STUDIES ON LIE GROUPS AND THEIR APPLICATIONS

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DECLARATION

I declare that this dessertation , " **STUDIES ON LIE GROUPS AND THEIR APPLICATIONS**", is based on bonafide work carried out by me under the guidance of **Prof.C.S.BAGEWADI**. This dissertation or a part thereof has not been previously submitted for any other diploma or degree.

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CERTIFICATE

I certify that this Dissertation " **STUDIES ON LIE GROUPS AND THEIR APPLICATIONS**", Presented by **Sri.GIRISH KUMAR.E**. for the award of degree of Master of Philosophy in Mathematics is carried out by him under my guidance. This dissertation or part thereof has not been previously submitted for any other diploma or degree.

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DEDICATED

TO

MY PARENTS

PREFACE

A Lie group is a group, which is also a 'manifold' ofcourse, to make sense of this definition, we must explain these two basic concepts and how they can be related. Group s arise as an algebraic abstraction of the notion of symmetry, an important example is the group of rotation s in the plane or three- dimensional space. Manifolds which form the fundamental objects in the field of differential geometry, generalize the familiar concepts of curves and surfaces in three dimensional space. In general, a manifold is a space that looks like locally Euclidian space, but whose global characters might be quite different. The conjunction of these two seemingly disparate mathematical ideas combines, and significantly extends both the algebraic methods of group theory and the multi-variable calculus used in analytic geometry. This resulting theory, particularly the powerful infinitesimal techniques, can then be applied to a wide range of physical and mathematical problem.

This dissertation consists of four chapters. The first chapter deals with the basic definitions and their properties of Groups, topology and manifolds with some examples. In the second chapter we defined the definition of Lie-groups and Lie sub-grups with some examples. Also we defined action of a Lie group on a manifold and by using this we defined quotient manifold . Further Lie transformation group and one parameter groups are defined. Also flow property of fluid is explained by the notion of one parameter group with some examples of flow. Further topological properties of Lie groups are stated.

In chapter three we study basic definition of Lie-algebra with some examples. Also examples of those relating to Lie-groups are explained. Left and right translations are defined and an inner automorphism is constructed as in group theory by using these translations. Further Lie-algebras of GL(n,R) and GL(n,C) are given .

Finally in the fourth chapter we give applications of Lie-groups. It is explained how relativity group can be constructed by Galilean transformations. This group is nothing but Lie-group and hence it can be seen that how Lie-groups are used in relativity theory, similarly Lorentz group which is also a Lie-group is used in quantam theory.

CONTENTS

CHAPTER - I

BASIC CONCEPTS OF GROUPS, TOPOLOGY AND DIFFERENTIABLE MANIFOLDS

1.0: Introduction: In this chapter we collect the basic definitions and their properties of groups, topology and manifolds with some examples.

1.1. Groups:

Definition 1.1.1:

A non empty set of elements G is said to form a group if in G there is defined a binary operation, denoted by $*$ such that

(a) for all a,b $\in G \Rightarrow a*b \in G$

(b) for all a,b,c $\in G \Rightarrow a*(b*c)= (a*b)*c$

(c) There exists an unique element $e \in G$ such that

$a*e=e*a=a$ for all $a \in G$

(d) for every $a \in G$ there exists an element $a^{-1} \in G$ such that

$a*a^{-1} = a^{-1}*a=e$

In addition to these properties if

(e) for every a,b $\in G$, $a*b=b*a$ then G is called commutative group

Notation:

The set G together with binary operation is denoted by (G,*)

If G consists of a finite number of elements then G is said to be finite otherwise in finite.

Example 1.1.1

(Z,+) , (R,+), (Q,+), (R,-{0}, X), (Q,-{0}, X), (C,+) (C,-{0}, X), are abelian groups.

Example 1.1.2

(R^n, +) is an abelian group.

Example 1.1.3

{(1,-1), X} is a finite abelian group

Example 1.1.4

The set of all n x n matrices whose entries are integers (rationals or reals) is an abelian group under addition .

Example 1.1.5:

Let M=GL(n,R) be the set of all n x n non singular matrices whose entries are real numbers is a non abelian group under matrix multiplication.

Definition 1.1.2:

A non empty sub set H of a group G is said to be a sub gorup of G if under the binary operation in G, H itself forms a group.

Example 1.1.6:

The set of Integers is a subgroup of real numbers under additive binary operation

Example 1.1.7:

The set of real number's is a sub group of complex number's under additive binary operation

Example 1.1.8:

The set of non-zero complex numbers of absolute value 1 (one) is a subgroup of (C^*, X) where C^*=(C-{0})

Example 1.1.9:

Let SL $(n,R)=\{A\in GL(n,R) : \det A=1\}$ is a subgroup of $GL(n,R)$

Example 1.1.10

Let $O(n,R)=\{A\in GL(n,R) : A^tA=1\}$ is a subgroup of $GL(n,R)$

Definition 1.1.3 :

Let $(G_1, .)$ and $(G_2, *)$ be groups.

