_2 + \partial_1 + b_2\partial_2 + c_2\partial_3$$

$$Z = a_3 + \partial_1 + b_3\partial_2 + c_3\partial_3$$

where $a_i, b_i, c_i \in R^+$(the set of all positive real numbers), $i = 1, 2, 3$. Then

$$\begin{aligned} R(X,Y)Z = \quad & [b_3(a_1b_2 - a_2b_1) + c_3(c_1a_2 - a_1c_2)]\partial_1 \\ & +[a_3(a_1b_2 - b_1a_2) + c_3(c_1b_2 - b_1c_2)]\partial_2 \\ & +[a_3(a_2c_1 - c_2a_1) + b_3(c_2b_1 - b_2c_1)]\partial_3, \end{aligned} \tag{4.53}$$

and

$$\begin{aligned} F(X,Y)Z = \quad & [b_3(b_1a_2 - a_1b_2) + 2c_3(c_2a_1 - a_2c_1)]\partial_1 \\ & +[a_3(a_2b_1 - b_2a_1) + 2c_3(c_2b_1 - b_2c_1)]\partial_2 \\ & +2[a_3(c_1a_2 - a_1c_2) + b3(b_1c_2 - c_1b_2)]\partial_3. \end{aligned} \tag{4.54}$$

From (4.53), we have the following

$$\begin{aligned} (\nabla_{\partial_1}R)(X,Y)Z = & \{b_3(c_2b_1 - b_2c_1) + 2a_3(c_1a_2 - a_1c_2)\}\partial_1 - 2a_1b_2c_3\partial_2 \\ & + \{2b_3(a_2b_1 - a_1b_2) + 2c_3(a_1c_2 - c_1a_2)\}\partial_3, \end{aligned} \tag{4.55}$$

$$\begin{aligned} (\nabla_{\partial_2}R)(X,Y)Z = & \{b_3(c_2b_1 - b_2c_1) + 2a_3(c_1a_2 - a_1c_2)\}\partial_2 \\ & + \{2c_3(c_1b_2 - b_1c_2) + 2a_3(a_1b_2 - b_1a_2)\}\partial_3, \end{aligned} \tag{4.56}$$

$$\begin{aligned} (\nabla_{\partial_3}R)(X,Y)Z = & 2a_3(b_1a_2 - b_2a_1)\partial_2 - 2a_1b_2b_3\partial_1 \\ & + \{2b_3(c_1b_2 - b_1c_2) + 2a_3(a_1c_2 - c_1a_2)\}\partial_3. \end{aligned} \tag{4.57}$$

Now considering (4.53) and (4.54) we have,

$$\varphi^2(R(X,Y)Z) = \gamma_1\partial_1 + \gamma_2\partial_2, \ \ and \ \ \varphi^2(F(X,Y)Z) = \delta_1\partial_1 + \delta_2\partial_2, \tag{4.58}$$

where

$$\gamma_1 = [b_3(a_1b_2 - a_2b_1) + c_3(c_1a_2 - a_1c_2)],$$

$$\gamma_2 = [a_3(a_1b_2 - b_1a_2) + c_3(c_1b_2 - b_1c_2)],$$

$$\delta_1 = [b_3(b_1a_2 - a_1b_2) + 2c_3(c_2a_1 - a_2c_1)],$$

$$\delta_2 = [a_3(a_2b_1 - b_2a_1) + 2c_3(c_2b_1 - b_2c_1)].$$

In the view of (4.55), (4.56) and (4.57) we obtain

$$\varphi^2(\nabla_{\partial_i}R(X,Y)Z) = m_i\partial_1 + n_i\partial_2 \quad for \quad i = 1,2,3 \tag{4.59}$$

namely one parts

$$m_1 = \{b_3(c_2b_1 - b_2c_1) + 2a_3(c_1a_2 - a_1c_2)\},$$

$$n_1 = -2a_1b_2c_3,$$

$$m_2 = 0,$$

$$n_2 = \{b_3(c_2b_1 - b_2c_1) + 2a_3(c_1a_2 - a_1c_2)\},$$

$$m_3 = -2a_1b_2b_3,$$

$$n_3 = 2a_3(b_1a_2 - b_2a_1).$$

Now let us consider the 1-form as

$$\Pi_1(\partial_1) = \frac{m_1\delta_2 - n_1\delta_1}{\delta_2\gamma_1 - \gamma_2\delta_1}, \qquad \Pi_2(\partial_1) = \frac{\gamma_2m_1 - \gamma_1n_1}{\delta_1\gamma_2 - \delta_2\gamma_1}, \tag{4.60}$$

$$\Pi_1(\partial_2) = \frac{\delta_1n_2}{\gamma_2\delta_1 - \gamma_1\beta_2}, \qquad \Pi_2(\partial_2) = \frac{\gamma_1n_2}{\gamma_1\delta_2 - \gamma_2\delta_1}, \tag{4.61}$$

$$\Pi_1(\partial_3) = \frac{\delta_2m_3 - \delta_1n_3}{\gamma_1\delta_2 - \delta_1\gamma_2}, \qquad \Pi_2(\partial_3) = \frac{\gamma_2m_3 - \gamma_1n_3}{\delta_1\gamma_2 - \gamma_1\delta_2}, \tag{4.62}$$

where $m_1\delta_2 - n_1\delta_1 \neq 0$, $\gamma_2 m_1 - \gamma_1 n_1$, $\delta_1 n_2 \neq 0$, $\gamma_1 n_2 \neq 0$, $\delta_2 m_3 - \delta_1 n_3 \neq 0$, $\gamma_2 m_3 - \gamma_1 n_3 \neq 0$ and $\gamma_1\delta_2 - \delta_1\gamma_2 \neq 0$. From (4.38), we obtain

$$\varphi^2 = (\nabla_{\partial_i} R(X,Y)Z) = \Pi_1(\partial_i)\varphi^2 R(X,Y)Z + \Pi_2(\partial_i)\varphi^2 F(X,Y)Z \tag{4.63}$$

for $i = 1, 2, 3$. By the view of (4.60), (4.61), (4.62) and (4.58), this shows that the manifold satisified (4.63). Hence the manifold is a 3-dimensional extended quasi generalized φ-recurrent para-Kenmotsu manifold, which is not φ-recurrent.

4.4 C-Bochner Pseudosymmetric para-Kenmotsu manifolds

A n-dimensional para-Kenmotsu manifold M is said to be C-Bochner pseudosymmetric if

$$(R(X,Y) \cdot B)(U,V)W = L_B[((X \wedge Y) \cdot B)(U,V)W], \tag{4.64}$$

holds on the set $U_B = \{x \in M : B \neq 0\}$ at x, where L_B is some function on U_B.