A mapping $\phi: G_1\rightarrow G_2$ is said to be a homomorphism if for all

$a,b \in G_1 : \phi(a.b)=\phi(a)*\phi(b)$

Moreover if ϕ is bijective then ϕ is called an isomorphism

Definition 1.1.4:

If ϕ is a homomorphism of G_1 into G_2 the Kernel of ϕ is given

by $K_\phi = \{ x \in G \mid \phi(x)=e$, where e is the Identity element of $G_2\}$

Definition 1.1. 5:

If N is a sub group of a group G & if $gNg^{-1}=N$ for all $g\in G$ then N is called **Normal sub group** of G or **self conjugate sub group**

Example 1.1.11:

Let $(R,+) = (G_1, \cdot)$ and $(R-\{0\}, X) = (G_2, X)$ be groups.

(i) A mapping $\phi: G_1 \rightarrow G_2$ defined by $\phi(x)=e^x$ is a homomorphism

(ii) A mapping $\psi: G_2 \rightarrow G_1$ defined by $\psi(x)=\log x$ is a homomorphism

1.2. Topology:

Definition 1.2.1:

A **topology** on a set X is a collection τ of sub sets of X having the following properties.

(a) ϕ and X are in τ

(b) Arbitrary union of the elements of τ is in τ

(c) The Intersection of the elements of any finite sub collection of τ is in τ

A set X for which a topology τ has been specified is called a **Topological space**.

Example 1.2.1:

Let X be a three element set X= {a,b,c} there are many possible topologies on X, some of which are indicated schematically in figure-1. The diagram in the upper right hand corner indicates the topology in which the open sets are X,Φ,{a,b},{b} and {b,c} the topology in the upper left hand corner contains only X and Φ, while the topology in the lower right hand corner contains every sub set of X you can get other topologies on X by permuting a, b and c

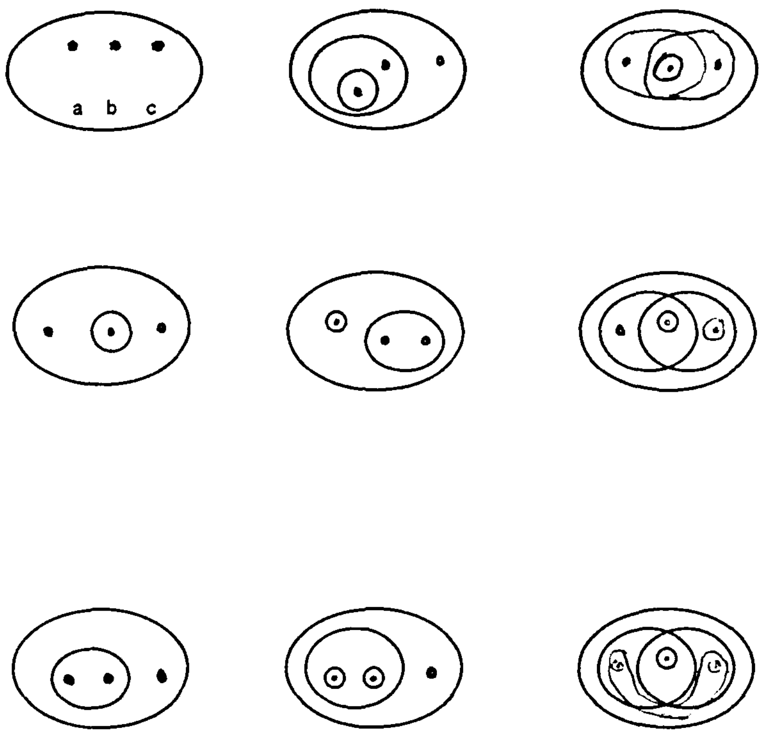

Figure -1

Example:1.2.2:

If X is any set, the collection of all sub sets of X is a topology on X, it is called the **discrete topology**. The collection consisting of X and Φ only is also a topology on X, we shall call it the **indiscrete topology** or **trivial topology**

Definition:1.2.2:

If X is a set, a **basis** of a topology on X is a collection β of sub sets of X such that

(a) for each $x \in X$, there is at least one basis element B containing x.

(b) If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$

Definition1.2.3:

Let X be a Topological space with topology τ if Y is a sub set of x, the collection $\tau_Y = \{ Y \cap U / U \in \tau \}$ is a topology on Y called **sub space topology**.

Definition1.2.4:

A topological space X is called a **Hausdorff** space if for each pair x,y of distinct points of X, there exists disjoint neighborhoods U and V of x & y respectively,

Definition 1.2.5:

Let X be a Topological space. A **Separation** of X is a pair U,V of disjoint non empty open sub sets of X whose union is X. The space X is said to be **connected** if there does not exist a separation of X.

Definition 1.2.6:

Given points x and y of the space X, a path in X from x to y is a continuous map $f : [a,b] \to X$ of some closed interval in the real line into X, such that f(a) = x and f(b) =y. A space X is said to be **path connected** if every pair of points of X can be joined by a path in X.

Difinition1.2.7:

A space X is said to be **locally connected at** x if for every neighborhood U of x, there is a connected neighborhood V of x contained in

U. If X is locally connected at each of its points, it is said simply to be **locally connected.**

Difinition1.2. 8:

A space X is said to be **locally path connected at x** if for every neighborhood U of x, there is a path-connected neighborhood V of x contained in U. If X is locally path connected at each of its points, then it is said to be **locally path connected.**

Example.1.2.3:

Each interval and each ray in the real line is both connected and locally connected.

Example.1.2.4:

R^n is locally path connected

1.3: DIFFERENTIABLE MANIFOLDS:

Definition:1.3.1:

Let M be a Hausdorff topological space. If each point p in M has a neighborhood U homeomorphic to an open set E in R^n , then M is called an n-dimensional topological manifold.

If U is an open set of M which is homeomorphic to an open set E of R^n i.e ψ:U $\rightarrow$E. then we call the pair (U, ψ) a co-ordinate neighborhood about p of M or chart about p of M. If p $\in$U, then ψ (p) is a point of R^n , so ψ (p) is an n-tuple of real numbers.

Let $x_i(p)$ be the i^{th} co-ordinate of ψ (p).

We have ψ (p)= (x_1(p),......,x_n(p)) Since ψ is continuous, each x^i is a continuous real valued function defined on the neighbourhood U of P and ψ is one-one. $X_i(p) = X_i(q)$ (i=1,2,3.............n) for p.q. $\in$U $\Rightarrow$ p=q, that is the point p of U is determined by the n-tuple of real numbers.

(x_1(p).......... x_n(p)) These are called the set of local co-ordinates of the point p of U with respect to the co-ordinate neighbourhood (U, ψ) and the n-tuple. (x_1, x_2............x_n) of functions of U is called co-ordinate system on (U, ψ). Since M is an n-dimensional topological manifold, there is a open covering $\{U_\alpha\}$ $\alpha \in A$ such that $M = \cup U_{\alpha\ \alpha \in A}$ where A is some index set. Let E_α be an open set of R^n homeomorphic to $U\alpha$, and let $\psi\alpha$ be a homeomorphism from U_α, onto E_α, then the collection $(U_\alpha, \psi_\alpha)_{\alpha \in A}$ is called a co-ordinate neighborhood system or an atlas.

Set $S = \{(U_\alpha, \psi_\alpha)\}_{\alpha \in A}$

Let (U_α, ψ_α) and $(U_\beta, \psi_\beta) \in S$ and $(U_\alpha \cap U_\beta) \neq \varphi$ then

$\psi_\alpha : U_\alpha \rightarrow E_\alpha$ and $\psi_\beta : U_\beta, \rightarrow E_\beta$,

are homeomorphisms respectively.

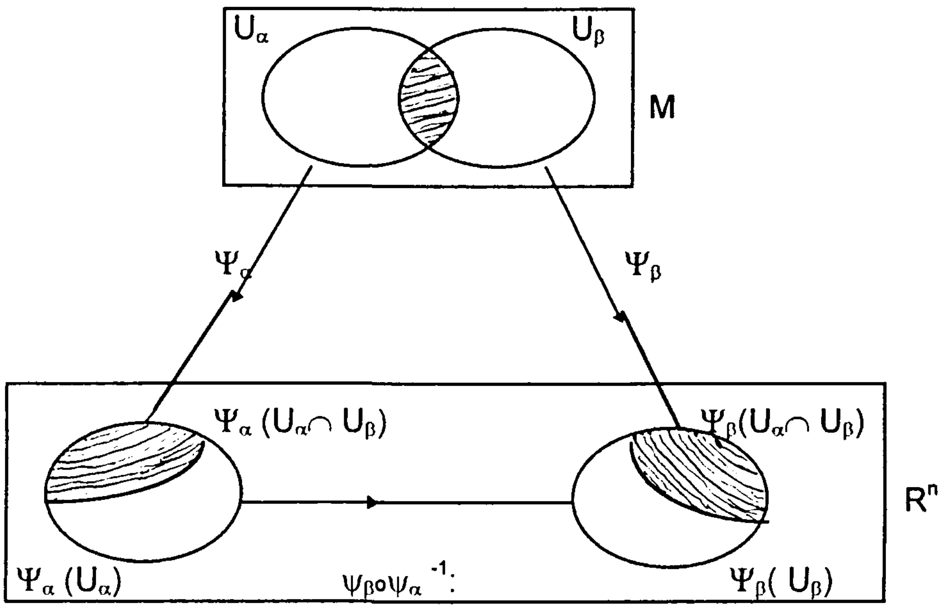

Figure -2

The transition maps (fig-2)