Let M be a C-Bochner pseudosymmetric para-kenmotsu manifold. Then from (4.64) we have

$$(R(X,\xi) \cdot B)(U,V)W = L_B[((X \wedge_g \xi) \cdot B)(U,V)W]. \tag{4.65}$$

Now from (4.8), the left-hand side of equation (4.65) becomes

$$\begin{aligned} & \{\eta(B(U,V)W)\xi - g(X,B(U,V)W)\xi - \eta(U)B(X,V)W \\ + & g(X,U)B(\xi,V)W - \eta(V)B(U,X)W + g(X,V)B(U,\xi)W \\ - & \eta(W)B(U,V)X + g(X,W)B(U,V)\xi\} = 0. \end{aligned}$$

which implies

$$\begin{aligned} & \{g(\xi,B(U,V)W)\xi - g(X,B(U,V)W)\xi - \eta(U)B(X,V)W \\ + & g(X,U)B(\xi,V)W - \eta(V)B(U,X)W + g(X,V)B(U,\xi)W \\ - & \eta(W)B(U,V)X + g(X,W)B(U,V)\xi\} = 0. \end{aligned} \tag{4.66}$$

Using (4.2), the right hand side of equation (4.65) turns into

$$\begin{aligned} & L_B\{g(\xi,B(U,V)W)\xi - g(X,B(U,V)W)\xi - \eta(U)B(X,V)W \\ + & g(X,U)B(\xi,V)W - \eta(V)B(U,X)W + g(X,V)B(U,\xi)W \\ - & \eta(W)B(U,V)X + g(X,W)B(U,V)\xi\} = 0. \end{aligned} \tag{4.67}$$

By virtue of (4.66) and (4.67), (4.65) give rise to

$$\begin{aligned} & (1-L_B)\{g(\xi,B(U,V)W)\xi - g(X,B(U,V)W)\xi - \eta(U)B(X,V)W \\ + & g(X,U)B(\xi,V)W - \eta(V)B(U,X)W + g(X,V)B(U,\xi)W \\ - & \eta(W)B(U,V)X + g(X,W)B(U,V)\xi\} = 0, \end{aligned} \tag{4.68}$$

which implies $L_B = 1$ or

$$\begin{aligned} &\{g(\xi, B(U,V)W)\xi - g(X, B(U,V)W)\xi - \eta(U)B(X,V)W \\ + \; &g(X,U)B(\xi,V)Z - \eta(V)B(U,X)W + g(X,V)B(U,\xi)W \\ - \; &\eta(W)B(U,V)X + g(X,W)B(U,V)\xi\} = 0. \end{aligned} \tag{4.69}$$

Putting $W = \xi$ in the above equation and simplifying we get

$$B(U,V)X = \{g(X,V)U - g(X,U)V\}. \tag{4.70}$$

Thus, we have the following assertion;

Theorem 4.4.1. *If a n-dimensional para-Kenmotsu manifold M is C-Bochner Pseudosymmetric then M is locally isometric to a sphere or $L_B = 1$.*

4.5 Para-Kenmotsu manifolds Satisfying $B(\xi, X) \cdot B = 0$

Let us consider a para-Kenmotsu manifold satisfying $B(\xi, X) \cdot B = 0$. Then we have,

$$\begin{aligned} &B(\xi,X)B(U,V)W - B(B(\xi,X)U,V)W \\ - \; &B(U,B(\xi,X)V)W - B(U,V)B(\xi,X)W = 0, \end{aligned} \tag{4.71}$$

In view of (4.8), (4.71) gives

$$\begin{aligned} &H[g(X,B(U,V)W)\xi - \eta(B(U,V)W)X - g(X,U)B(\xi,V)W \\ + \; &\eta(U)B(X,V)W - g(X,V)B(U,\xi)W + \eta(V)B(U,X)W \\ - \; &g(X,W)B(U,V)\xi + \eta(W)B(U,V)X] = 0. \end{aligned} \tag{4.72}$$

Setting $V = \xi$ in (4.72) and making use of (4.13), we get

$$B(U, X)W = -\{g(X, W)U - g(U, W)X\}. \tag{4.73}$$

Hence, we can state the following:

Theorem 4.5.1. *If a n-dimensional para-Kenmotsu manifold M satisfies $B(\xi, X) \cdot B = 0$ then M is isometric to a hyperbolic space.*

4.6 Para-Kenmotsu manifold Satisfying $B(\xi, X) \cdot R = 0$

Suppose a para-Kenmotsu manifold M_n satisfies $B(\xi, X) \cdot R = 0$. The condition $B(\xi, U) \cdot R = 0$ implies that

$$\begin{aligned} & B(\xi, U)R(X, Y)Z - R(B(\xi, U)X, Y)Z \\ - \quad & R(X, B(\xi, U)Y)Z - R(X, Y)B(\xi, U)Z = 0. \end{aligned} \tag{4.74}$$

By virtue of (4.14), (4.74) turns into

$$\begin{aligned} & H[g(U, R(X, Y)Z)\xi - \eta(R(X, Y)Z)U - g(U, X)R(\xi, Y)Z \\ + \quad & \eta(X)R(U, Y)Z - g(U, Y)R(X, \xi)Z + \eta(Y)R(X, U)Z \\ - \quad & g(U, Z)R(X, Y)\xi + \eta(Z)R(X, Y)U] = 0. \end{aligned} \tag{4.75}$$

Plugging $Z = \xi$ in (4.75) and using (4.9), one can get

$$H\{-g(U, X)Y + g(U, Y)X - R(X, Y)U\} = 0. \tag{4.76}$$

which yields, either $H = 0 \Longrightarrow \tau = 2n$,

or

$$R(X,Y)U = [g(Y,U)X - g(X,U)Y]. \tag{4.77}$$

Thus, we can state the following theorem;

Theorem 4.6.1. *An n-dimensional para-Kenmotsu manifold M_n satisfying the condition $B(\xi, X) \cdot R = 0$ is locally isometric to a Sphere or $\tau = 2n$.*

4.7 Para-Kenmotsu manifolds Satisfying $B(\xi, X) \cdot S = 0$

Consider a para-Kenmotsu manifolds M_n satisfying $B(\xi, X) \cdot S = 0$. Then we have

$$S(B(\xi, X)Y, \xi) + S(Y, B(\xi, X)\xi) = 0. \tag{4.78}$$

Using (4.14) and (4.13) in (4.78), we get

$$S(X,Y) = -(n-1)g(X,Y). \tag{4.79}$$

Now we can state the following;

Theorem 4.7.1. *A n-dimensional para-Kenmotsu manifold satisfying $B(\xi, X) \cdot S = 0$ is an Einstein manifold.*

4.8 Conclusion

In this chapter, we studied some geometric properties of extended quasi generalized φ-recurrent para-Kenmotsu manifolds. And a proper example is also provided to demonstrate the existence of an extended quasi-generalized φ-recurrent para-Kenmotsu manifold. Also we study C-Bochner pseudosymmetric para-Kenmotsu manifold. We have obtained the following results;

- Let M be a para-Kenmotsu manifold. If M is an extended quasi φ- recurrent manifold, then M is super generalized Ricci-recurrent manifold.

- Let M be a para-Kenmotsu manifold is an extended quasi φ- recurrent manifold, then M is an Einstein manifold. Moreover, the associated vector fields χ_1 and χ_2 of 1-forms Π_1 and Π_2 respectively are co-directional.

- Let a para-Kenmotsu manifold M admitting an extended quasi generalized φ-recurrent, then M is of constant sectional curvature -1.

- Let M be a para-Kenmotsu manifold and if M is an extended quasi φ- recurrent manifold, then the 1-forms Π_1 and Π_2 are related by the equation $dr(W) = [2n(2n+1) + r]\Pi_1(W) - 2(n+1)(2n+1)\Pi_2(W)$.

- Let M be a n-dimensional para-Kenmotsu manifold is C-Bochner Pseudo-symmetric then M_n is locally isometric to a sphere or $L_B = 1$.

- Let M be a n-dimensional para-Kenmotsu manifold satisfies $B(\xi, X) \cdot B = 0$ then M is isometric to a hyperbolic space.

- Let M be a n-dimensional para-Kenmotsu manifold satisfying the condition $B(\xi, X) \cdot R = 0$ is locally isometric to a sphere or $\tau = 2n$.

- Let M be a n-dimensional para-Kenmotsu manifold satisfying $B(\xi, X) \cdot S = 0$ is an Einstein manifold.

Chapter 5

$(LCS)_n$- Manifolds

5.1 Introduction

The Lorentzian concircular structure n-manifold or simply an $(LCS)_n$-manifold is a generalization of LP-Sasakian manifold [71]. A n-dimensional Lorentzian manifold M is a smooth, connected, paracontact and Hausdorff manifold with a Lorentzian metric g. i.e., M admits a smooth symmetric tensor field g of type $(0, 2)$ such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \longrightarrow \mathbb{R}$ is a non degenerate inner product of signature $(-, +, ...-, +)$, where T_pM denotes the tangent space of M at p and $\mathbb{R}$ is the real number space.

In a Lorentzian manifold (M, g), a vector field P defined by

$$g(X, P) = A(X), \tag{5.1}$$

for any vector field $X \in (T_pM)$ is said to be concircular vector field [99], if

$$(\nabla_X A)(Y) = \alpha \; \{g(X, Y) + \omega(X)\omega(Y)\}, \tag{5.2}$$

where α is a non zero scalar function, A is a 1-form and ω is a closed 1-form.

Since ξ is the unit concircular vector field, there exists a non-zero 1-form η such that

$$g(X, \xi) = \eta(X), \tag{5.3}$$

and hence the equation

$$(\nabla_X \eta)(Y) = \alpha\{g(X, Y) + \eta(X)\eta(Y)\}, \tag{5.4}$$

holds for all vector fields X and Y. Here ∇ denotes the operator of covariant differentiation with respect to Lorentzian metric g and α is a non zero scalar function satisfying

$$(\nabla_X \alpha) = X(\alpha) = \rho\eta(X), \tag{5.5}$$

where ρ being a scalar function. If we put

$$\phi X = \frac{1}{\alpha}\nabla_X \xi, \tag{5.6}$$

then from (5.4) and (5.6), we have

$$\phi X = X + \eta(X)\xi, \tag{5.7}$$

from which it follows that ϕ is a symmetric $(1, 1)$-tensor. Thus the Lorentzian manifold M together with unit time like concircular vector field ξ, an associated 1-form η and a $(1, 1)$-tensor field ϕ is said to be Lorentzian concircular structure manifold [70] or more briefly $(LCS)_n$- manifold. In particular, if $\alpha = 1$, then the manifold reduces to LP-Sasakian manifold [50].

A n-dimensional differentiable manifold M is called an $(LCS)_n$ manifold if it admits a $(1, 1)$ tensor field, a contravariant vector field ξ, a 1-form η, and a Lorentzian metric g

such that [71]:

$$\phi^2 = X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = -1, \tag{5.8}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{5.9}$$

$$(\nabla_X \phi)(Y) = \alpha\{g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi\}. \tag{5.10}$$

It is easy to see that the following relations hold in an $(LCS)_n$ manifold [70]:

$$R(X, Y)Z = (\alpha^2 - \rho)\{g(Y, Z)X - g(X, Z)Y\}, \tag{5.11}$$

$$R(\xi, Y)Z = (\alpha^2 - \rho)\{g(Y, Z)\xi - \eta(Z)Y\}, \tag{5.12}$$

$$R(X, Y)\xi = (\alpha^2 - \rho)\{\eta(Y)X - \eta(X)Y\}, \tag{5.13}$$

$$\eta(R(X, Y)Z) = (\alpha^2 - \rho)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}, \tag{5.14}$$

$$S(X, \xi) = (n - 1)(\alpha^2 - \rho)\eta(X), \tag{5.15}$$

$$S(\phi Y, \phi Z) = S(Y, Z) + (n - 1)(\alpha^2 - \rho)g(Y, Z). \tag{5.16}$$

The authours in [55] investigated a new curvature tensor of type $(1, 3)$ in an n-dimensional Riemannian manifold and is called as Q-curvature tensor, and is given by

$$Q(X, Y)Z = R(X, Y)Z - \frac{\Psi}{(n - 1)}\{g(Y, Z)X - g(X, Z)Y\}, \tag{5.17}$$

where Ψ is the arbitrary scalar function. If $\Psi = \frac{r}{2n+1}$, then it converts into concircular curvature tensor. Recently authors in [25] studied the generalized Sasakian space forms with Q-curvature tensor. The authors in [55] introduced the Z-tensor of type $(0, 2)$ and it is a new kind of Riemannian manifold that generalize the concept of both pseudo Ricci symmetric manifold and pseudo projective Ricci symmetric manifold. Here the Z-tensor is a general notion of the Einstein gravitational tensor in general relativity. B-tensor is the generalisation of Z-tensor and it is written as (see[74]),

$$B(X,Y) = aS(X,Y) + brg(X,Y). \tag{5.18}$$

As a generalization of spaces of constant curvature locally symmetric spaces were introduced by Cartan [12]. Every locally symmetric space satisfies $R \cdot R = 0$, where by the first R stands for the curvature operator which acts as a derivation on the second R which stands for the Riemanian curvature tensor. Manifold satisfying the condition $R \cdot R = 0$ are called semisymmetric manifolds and were classified by Szabo [75]. The condition of semisymmetry was weakened by Deszcz as pseudosymmetry which is characterized by the condition $R \cdot R = LQ(g,R)$, where by L is a real function on M and $Q(g,R)$ is the Tachibana tensor of M.

A Riemannian manifold M is said to be pseudosymmetric, in the sense of Deszcz [32] if

$$(R(X,Y) \cdot R)(U,V)W = L_R\{((X \wedge Y) \cdot R)(U,V)W\},$$

holds. Where L_R is some smooth function on $U_R = \{x \in M | R - \frac{r}{n(n-1)}G \neq 0 \text{ at } x\}$, G is the $(0,4)$-tensor defined by $G(X_1, X_2, X_3, X_4) = g((X_1 \wedge X_2)X_3, X_4)$ and $(X_1 \wedge X_2)X_3$ is the endomorphism and it is defined as,

$$(X_1 \wedge X_2)X_3 = g(X_2, X_3)X_1 - g(X_1, X_3)X_2.$$

A Riemannian manifold M is said to be Ricci pseudosymmetric if $R \cdot S$ and $Q(g,S)$

on M are linearly dependent. This is equivalent to

$$R \cdot S = f_s Q(g, S),$$

holds on U_S, where $U_S = \{x \in M : S - \frac{k}{n} \neq 0 \ at \ x\}$ and f_S is a function defined on U_S. Every pseudosymmetric manifold is Ricci-pseudosymmetric, but the converse statement is not true. If $R \cdot S = 0$ then M is called Ricci-semisymmetric. Every semisymmetric manifold is Ricci-semisymmetric but the converse statement is not true. Every Ricci-semisymmetric manifold is Ricci-pseudosymmetric, but the converse statement is not true.

5.2 B-pseudosymmetric $(LCS)_n$-manifold

Definition 5.2.1. An $(LCS)_n$-manifold M is said to be B-pseudosymmetric if it satisfies

$$(R(X,Y) \cdot B)(Z,W) = L_B Q(g,B)(Z,W,X,Y). \tag{5.19}$$

for all vector fields X, Y, Z, W.