$\psi_\beta o \psi_\alpha^{-1}$: $\psi_\alpha(U_\alpha \cap U_\beta) \rightarrow \psi_\beta(U_\alpha \cap U_\beta)$ and

$\psi_\alpha o \psi_\beta^{-1}$: $\psi_\beta(U_\alpha \cap U_\beta) \rightarrow \psi_\alpha(U_\alpha \cap U_\beta)$ are also homeomorphic on the open sets $\psi_\alpha(U_\alpha \cap U_\beta)$ and $\psi_\beta(U_\alpha \cap U_\beta)$

An atlas S = $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in A}$ of an *n*-dimensional topological manifold M is called a co-ordinate neighborhood system of class C^r or an " atlas of class C^r", if for $\alpha, \beta \in A$, such that $U_\alpha \cap U_\beta \neq \Phi$, then the transition maps $\psi_\alpha O \psi_\beta^{-1}$ and $\psi_\beta O \psi_\alpha^{-1}$ on n-variables, determing the transformations between the local co-ordinate systems (U_α, ψ_α) and (U_β, ψ_β) are continuously r-times differentiable on $\psi_\beta(U_\alpha \cap U_\beta)$ and $\psi_\alpha(U_\alpha \cap U_\beta)$.

Definition : 1.3.2 :

If the transition maps $\psi_\alpha o \psi_\beta^{-1}$ and $\psi_\beta o \psi_\alpha^{-1}$ are all real analytic, then S is called a co-ordinate neighborhood system of class C^w [or atlas of class C^w i.e. analytic atlas]. If S is an atlas of class C^r (or C^w) on M, then we say that S defines a differentiable structure of class C^r (or C^w) on the topological manifold M.

Definition: 1.3.3

If an n-dimensional topological manifold M has a co-ordinate neighborhood system S of class C^r, then M is called an n-dimensional differentiable manifold of class C^r or a C^{r-} manifold.

If r= w, then M is called an "**analytic manifold**"

Example 1.3.1

R^n is an n-dimensional manifold, $\{(R^n,i)\}$ is an atlas on R^n

Example 1.3.2

$S^1 = \{(x,y) \in R^2 / x^2 + y^2 = 1\}$ is a 1-dimensional manifold.

Let $U_1 = S^1 - (0,1)$ $U_2 = S^1 - (0,-1)$

Define $\phi_1 : U_1 \to R$ by $\phi_1(x,y) = x / 1-y$

and $\phi_2 : U_2 \to R$ by $\phi_2(x,y) = x / 1+y$

(U_1, ϕ_1) and (U_2, ϕ_2) are charts and $\{(U_1,\phi_1)\ (U_2,\phi_2)\}$ is an atlas on S^1

Example 1.3.3 :

Let $S^2=\{(x,y,z) \in R^3 : x^2+y^2+z^2 =1\}$ is a two dimensional manifold

Let $U_1=S^2 -(0,0,1)$ $U_2=S^2-(0,0,-1)$

$\phi_1(x,y,z) = (x/1-z,\ y/1-z)$, $\phi_2(x,y,.z)= (x/1+z,\ y/1+z)$

Example :1.3.4

Let M= GL(n,R) the set of non-singular nxn matrices and is an open submanifold of $M_n(R)$ the set of all nxn real matrices identified with R^{n^2}.

Example :1.3.5

Let C-{o} = C * be non-zero complex numbers and is a one-dimensional manifold.

Definition: 1.3.4:

Let M and M^1 be differentiable manifolds of dimensions m and n respectively. If the derivative map of f: $M \to M^1$ i.e.

$f_{*p}: T_p(M) \to T_{f(p)}(M^1)$

is one-one for all $p \in M$, then f is said to be an '**Immersion**' of M into M^1. In addition if f is one-one then f is said to be an '**Imbedding**' of M into M^1

Definition : 1.3.5 :

Let M and M^1 be differentiable manifolds. M is said to be "submanifold of M^1, if the following two conditions are satisfied :

I) the set M is a subset of M'

ii) the inclusion map $i: M \to M^1$ is an imbedding of M into M^1

Example:1.3.5:

S^n is a submanifold of R^{n+1}

CHAPTER - 2

Basic concepts of Lie Groups

2.0:Introduction: In this chapter we give the definition of Lie groups & its sub-goup with some examples. Also action of a Lie group on a manifold is given and by using this quotient manifold is defined. Further Lie transformation groups and one parameter groups are defined. Also flow property of fluid is explained by the notion of one parameter group and some examples of flow are given. Further topological properties of Lie groups are stated.