The above equation can be written as

$$\begin{aligned} B(R(X,Y)Z,W) + B(Z,R(X,Y)W) = L_B\{g(Y,Z)B(X,W) \\ -g(X,Z)B(Y,W) + g(Y,W)B(Z,X) - g(X,W)B(Z,Y)\}. \end{aligned} \tag{5.20}$$

Setting $X = W = \xi$ in (5.20), we get

$$\begin{aligned} B(R(\xi,Y)Z,\xi) + B(Z,R(\xi,Y)\xi) = L_B\{g(Y,Z)B(\xi,\xi) \\ -g(\xi,Z)B(Y,\xi) + g(Y,\xi)B(Z,\xi) - g(\xi,\xi)B(Z,Y)\}. \end{aligned} \tag{5.21}$$

The equation (5.21) implies that

$$(\alpha^2-\rho)\{g(Y,Z)B(\xi,\xi)-\eta(Z)B(Y,\xi)+\eta(Y)B(Z,\xi)+B(Z,Y)\}$$

$$=L_B\{g(Y,Z)B(\xi,\xi)-\eta(Z)B(Y,\xi)+\eta(Y)B(Z,\xi)+B(Z,Y)\}. \tag{5.22}$$

Then either

$$L_B-(\alpha^2-\rho)=0 \quad or \tag{5.23}$$

$$g(Y,Z)B(\xi,\xi)-\eta(Z)B(Y,\xi)+\eta(Y)B(Z,\xi)+B(Z,Y)=0. \tag{5.24}$$

If we assume that $L_B\neq(\alpha^2-\rho)$, then we have

$$g(Y,Z)B(\xi,\xi)-\eta(Z)B(Y,\xi)+\eta(Y)B(Z,\xi)+B(Z,Y)=0. \tag{5.25}$$

In view of (5.18), the equation (5.25) reduces to

$$g(Y,Z)\{aS(\xi,\xi)+brg(\xi,\xi)\}-\eta(Z)\{aS(Y,\xi)+brg(Y,\xi)\}$$

$$+\eta(Y)\{aS(Z,\xi)+brg(Z,\xi)\}+aS(Z,Y)+brg(Z,Y)=0 \tag{5.26}$$

The above equation implies that

$$g(Y,Z)\{(n-1)(\alpha^2-\rho)a\eta(\xi)+br\eta(\xi)\}$$

$$-\eta(Z)\{(n-1)(\alpha^2-\rho)a\eta(Y)+br\eta(Y)\}$$

$$+\eta(Y)\{a(n-1)(\alpha^2-\rho)\eta(Z)+br\eta(Z)\}$$

$$+aS(Y,Z)+brg(Z,Y)=0 \tag{5.27}$$

By virtue of $\phi^2=X+\eta(X)\xi$, (5.27) yields

$$S(Y,Z)=(n-1)(\alpha^2-\rho)g(Y,Z) \tag{5.28}$$

Hence, we have,

Theorem 5.2.1. *A B-pseudosymmetric $(LCS)_n$-manifold is an Einstein manifold if $L_B \neq (\alpha^2 - \rho)$.*

We know that B-tensor reduces to Z-tensor if $a = 1$ and $b = \frac{\Psi}{r}$. Therefore we can state the following:

Corollary 5.2.2. *A Z-pseudosymmetric $(LCS)_n$-manifold is an Einstein manifold if $L_Z \neq (\alpha^2 - \rho)$.*

If $a = 1$ and $b = 0$, then B-tensor reduces to Ricci-tensor and hence we have

Corollary 5.2.3. *A Ricci-pseudosymmetric $(LCS)_n$-manifold is an Einstein manifold if $L_S \neq (\alpha^2 - \rho)$.*

If $L_B = 0$ then B-pseudosymmetric $(LCS)_n$manifold reduces to B-semisymmetric $(LCS)_n$ manifold. Therefore for $\alpha^2 \neq \rho$, one can get

Corollary 5.2.4. *A B-semisymmetric $(LCS)_n$-manifold is an Einstein manifold.*

Corollary 5.2.5. *A Z-semisymmetric $(LCS)_n$-manifold is an Einstein manifold.*

Corollary 5.2.6. *A Ricci-semisymmetric $(LCS)_n$-manifold is an Einstein manifold.*

5.3 $(LCS)_n$-manifold satisfying $Q(\xi, X) \cdot Q(Y, U)Z = 0$.

Suppose $(LCS)_n$-manifold satisfying $Q(\xi, X) \cdot Q(Y, U)Z = 0$ for any vector fields $X, Y, U, Z = 0$. Then

$$\begin{aligned} &Q(\xi, X)Q(Y, U)Z - Q(Q(\xi, X)Y, U)Z \\ &- Q(Y, Q(\xi, X)U)Z - Q(Y, U)Q(\xi, X)Z = 0. \end{aligned} \tag{5.29}$$

Using (5.12) and (5.17), we can write the following:

$$Q(\xi, X)Q(Y, U)Z = \{A\}\left\{g(X, Q(Y, U)Z)\xi - \eta(Q(Y, U)Z)X\right\}, \tag{5.30}$$

$$\begin{aligned} Q(Q(\xi, X)Y, U)Z =& \{A\}\{\{A\}g(X, Y)g(U, Z)\xi - \eta(Z)g(X, Y)U\}, \\ & - \{A\}\eta(Y)Q(X, U)Z \end{aligned} \tag{5.31}$$

$$\begin{aligned} Q(Y, Q(\xi, X)U)Z =& \{A\}\{\{A\}g(X, Z)\eta(U)Y - g(X, Z)g(Y, U)\xi\} \\ & - \{A\}\eta(Z)Q(Y, X)U, \end{aligned} \tag{5.32}$$

$$\begin{aligned} Q(Y, U)Q(\xi, X)Z =& \{A\}\{A\{g(X, U)\eta(Z)Y - g(X, U\eta(Y))Z\} \\ & - \{A\}\eta(Y)Q(Y, Z)X, \end{aligned} \tag{5.33}$$

where $A = \left\{(\alpha^2 - \rho) - \frac{\psi}{(n-1)}\right\}$. Using (5.30)-(5.32) in (5.29) and then inserting $X = Y = e_i$, where e_i is an orthogonal basis of the tangent space at each point of the manifold and taking summation over i, we obtain

$$S(Z, U) = A'g(Z, U) + B'\eta(Z)\eta(U) \tag{5.34}$$

where $A' = \left\{(n+1)\left\{(\alpha^2 - \rho) - \frac{\psi}{(n-1)}\right\} + \psi\right\}$ and
$B' = \left\{(n+1)\left\{(\alpha^2 - \rho) - \frac{\psi}{(n-1)}\right\}\right\}$, From (5.33), we can state the following:

Theorem 5.3.1. *An $(LCS)_n$-manifold satisfying $Q(\xi, X)Q(Y, U)Z = 0$ is an η-Einstein manifold.*

5.4 Q-pseudosymmetric $(LCS)_n$-manifold

Definition 5.4.1. An n-dimensional $(LCS)_n$-manifold is said to be Q-pseudosymmetric, if

$$(R(X, Y) \cdot Q)(U, V)W = L_Q\{((X \wedge Y) \cdot Q)(U, V)W)\}. \tag{5.35}$$

where L_R is some smooth function on $U_Q = \{x \in M | Q - \frac{r}{n(n-1)} G \neq 0 \text{ at } x\}$, where G is the $(0,4)$-tensor defined by $G(X_1, X_2, X_3, X_4) = g((X_1 \wedge X_2)X_3, X_4)$ and $(X \wedge Y)Z$ is an endomorphism defined as

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y. \tag{5.36}$$

Inserting $X = \xi$ in (5.35), we obtain

$$(R(\xi, Y) \cdot Q)(U, V)W = L_Q\{((\xi \wedge Y) \cdot Q)(U, V)W)\}. \tag{5.37}$$

Now left hand side of (5.37) can be written as

$$\begin{aligned} &R(\xi, Y)Q(U, V)Z - Q(R(\xi, Y)U, V)Z \\ &-Q(U, R(\xi, Y)V)Z - Q(U, V)R(\xi, Y)Z = 0. \end{aligned} \tag{5.38}$$

By virtue of (5.12), the above expression becomes

$$\begin{aligned} &(\alpha^2 - \rho)\{g(Y, Q(U, V)Z)\xi - \eta(Q(U, V)Z)Y \\ &- g(Y, U)Q(\xi, V)Z + \eta(U)Q(Y, V)Z \\ &- g(Y, V)Q(U, \xi)Z + \eta(V)Q(U, Y)Z \\ &- g(Y, Z)Q(U, V)\xi + \eta(Z)Q(U, V)Y\} = 0. \end{aligned} \tag{5.39}$$