2.1:An Introduction to Lie Groups:

2.1.1 Definition of Topological group:

If a non-empty set G has the following properties, G is called a topological Group:

(a)The set G is a group and a topological space simultaneously.

(b) the map (x,y) $\rightarrow$ xy from the direct product space G x G to G is continuous. Here xy denotes the product of two elements x and y of the group G.

(c) The map $x \to x^{-1}$ from G to G is continuous. Here x^{-1} is the inverse element of x in G.

Example 2.1.1

R^n is a topological group with respect to addition

Example 2.1.2

Let GL (n,R) be the set of nonsingular n x n matrices with real entries, GL (n,R) is a group with respect to multiplication of matrices. The n x n unit matrix is the identity element, and the inverse of A is the inverse matrix of A. On the other hand, we can indentify the set of all n x n real matrices with R^{n2} The determinant det a of a matrix a is a continuous function of $A \in R^{n2}$. We have GL (n,R) = $\{ A \in R^{n2} :\det A \neq 0\}$. Hence GL (n,R) is an open set of R^{n2}, and hence can be considered to be a topological space. The group GL (n,R) is a topological group with respect to this topology. Similarly, GL (n,C) is also a topological group.

2.1.2 Definition of Lie Group:

In definition 2.1.1, if we replace topology by a differentiable manifold and the maps (b) and (c) defined in definition 2.1. 1 are differentiable, then G is called a Lie group

Example 2.1.2

The space R^n is a C^∞ manifold and at the same time an abelian group with group operation given by componentwise addition, moreover the algebraic and differentiable structures are related by

$(x,y) \to x+y$ i.e $(R^n x R^n) \to R^n$

is differentiable and infact C^∞, Hence R^n is a Lie group. In particular R^1 is a Lie group

Example 2.1.3

Let C^* be the non -zero complex numbers It is a group with respect to multiplication and we know that C^* is a 1-dimension al manifold. the algebraic and differentiable structures are related by

$((x,y), (x^1,y^1)) \to (xx^1-yy^1 , xy^1+yx^1)$ i.e $C^* x C^* \to C^*$

and $z \to \bar{z}^-$ i.e $(x,y) \to \{ \frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2} \}$ i.e. $C^* \to C^*$

are C^∞, hence C^* is a Lie group.

Example:2.1.4:

The unit circle can be identified with the complex numbers z: $|z| = 1$ and can be regarded as a subgroup of C^* and so S^1 is a group. The algebraic and differentiable structures are related by

$s^1 \times s^1 \rightarrow s^1$ and $s^1 \rightarrow s^1$ as

$((\cos\theta, \sin\theta), (\cos\Phi, \sin\Phi)) \rightarrow ((\cos(\theta+\Phi), \sin(\theta+\Phi)), \theta + \Phi \leq 2\Pi$

and $(\cos\theta, \sin\theta), \rightarrow (\cos\theta, -\sin\theta)$ are C^{∞}

hence S^1 is a Lie group.

Example: 2 .1.5:

M= GL(n,R)- the set of all non-singular nxn matrices is a manifold of dimension n^2 and also a group with respect to matrix multiplication. The maps GL(n,R) X GL(n,R) $\rightarrow$ GL(n,R) and GL(n,R) $\rightarrow$ GL(n,R) defined by (A,B) $\rightarrow$ AB and A$\rightarrow$ A^{-1} are C^{∞}. The product has entries which are polynomials in the entries of A and B and these entries are exactly ' the expressions in local co-ordinates of the product which is thus C^{∞} The inverse of $A=(a_{ij})$ may be written as $A^{-1} = (a_{ij})/\det A$ where the (a_{ij}) are the cofactors of A and thus polynomials in the entries of A & where det A is a polynomial in these entries which does not vanish on GL (n,R). It follows that the entries of A^{-1} are rational functions on GL (n,R) with non-vanishing denominators and hence C^{∞}. Therefore GL (n,R) is a Lie group. For n=1, GL(1,R) = R* - the multiplicative group of non-zero reals is a Lie group.

2.1.3 Definition of Lie Subgroup:

A subgroup H which is also a submanifold of a Lie group G is called Lie subgroup of G when it is a Lie group with respect to C^{∞} structure of H as a submanifold of G.

Remark: To make H into a Lie group all what we require C^{∞} mappings

$H \times H \rightarrow H$ & $H \rightarrow H$ as defined in Lie groups

Example:2.1.6:

The Lie group on S^1 can be considered as a Lie subgroup of $C^*=C-\{0\}$ Lie group of non-zero complex numbers i.e. ($z \in S^1$ means $|z|=1$)

Example: 2.1.7:

$SL(n,R) = \{A \in GL(n,R) : \det A = 1\}$ is a Lie subgroup of GL(n,R)

Example:2.1.8:

Let O(n) denote the set of all non-singular orthogonal linear. transformations of R^n & clearly $O(n)= \{A\in GL(n,R): AA^t =1\}$ is a Lie subgroup of GL(n,R).

2.2: HOMOMORPHISM OF LIE GROUPS

Definition 2.2.1:

If G_1 and G_2 are two Lie groups then the direct product $G_1 \otimes G_2$ of these groups is also a Lie group.

Proof: $G_1 \times G_2$ is a group , $G_1 \times G_2$ is a C^∞ - manifold since G_1 and G_2 are Lie groups there are C^∞ maps ϕ_1 and ϕ_2 such that $\phi_1(a_1,b_1)=a_1b_1^{-1}$ and $\phi_2(a_2,b_2)=a_2b_2^{-1}$ in case of product $G_1 \times G_2 =G$ we should show that there exist a map $\phi : G \times G \to G$ defined by $\phi((a_1,a_2), (b_1,b_2)= ((a_1,a_2), (b_1,b_2)^{-1})$ $= ((a_1,a_2), (b_1^{-1},b_2^{-1})) =(a_1b_1^{-1} , a_2b_2^{-1})$ are C^∞

Example:2.2.1:

$S^1 \times S^1$ - is a toral grpup

Example 2.2.2:

S X R is a cylinderical group, where $S=\{x^2+y^2=a^2\}$ and R is not the extended real line.

Definition 2.2.2:

Let $\Phi : G_1 \rightarrow G_2$ be an algebraic homomorphism of Lie groups G_1 and G_2 We call Φ a homomorphism of Lie groups if Φ is also C^{∞}.

Example:2.2.3:

Let $G_1 = GL(n,R)$ and $G_2 = GL(1,R)=R^*$

Define $\Phi : G_1 \rightarrow G_2$ by $\Phi(A) = \det A$ and $\Phi(AB)= \det (AB)=\det A. \det B = \Phi(A)\ \Phi(B)$. Also Φ is a real valued function whose values are polynomials in entries of A and So it is C^{∞}. Hence Φ is a Lie homomorphism.

Karnel of $\Phi=\{A\in GL(n,R): \Phi (A)=1\}$

$=SL(n,R) \subset GL(n,R)$ but by the first fundamental theorem of isomorphism

$GL(n,R) /SL(n,R) \approx GL(1,R)$

Where GL(1,R) is a Lie Subgroup of GL(n,R)

Example: 2.2.4:

Set G_1 = R the additive group of reals & G_2 S^1

Define $\Phi : G_1 \rightarrow G_2$ by $\Phi(t) = e^{2\Pi it}$

$\Phi(t_1+t_2)= e^{2\Pi i\,(t1+t2)} = e^{2\Pi it1}. e^{2\Pi it2} = \Phi(t_1)\, \Phi(t_2)$

Also Φ is C^{∞} Hence Φ is a Lie homomorphism.

Kernel of $\Phi = \{t \in R: e^{2\Pi it} = 1\}$ which impLies Kernel of Φ=Z i.e. the set of all integers . But Z is not a Lie Subgroup of R

Example:2.2.5:

Let G_1 =R X R and $G_2 = S^1 \times S^1$

Define $\Phi : G_1 \rightarrow G_2$ by $\Phi\,(t_1, t_2) = (e^{2\Pi it1}, e^{2\Pi it2})$

Clearly Φ is a Lie homomorphism

Kernel of Φ= Z^2 = integral lattice of R^2

ACTION OF A LIE GROUP G ON A MANIFOLD M

Defnition:2.2.3:

Let G be a group and M a set then G is said to act on M (on the left) if there is a mapping $\theta : G \times M \to M$ satisfying

(a) if e is the identity element of G then $\theta(e,x) = x$ for all $x \in M$

(b) if $g_1, g_2 \in G$ then $\theta(g_1, \theta(g_2, x)) = \theta(g_1 g_2, x)$ for all $x \in M$

when G is a Lie group and M is a C^∞ manifold then θ is C^∞ and we say θ is C^∞ action

Notation: $\theta(g,x) = gx$. Thus (a) and (b) may be written as $ex = x$, $g_1(g_2x) = (g_1g_2)x$

Let $g \in G$ be fixed, Then difine $\theta_g : M \to M$ by $\theta_g(x) = \theta(g,x)$

Thus $\theta_{g1} o \theta_{g2} = \theta_{g1g2}$ and $\theta_{g^{-1}} = (\theta_g)^{-1}$

Theorem:

If G acts on a set M then the map $\theta \to \theta_g$ is a homomorphism of G into S (X), where S (X) denote the set of all bijective maps.