Next the right hand side of (5.37) is

$$\begin{aligned} &L_Q\{(\xi \wedge Y)Q(U, V)Z - Q((\xi \wedge Y)U, V)Z \\ &- Q(U, (\xi \wedge Y)V)Z - Q(U, V)(\xi \wedge Y)Z\} = 0. \end{aligned} \tag{5.40}$$

By virtue of (5.36), (5.40) becomes

$$
\begin{aligned}
& L_Q\{g(Y,Q(U,V)Z)\xi - \eta(Q(U,V)Z)Y \\
- & \ g(Y,U)Q(\xi,V)Z + \eta(U)Q(Y,V)Z \\
- & \ g(Y,V)Q(U,\xi)Z + \eta(V)Q(U,Y)Z \\
- & \ g(Y,Z)Q(U,V)\xi + \eta(Z)Q(U,V)Y\} = 0. \qquad (5.41)
\end{aligned}
$$

Using expressions (5.39) and (5.41) in (5.37) and taking inner product with ξ, we obtain

$$
\begin{aligned}
& \{(\alpha^2-\rho) - L_Q\}\{-Q(U,V,Z,Y) - \eta(Q(U,V)Z)\eta(Y) \\
& - g(Y,U)\eta(Q(\xi,V)Z) + \eta(U)\eta(Q(Y,V)Z) \\
& - g(Y,V)\eta(Q(U,\xi)Z) + \eta(V)\eta(Q(U,V)Z) \\
& - g(Y,Z)\eta(Q(U,V)\xi) + \eta(Z)\eta(Q(U,V)Z)\} = 0. \qquad (5.42)
\end{aligned}
$$

Let $\{e_1, e_2, ...e_n\}$ be an orthonormal basis of the tangent space at each point of the manifold. Setting $U = Y = e_i$ in (5.42) and taking summation over i and using (5.12), (5.13), we get

$$S(V,Z) = \left\{\psi + (n-1)\left[(\alpha^2-\rho) - \frac{\psi}{(n-1)}\right]\right\} g(V,W).$$

Now we can state the following:

Theorem 5.4.1. *If $(LCS)_n$-manifold is Q-pseudosymmetric, then it is an Einstein manifold.*

Corollary 5.4.2. *A Q-pseudosymmetric $(LCS)_n$-manifold is Q-semisymmetric if and only if $L_Q = 0$.*

5.5 Q-Ricci semisymmetric $(LCS)_n$ manifold

Definition 5.5.1. An $(LCS)_n$-manifold M is Q-Ricci semisymmetric if

$$(Q(X,Y)\cdot S)(Z,W)=0, \tag{5.43}$$

for all vector fields $X,Y,Z,W \in T_PM$

The equation (5.43) can be written as

$$S(Q(X,Y)Z,U)+S(Z,Q(X,Y)U)=0. \tag{5.44}$$

Putting $Z=\xi$ in (5.44), we have

$$S(Q(X,Y)\xi,U)+S(\xi,Q(X,Y)U)=0. \tag{5.45}$$

Using (5.17), (5.13) and (5.14) in (5.45), we get

$$\begin{aligned}&\left\{(\alpha^2-\rho)-\frac{\Psi}{(n-1)}\right\}\{\eta(Y)S(X,U)-\eta(X)S(Y,U)\}\\&+(\alpha^2-\rho)(n-1)\left\{(\alpha^2-\rho)-\frac{\Psi}{(n-1)}\right\}\{g(Y,U)\eta(X)-g(X,U)\eta(Y)\}=0.\end{aligned} \tag{5.46}$$

Inserting $Y=\xi$ in (5.46), we obtain

$$\left\{(\alpha^2-\rho)-\frac{\Psi}{(n-1)}\right\}\{(\alpha^2-\rho)(n-1)g(X,U)-S(X,U)\}=0. \tag{5.47}$$

This implies that either

$$(\alpha^2-\rho)-\frac{\Psi}{(n-1)}=0 \quad or \quad S(X,U)=(\alpha^2-\rho)(n-1)g(X,U). \tag{5.48}$$

From (5.48), we can state the following:

Theorem 5.5.1. *A Q-Ricci semisymmetric $(LCS)_n$-manifold is an Einstein manifold if $\Psi\neq(\alpha^2-\rho)(n-1)$.*

5.6 ϕ-Q-flat $(LCS)_n$ manifold

Definition 5.6.1. An $(LCS)_n$-manifold M is called $\phi - Q$-flat if

$$Q(\phi X, \phi Y, \phi Z, \phi W) = 0. \tag{5.49}$$

for all the vector fields $X, Y, Z, W \in T_P M$

According to (5.49), the equation (5.17) can be written as

$$\begin{aligned} R(\phi X, \phi Y, \phi Z, \phi W) = \frac{\Psi}{(n-1)} \{ & g(\phi Y, \phi Z) g(\phi X, \phi W) \\ & - g(\phi X, \phi Z) g(\phi Y, \phi W)\}. \end{aligned} \tag{5.50}$$

Let $\{e_1, ..., e_n\}$ be a local orthonormal basis of the vector fields on M. Then, by putting $X = W = e_i$ in (5.50) and taking summation over i $(1 \leq i \leq n)$, we get

$$\sum_{i=1}^{n-1} R(\phi X, \phi Y, \phi Z, \phi W) = \frac{\Psi}{(n-1)} \left[\sum_{i=1}^{n-1} \{g(\phi Y, \phi Z) g(\phi X, \phi W) - g(\phi X, \phi Z) g(\phi Y, \phi W)\} \right]. \tag{5.51}$$

We know that

$$\sum_{i=1}^{n-1} R(\phi e_i, \phi Y, \phi Z, \phi e_i) = S(\phi Y, \phi Z) + g(\phi Y, \phi Z), \tag{5.52}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = (n-1), \tag{5.53}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi Z) g(\phi Y, \phi e_i) = g(\phi Y, \phi Z). \tag{5.54}$$

Using (5.52)-(5.54) in (5.50), we obtain

$$S(\phi Y, \phi Z) + g(\phi Y, \phi Z) = \frac{\Psi}{(n-1)} \{g(\phi Y, \phi Z)(n-1) - g(\phi Y, \phi Z)\}. \tag{5.55}$$

With the help of (5.9) and (5.18), we get

$$\begin{aligned} S(Y,Z) = &\left\{\frac{(n-2)\Psi}{(n-1)} - (n-1)(\alpha^2-\rho) - 1\right\} g(Y,Z) \\ &+ \left\{\frac{(n-2)\Psi}{(n-1)} - 1\right\} \eta(Y)\eta(Z) \end{aligned}$$

Hence we can state the following:

Theorem 5.6.1. *A ϕ-Q-flat $(LCS)_n$-manifold is an η-Einstein manifold.*

5.7 Lorentzian Para-Sasakian manifold

In an Lorentzian Para-Sasakian manifold, the 1-form η is closed. In recent years, Lorentzian para-Sasakian manifold has been studied by many authors [53], [56], [57], [69], [78], [73]. So we have the following expressions

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{5.56}$$

$$R(\xi,Y)Z = g(Y,Z)\xi + \eta(Z)Y + 2\eta(Y)\eta(Z)\xi, \tag{5.57}$$

$$Q\xi = (n-1)\xi, \tag{5.58}$$

$$\begin{aligned} R(X,Y)\varphi Z - \varphi R(X,Y)Z = &2\{\eta(Y)\eta(Z)\varphi X - \eta(X)\eta(Z)\varphi Y\} \\ &+ g(Y,Z)\varphi X - g(X,Z)\varphi Y + g(X,\varphi Z)Y \\ &- g(Y,\varphi Z)X + 2\eta(Y)g(X,\varphi Z)\xi \\ &- 2\eta(X)g(Y,\varphi Z)\xi + 2\eta(Z)g(X,\varphi Y)\xi. \end{aligned} \tag{5.59}$$

Moreover ξ is never be a Killing vector on M i.e.,

$$(\pounds_\xi g)(X,Y) = 2g(X,\varphi Y), \tag{5.60}$$

φ is linear and rank of φ is $n-1$, so $\pounds_\xi g \neq 0$ for all vector fields on $\mathcal{X}(M)$.