Example: 2.2.6

Let $G = \begin{vmatrix} a & b \\ 0 & 1 \end{vmatrix}$: $a > 0$ & $b \in R$ Shows that G is a Lie Group and acts on R by $\theta\left(\begin{vmatrix} a & b \\ 0 & 1 \end{vmatrix}, x\right) = ax + b$.

Example:2.2.7

Let $G = GL(n,R)$ and $M=R^n$, Define θ: $G \times M \rightarrow M$ by θ (A, x) =Ax Matrix multiplication of the n x n matrix A and (nx1) column vector. shows that θ is C^{∞} action.

Example: 2.2.8

Let H,G be Lie Groups and ψ: $H \rightarrow G$ be a homomorphism Let $\theta : H \times G \rightarrow G$ be defined by $\theta (h,x)= \psi(h) x$ & is a left action. If H and G are Lie Groups and ψ is a Lie homomorphism then, θ is C^{∞}. This may be applied to the case when H is a Lie subgroup of G & even when H=G.

Example : 2.2.9

The upper half plane $H=\{z=x+iy \in C: y >0\}$ is a homogeneous space of SL(2,R) an element $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ of SL (2,R) act's transitively on H by the action

$Z \rightarrow az+b/cz+d$, $\theta \left(\begin{vmatrix} a & b \\ c & d \end{vmatrix}, z\right) = az+b/cz+d$

Definition:2.2.4

Let a group G act on a set M and suppose that $A \subset M$.

$GA=\{ga: g \in G \text{ and } a \in A\}$. The orbit of $x \in M$ is the set Gx. If Gx=x then x is a fixed point of G and if Gx=M for some x then G is said to be transitive on M.

Example:2.2.10

Consider θ : GL(n,R) x $R^n \rightarrow R^n$ by $\theta(A,x)=Ax$. The origin is a fixed point of GL(n,R) and GL(n,R) is transitive on R^n {0}

Example:2.2.11

Let G=O(n)= The group of nxn orthogonal matrices & is a subgroup of GL(n,R), the orbits are concentric spheres with origin being a fixed point

Definition:2.2.5

Let G denote a Lie group & M a C^∞ manifold, Let θ: GXM $\rightarrow$M be a C^∞ action. We define a relation $\sim$ on M by $p \sim q$ if for some $g \in G$, $q=\theta_g(p)=gp$. This is an equivalence relation. The equivalence classes coincide with orbits and denote the set of all equivalence classes by M/G.

Definition:2.2.6

An Equivalence relation R on a manifold M is called regular if the quotient space M/R carries a manifold structure such that the cannonical projection $\Pi: M \to M/R$ is a submersion, If R is a regular equivalence relation, then M/R is called the Quotient manifold of M by R

Example:2.2.12

Let G be a Lie group and H a Lie subgroup of G Define a C^{∞} action $\theta: HxG \to G$ by $\theta(h,x)= \theta(h)x$, The set G/H of left cosets coincides with the orbits of this action and thus is a quotient monifold.

Example:2.2.13

Let G=O(n) and $M=R^n$, Then $R^n/O(n) \approx$ the ray $0 \leq r < \infty$

Example:2.2.14

Let $G= R^*$, and $M= R^2 -\{0\}$. Define $\theta : R^* X R^2 -\{0\} \to R^2 -\{0\}$. by $\theta(t,x)=tx$. Then $R^2 -\{0\}/ R^* \approx P^1(R)$.

Definition:2. 2.7

Let G be a Lie group and let M be a monifold. The group G is called a Lie transformation group of M if there is a differentiable map φ:G XM $\to$M defined by $\varphi(g,p)$= gp (i) ep=p for the identity e of G and p$\in$ M (ii) (gh) p= g(hp) for g, h$\in$G and p$\in$ M are satisfied and φ is C^{∞} if a Lie transformation group G acts transitively on M then M is called a homogeneous space of G. If N is the set of all element's of G suchthat gp=p for all point's p of M then N is a Normal Sub group of G.

If G is abelian then G is abelian Lie transformation Group.

Definition:2.2.8

If G=R in definition **2.2.7** then $\varphi(t,p)$= tp. Clearly φ (0,p)=0 and φ (t, φ(s, p))= φ(t+s,p) for t$\in$R, φ_t : M$\to$ M satisfies the following conditions.