Definition 5.7.1. A vector field V on an n-dimensional pseudo-Riemannian manifold M is said to be a conformal vector field if [72]

$$\pounds_V g = 2\rho g, \tag{5.61}$$

for a smooth function ρ on M.

Definition 5.7.2. A vector field V on a pseudo-Riemannian manifold M is said to be holomorphically planar conformal vector field if it satisfies [43]

$$\nabla_X V = aX + b\varphi X, \tag{5.62}$$

where a and b are some smooth functions on M.

Definition 5.7.3. On a pseudo-Riemannian manifold M, any vector field V is said to be an infinitesimal contact transformation if it satisfies

$$\pounds_V \eta = \sigma\eta \tag{5.63}$$

where σ is the smooth function on M. If $\sigma = 0$ then V is called to be strict.

Theorem 5.7.1. *On a Lorentzian para-Sasakian manifold M, a Reeb vector field ξ is never a conformal vector field.*

Proof. If we assume that a Reeb vector field ξ is a conformal vector field on a Lorentzian para-Sasakian M, then the equation (5.61), on (X, Y) gives

$$2g(X, \varphi Y) = \rho g(X, Y). \tag{5.64}$$

Choosing $X = Y = \xi$ in the foregoing relation we get $\rho = 0$. But this in the above equation leads to the contradiction. $\square$

Theorem 5.7.2. *If an orthogonal vector field V on a Lorentzian para-Sasakian manifold M is a conformal vector field, then it is Killing on M.*

Proof. Suppose that an orthogonal vector field V on a Lorentzian para-Sasakian M is a conformal vector field, then the equation (5.61) on (X, ξ) gives

$$g(\nabla_X V, \xi) + g(\nabla_\xi V, X) = \rho\eta(X). \tag{5.65}$$

Since, $g(V, \xi) = 0$, which finds $g(\nabla_X V, \xi) = -g(V, \varphi X)$. Next, choosing $X = \xi$ in the above expression, we find

$$2g(\nabla_\xi V, \xi) = \rho = 0. \tag{5.66}$$

This completes the proof. □

Theorem 5.7.3. *If an infinitesimal contact transformation on a Lorentzian para-Sasakian manifold is a holomorphically planar conformal vector field, then it is either collinear with ξ, or strictly infinitesimal contact transformation of M.*

Proof. Suppose a vector field V on a Lorentzian para-Sasakian manifold M is an infinitesimal contact transformation and holomorphically planar conformal vector field. Then taking the co-derivative of (5.62) along the vector Y, we obtain

$$\nabla_Y \nabla_X V = (Ya)X + a\nabla_Y X + (Yb)\varphi X + b\nabla_Y \varphi X. \tag{5.67}$$

In a similar way, we also derive

$$\nabla_X \nabla_Y V = (Xa)Y + a\nabla_X Y + (Yb)\varphi Y + b\nabla_X \varphi Y. \tag{5.68}$$

Later, in (5.62) by replacing X by $[X, Y]$, we find

$$\nabla_{[X,Y]} V = a[X, Y] + b\varphi[X, Y]. \tag{5.69}$$

On combining the last three expressions in the well-known formula $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_Y Z - \nabla_{[X,Y]} Z$ provides

$$g(R(X, Y)V, \xi) = (Xa)\eta(Y) - (Ya)\eta(X). \tag{5.70}$$

As we know η is closed on M i.e., $d\eta = 0$, and therefore applying d on both sides of relation (5.63) gives

$$(d\sigma \wedge \eta)(X, Y) = 0. \tag{5.71}$$

In the above equation for $X = \xi$, we get $Y\sigma = -(\xi\sigma)\eta(Y)$. This implies, $X\sigma = 0$ for all X orthogonal to ξ. As a result, we can easily determine that $a = \sigma$ using the equation (5.63) in (5.62). Because $X\sigma = 0$ for an orthogonal vector field X, it follows that Xa is also a zero. Next, with the help of equation (5.56) in condition (5.70), we find

$$\eta(Y)g(X,V) - \eta(X)g(V,Y) = (Ya)\eta(X) - (Xa)\eta(Y). \tag{5.72}$$

Putting $X = \xi$ and $Y = \varphi Y$ in the foregoing relation leads to

$$g(\varphi Y, V) = -(\varphi Y a). \tag{5.73}$$

As we know a is constant along an orthogonal vector field ξ. This in the above relation shows $g(\varphi Y, V) = 0$, this implies $V = -\eta(V)\xi$. Hence this proves either part of the theorem. Next, if we suppose $\eta(V)$ is constant then from (5.61) we have that $-\eta(V)\varphi X = aX + b\varphi X$. From this it is easy to find $a = 0$. Lastly, this in (5.63) shows $\sigma = 0$ and again from (5.63) we have

$$\pounds_V \eta = 0. \tag{5.74}$$

This completes the theorem. □

Theorem 5.7.4. *Let M be a Lorentzian para-Sasakian manifold and V be an orthogonal vector field which is non-zero. Then V never be a holomorphically planar conformal vector field on M.*

Proof. Suppose V is a non-zero orthogonal vector field on a Lorentzian para-Sasakian manifold M and satisfies (5.62), then from taking inner product with ξ and $X = \xi$ gives

$$g(\nabla_\xi V, \xi) = a. \tag{5.75}$$

Since, $g(V,\xi) = 0$ this implies in getting $g(\nabla_\xi V, \xi) = 0$. Therefore, this in the above equation finds $a = 0$. Also, the condition $g(V,\xi) = 0$ provides

$$g(\nabla_X V, \xi) + g(V, \nabla_X \xi) = 0. \tag{5.76}$$

As we know, V is holomorphically planar conformal vector field on M, then by substituting $\nabla_X V = b\varphi X$ in the preceding equation results in

$$g(V, \varphi X) = 0. \tag{5.77}$$

And this shows $V = -\eta(V)\xi = 0$. Hence, there is no non-zero orthogonal vector field on a Lorentzian para-Sasakian manifold which is holomorphically planar conformal. $\square$

Now we are going to construct following example to justify our above mentioned results

Example 5.7.1. Here we construct the 5-dimensional Lorentzian para-Sasakian manifold M. We consider $M = \{(u, v, w, x, y) \in R^5\}$, where (u, v, w, x, y) are the standard coordinates in R^5.

Let $\{x_1, x_2, x_3, x_4, x_5\}$ be the basis for M and the Lorentzian metric g is defined as

$$g(x_i, x_j) = \begin{cases} 0 & for \quad i \neq j, \\ 1 & for \quad i = j \quad and \quad i \neq 3, \\ -1 & for \quad i = j = 3. \end{cases} \tag{5.78}$$

Let ∇ be the Levi-Civita connection corresponding to g and we have

$$[x_1, x_2] = 0, \quad [x_1, x_3] = -x_1, \quad [x_1, x_4] = 0,$$

$$[x_1, x_5] = x_1, \quad [x_2, x_3] = -x_2, \quad [x_2, x_4] = x_2,$$

$$[x_2, x_5] = x_2, \quad [x_3, x_4] = x_4, \quad [x_3, x_5] = x_5, \quad [x_4, x_5] = -x_5.$$

Let the $(1,1)$ tensor field φ be defined by

$$\varphi x_1 = -x_1, \quad \varphi x_2 = -x_2, \quad \varphi x_3 = 0, \quad \varphi x_4 = -x_4, \quad \varphi x_5 = -x_5. \tag{5.79}$$

Let η is the 1-form defined by $\eta(X) = g(X, x_3)$, for any vector field X on $\mathcal{X}(M)$. Then, by the linearity of φ and g, we find

$$\eta(x_3) = -1, \tag{5.80}$$

$$\varphi^2 = I + \eta \otimes \xi, \tag{5.81}$$

$$g(\varphi\cdot, \varphi\cdot) = (g + \eta \otimes \eta)(\cdot, \cdot). \tag{5.82}$$

By the Koszul's formula, we find

$$\nabla_{x_1}x_1 = -x_3 - x_5, \quad \nabla_{x_1}x_2 = 0, \quad \nabla_{x_1}x_3 = -x_1, \quad \nabla_{x_1}x_4 = 0, \quad \nabla_{x_1}x_5 = x_1,$$
$$\nabla_{x_2}x_1 = 0, \quad \nabla_{x_2}x_2 = -x_3 - x_4 - x_5, \quad \nabla_{x_2}x_3 = -x_2, \quad \nabla_{x_2}x_4 = x_2, \quad \nabla_{x_2}x_5 = x_2,$$
$$\nabla_{x_3}x_1 = 0, \quad \nabla_{x_3}x_2 = 0, \quad \nabla_{x_3}x_3 = 0, \quad \nabla_{x_3}x_4 = 0, \quad \nabla_{x_3}x_5 = 0,$$
$$\nabla_{x_4}x_1 = 0, \quad \nabla_{x_4}x_2 = 0, \quad \nabla_{x_4}x_3 = -x_4, \quad \nabla_{x_4}x_4 = -x_3, \quad \nabla_{x_4}x_5 = 0,$$
$$\nabla_{x_5}x_1 = 0, \quad \nabla_{x_5}x_2 = 0, \quad \nabla_{x_5}x_3 = -x_5, \quad \nabla_{x_5}x_4 = x_5, \quad \nabla_{x_5}x_5 = -x_3 - x_4.$$

Hence, we can conclude that (φ, x_3, η, g) defines a Lorentzian para-Sasakian structure on M and so M is a Lorentzian para-Sasakian manifold. If $\xi = x_3$ is a coformal vector field then the equation (5.61) over (x_1, x_1) gives $\rho = -1$ and again equation (5.61) over finds $\rho = 0$. Which is a contradiction.

Example 5.7.2. Let us consider a manifold $M = \{(x, y, z) \in R^3\}$ and the orthonormal basis $\{y_1, y_2, y_3\}$ on M, with the Lorentzian metric g satisfying

$$g(y_i, y_j) = 0, \qquad for \qquad i \neq j$$
$$g(y_1, y_1) = g(y_2, y_2) = 1,$$
$$g(y_3, y_3) = -1.$$

Define 1-form η and the vector field ξ by

$$\eta(X) = g(X, y_3), \qquad \xi = y_3.$$

Let ∇ be the Levi-Civita connection corresponding to g and is defined by

$$[y_1, y_2] = 0, \quad [y_1, y_3] = -y_1, \quad [y_2, y_3] = -y_2,$$

and the tensor field φ is defined by

$$\varphi y_1 = -y_1, \quad \varphi y_2 = -y_2, \quad \varphi y_3 = 0.$$

Use of Koszul's formula gives the following relations

$$\nabla_{y_1}y_1 = -y_3, \quad \nabla_{y_1}y_2 = 0, \quad \nabla_{y_1}y_3 = -y_1, \tag{5.83}$$
$$\nabla_{y_2}y_1 = 0, \quad \nabla_{y_2}y_2 = -y_3, \quad \nabla_{y_2}y_3 = -y_2,$$
$$\nabla_{y_3}y_1 = 0, \quad \nabla_{y_3}y_2 = 0, \quad \nabla_{y_3}y_3 = 0.$$

From the above relations, it is clear that $(\nabla_X \varphi)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi$ and $\nabla_X \xi = \varphi X$, for any vector fields X, Y. Hence, the defined structure $(\varphi, \xi = y_3, \eta, g)$ is a Lorentzian para-Sasakian structure on M.

Case (i): Suppose a vector $V = a_1 y_1 + a_2 y_2$, where a_1 and a_2 are constants on M satisfying (5.62), then in (5.61) for $X = y_1$ and inner product with y_2 results in $a_1 = 0$. Similarly, equation (5.62) for $X = y_2$ and inner product with y_1 gives $a_2 = 0$. This shows $V = a_1 y_1 + a_2 y_2 = 0$.

Case(ii): Suppose a vector $V = a_1 y_1 + a_2 y_2$, where a_1 and a_2 are constants on M admitting (5.61), then for $X = Y = y_1$ yields

$$2a_1 g(\nabla_{y_1} y_1, y_1) + 2a_2 g(\nabla_{y_1} y_2, y_1) = 2\rho. \tag{5.84}$$

Hence by the use of (5.83) in the above relation we find $\rho = 0$. This proves V is Killing

5.8 Conclusion

In this chapter, we have obtained the following results:

- A B-pseudosymmetric $(LCS)_n$-manifold is an Einstien manifold if $L_B \neq (\alpha^2 - \rho)$

- If a $(LCS)_n$ manifold is Q-pseudosymmetric, then it is an Einstein manifold.

- A Q-pseudosymmetric $(LCS)_n$-manifold is Q-semisymmetric if and only if $L_Q = 0$.

- A Q-Ricci semisymmetric $(LCS)_n$-manifold is an Einstien manifold if $\Psi \neq (\alpha^2 - \rho)(n-1)$. In addition to this we studied conditions $Q(\xi, X) \cdot Q(Y,U)Z = 0$ and Q-pseudosymmetric and ϕ-Q-flat on $(LCS)_n$-manifolds.

- We showed that if the geometric aspects of a Reeb vector field ξ and an orthogonal vector field V on a Lorentzian para-Sasakian manifold M is a conformal vector field, then it is Killing on M.

- If an infinitesimal contact transformation on a Lorentzian para-Sasakian manifold is a holomorphically planar conformal vector field, then it is either collinear with ξ, or strictly infinitesimal contact transformation of M.

- And finally we showed that if M is a Lorentzian para-Sasakian manifold and V is a orthogonal vector field which is non-zero, then V never be a holomorphically planar conformal vector field on M.

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1. N. Bhanumathi and V. Venkatesha: *"Some Symmetric properties on* $(LCS)_n$ *Manifolds"*, Annals Mathematics and Computer Science. Vol. 2, 2021, 1-9.

2. V. Venkatesha, N. Bhanumathi and C. Shruthi : *"A study on* K*-paracontact and* (κ,μ)*-paracontact manifold admitting vanishing Cotton tensor and Bach tensor"*, Annals University of Craiova. Mathematics and Computer Science. 49 (1), 2022, 1-10 (Scopus, SCIE).

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7. N. Bhanumathi and V. Venkatesha: *"*C*-Bochner Curvature Tensor on Para-Kenmotsu manifold"*, (Communicated).

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Preface

On the 10th of June 1854, Riemann gave his famous inaugural lecture at Gottingen and discussed the foundations of geometry, introduced n-dimensional manifolds, formulated the concept of Riemannian manifolds and defined their curvature. Since every manifold admits a Riemannian metric, Riemannian geometry often helps us to solve problems of differential topology. Most remarkably, by applying Riemannian geometry, Perelman solved the famous Poincare's conjecture posed in 1904.

Under the impetus of Einstein's theory of general relativity (1915) a further generalization appeared; the positiveness of the inner product was weakened. Consequently, one has the notion of pseudo-Riemannian manifolds which is a generalization of a Riemannian manifold in which the metric tensor need not be positive-definite, but need only be a non-degenerate bilinear form, which is a weaker condition.

The theory of structures on manifolds is a very interesting and very fruitful fields of Riemannian geometry. In this thesis, we investigate Riemannian and pseudo-Riemannian manifolds admitting different types of structures. In particular, we study contact Riemannian structures, almost Kenmotsu structures, almost coKaehler structures, almost contact pseudo-Riemannian structures and almost paracontact metric structures under several geometric points of view. The entire work in the thesis has been partitioned into five chapters and are summarized as follows:

Chapter 1 gives a brief summary of the main concepts and results about almost contact manifolds and Paracontact manifolds which will be used widely in the rest of chapters.

Chapter 2 we study $\mathbb{H}$-Curvature tensor on almost Kenmotsu manifold with nullity distibution. Also we investigate Generalized Ricci Soliton on Almost Kenmotsu Manifolds.

In the beginning, we proved that if M is a locally ϕ-$\mathbb{H}$ symmetric alomost Kenmotsu manifold with characteristic vector field ξ belonging to the $(\kappa,\mu)'$-nullity distribution and $h\neq 0$, then the manifold M is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4)\times\mathbb{E}^n$. Next we showed that if M is a locally ϕ-$\mathbb{H}$ symmetric alomost Kenmotsu manifold with characteristic vector field ξ belonging to the generalized $(\kappa,\mu)'$-nullity distribution and $h\neq 0$ then M is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4)\times\mathbb{E}^n$. Also we proved that if M is a locally ϕ-$\mathbb{H}$ symmetric almost Kenmotsu manifold with the characteristic vector field ξ belonging to the (κ,μ)-nullity distribution and $h\neq 0$, then the manifold M is an Einstein manifold. And if M is a locally $\phi-\mathbb{H}$ symmetric almost Kenmotsu manifold with the characteristic vector field ξ belonging to the generalized (κ,μ)-nullity distribution and $h\neq 0$, then the manifold M is Einstein. Finally, we study the two classes of almost Kenmotsu manifolds. Firstly, we study a closed generalized Ricci soliton on the Kenmotsu manifold. Secondly, we prove that if a Kenmotsu manifold M admits a generalized Ricci soliton with conformal vector field V, then M is Einstein. Next, we show that a non-Kenmotsu almost Kenmotsu $(\kappa,\mu)'$-manifold admitting a closed generalized Ricci soliton is locally isometric to the Riemannian product $\mathbb{H}^{n+1}\times\mathbb{R}^n$, provided that $\lambda-\frac{\kappa}{\beta}(2n\alpha\beta-1)=-\frac{2}{\beta}$.

In Chapter 3, we devoted to the study of K-paracontact manifold admitting parallel Cotton tensor, vanishing Cotton tensor and to study Bach flatness on K-paracontact manifold. Also we study vanishing Cotton tensor on (κ,μ)-paracontact manifold for both $\kappa>-1$ and $\kappa<-1$. Further we study Yamabe and Quasi Yamabe soliton on (κ,μ)-paracontact manifold and K-paracontact manifold. First we consider M to be a K-paracontact manifold. Then M has constant scalar curvature if and only if $C(X,\xi)\xi=0$. Next, we show that if M is a K-paracontact metric manifold, then M has parallel Cotton tensor if and only if M is an η-Einstein manifold and $r=-2n(2n+1)$. Also we proved that if M is an η-Einstein K-paracontact manifold, and is Bach flat then M is an Einstein manifold. Also, we prove that if M is a (κ,μ)-paracontact manifold for $\kappa\neq 1$, and if M has vanishing Cotton tensor for $\mu\neq\kappa$ then M is an η-Einstein manifold. Next, we study Yamabe and quasi Yamabe soliton on (κ,μ)-paracontact manifold and K-paracontact manifold. Here we prove that, if M is non-para-Sasakian manifold and admits Yamabe soliton for the potential vector field V, then either V is Killing, or M is locally isometric to the product of a flat $(n+1)$-dimensional manifold and n-dimensional manifold of constant negative curvature equal to -4. Next we prove that if a non-para-Sasakian (κ, μ)-paracontact manifold admits a quasi Yamabe gradient soliton then for $\kappa > -1$, M is either $N(\frac{1-n}{n})$-manifold, or M is locally isometric to the product of a flat $(n+1)$-dimensional manifold and n-dimensional manifold of constant negative curvature equal to -4, or the potential function f is constant on M. For $\kappa < -1$ either $\mu \neq \frac{-4}{n+1}$ or the potential function f is constant on M. Lastly, we show that, if a K-paracontact metric g with $Q\varphi = \varphi Q$ represents a quasi Yamabe gradient soliton then either the scalar curvature $r = -2n(2n+1)$, or the potential function f is a constant.

In Chapter 4, we study some geometric properties of extended quasi generalized φ-recurrent para-Kenmotsu manifolds. And a proper example is also provided to demonstrate the existence of an extended quasi-generalized φ-recurrent Kenmotsu manifold. Also we study C-Bochner pseudosymmetric para-Kenmotsu manifold. Firstly we proved that if M is a para-Kenmotsu manifold and if M is an extended quasi φ- recurrent manifold, then M is super generalized Ricci-recurrent manifold. Also we show that if a para-Kenmotsu manifold M is an extended quasi φ- recurrent manifold, then M is an Einstein manifold. Moreover, the associated vector fields χ_1 and χ_2 of 1-forms Π_1 and Π_2 respectively are co-directional. And if a para-Kenmotsu manifold M admitting an extended quasi generalized φ-recurrent, then M is of constant sectional curvature -1. Next we prove that if M is a para-Kenmotsu manifold and if M is an extended quasi φ- recurrent manifold, then the 1-forms Π_1 and Π_2 are related by the equation $dr(W) = [2n(2n+1) + r]\Pi_1(W) - 2(n+1)(2n+1)\Pi_2(W)$. Finally we showed that if a n-dimensional para-Kenmotsu manifold M is C-Bochner Pseudo-symmetric then M_n is locally isometric to a sphere or $L_B = 1$. And we examine if a n-dimensional para-Kenmotsu manifold M satisfies $B(\xi, X) \cdot B = 0$ then M is isometric to a hyperbolic space. Later we showed that an n-dimensional para-Kenmotsu manifold satisfying the condition $B(\xi, X) \cdot R = 0$ is locally isometric to a sphere or $\tau = 2n$. Also we proved that a n-dimensional para-Kenmotsu manifold satisfying $B(\xi, X) \cdot S = 0$ is an Einstein manifold.

in the final Chapter 5 focuses on the study some symmetric properties on $(LCS)_n$-Manifolds. First, we show that a B-pseudosymmetric $(LCS)_n$-manifold is an Einstien manifold if $L_B \neq (\alpha^2 - \rho)$ and we show that if $(LCS)_n$ manifold is Q-pseudosymmetric, then it is an Einstein manifold. Also show that a Q-pseudosymmetric $(LCS)_n$-manifold is Q-semisymmetric if and only if $L_Q = 0$. Finally we show that a Q-Ricci semisymmetric $(LCS)_n$-manifold is an Einstien manifold if $\Psi \neq (\alpha^2 - \rho)(n-1)$. In addition to this we study conditions $Q(\xi, X) \cdot Q(Y, U)Z = 0$ and Q-pseudosymmetric and ϕ-Q-flat on $(LCS)_n$-manifolds. Next, we show that the geometric aspects of a Reeb vector field ξ and an orthogonal vector field V on a Lorentzian para-Sasakian manifold M is a conformal vector field, then it is Killing on M. Next, we prove that if an infinitesimal contact transformation on a Lorentzian para-Sasakian manifold is a holomorphically planar conformal vector field, then it is either collinear with ξ, or strictly infinitesimal contact transformation of M. And finally we showed that if M is a Lorentzian para-Sasakian manifold and V is a orthogonal vector field which is non-zero, then V never be a holomorphically planar conformal vector field on M.