(1) $\varphi_s \, o \varphi_t = \varphi_{s+t}$ (s,t$\in$R)

(2) (t,p)$\to$ φ_t (p) gives a differentiable map from R x M $\to$M. Then the family $\{\varphi_t : t\in R\}$ φ_t is called a one parameter group of tranformation's.

Definition: 2.2.9

Let φ: R x M$\rightarrow$M be a one parameter group of transformations and p$\in$M such that $\varphi_t(p) = \vartheta_p(t)$ then ϑ_p is a curve in M passing through p.

An integral curve of a vector field V is a smooth curve whose tangent vector at any point coincides with the value of V at the same point

Definition:2.2.10

If V is a Vector field, we denote the integral curve passing through x in M by φ **(t,x)** and call φ the flow genereted by V, Thus for each x$\in$M and I is some interval containing 0 (zero) φ (t,x) will be a point on the integral curve passing through x in M. The flow of a vector field has the basic properties.

φ (t, φ (s,x))= φ (s+t,x) ------------(1) for all t, s $\in$R

φ (0, x) = x------------(2)

From (1) and (2) we see that the flow genereted by a vector field is same as a group action of the Lie group R on the manifold M and often called as one parameter group of transformations.

Remark: Every Lie group satisfies the conditions of one parameter group, hence every one parameter group is a Lie group.

Thus one parameter group is related to Lie group, as seen above

Example:2.2.15

Let M= GL(2,R) and ψ: RXM$\rightarrow$M defined by

$$\psi\ (t,A)=\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix} A \text{ where } A=\begin{pmatrix}x_1 & x_2\\ x_3 & x_4\end{pmatrix}$$

$$\psi\ (t,A)=\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\begin{pmatrix}x_1 & x_2\\ x_3 & x_4\end{pmatrix} = \begin{pmatrix}x_1+t\,x_3 & x_2+t\,x_4\\ x_3 & x_4\end{pmatrix}$$

is a flow field

For example: (a) Flow of water in a canal or River

(b) Motion of Electron or flow of current in a circuit

Example:2.2.16

Let M=R^2, ψ: R x R^2 $\rightarrow$ R^2

by $\psi\ (t,(x_1,x_2))= (x_1,\ e^{at}, X_2\, e^{bt})$

Is an exponential function. This is a one parameter group & is flow in R^2

For example: (a) Exhaust smoke from the Chimminey

(b) Growth of population

Example:2.2.17

Let $M=R^2$, $\psi: R \times R^2 \rightarrow R^2$

ψ $(t,(x_1,x_2))=(-x_1,\ Sint + X_2\,Cost),\ x_1\ Cost + x_2\ Sint)$

is not a one parameter group and not a flow

Example:2.2.18

ψ $(t,(x_1,x_2))=(x_1,\ Cost - X_2\,Sint,\ x_1\ Sint + x_2\ Cost)$

ψ $(0,(x_1,x_2))=(x_1,\ x_2)$

ψ $(t,\ \psi(s,(x_1,x_2))=\psi(t+s(\ x_1,x_2))$

This is a Lie group of rotations in the plane we can quote the example of rotating of earth around the sun.

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2.3:TOPOLOGICAL PROPERTIES OF LIE GROUPS

Definition:2.3.1

Let G be a Lie group, If G is connected as a topological space then we call G a connected Lie group

Example:2.3.1

R is a Lie group, (**R**-Real Line) **R** is also connected, hence **R** is a **connected** Lie Group.

Definition:2.3.2

Let G be a Hausdorff Lie group.

(a) If G is locally compact as a topological space, then G is called a locally compact group.

(b) If G is compact as a topological space then G is called a compact group

Example:2.3.2

(i) R is locally compact

(ii)Consider the Set $S^1=\{(x,y) \in R^2 / x^2+y^2=1\}$ is compact, If we Induce a topology S^1 becomes a manifold. S^1 is a compact group.

Definition:2.3.3

Connected Lie sub group H of G is a sub set H of G and iscalled a connected Lie sub group of G if H is a connected Sub manifold of G and a sub group of G.

Example: 2.3.3

GL(n,R) is a connected Lie Sub group of GL(n,C).

CHAPTER -3

BASIC CONCEPTS OF LIE - ALGEBRAS

3.0:Introduction: In this chapter we study basic definition of Lie-algebra with some examples. Also examples of those relating to Lie-groups are explained. Left and right translations are defined and an inner automorphism is constructed as in group theory by using these translation. Further Lie-algebras of GL(n,R) and GL(n,C) are given .

3.1:AN INTRODUCTION TO LIE-ALGEBRAS

Definition: 3.1.1:

Let X be a vector space over a filed F. If for any two elements $a,b\in X$ there is an other element aob of X and if the conditions.

a) $\lambda(aob) = (\lambda a)ob = ao(\lambda b)$

b) $(a+b)oc = (aoc) + (boc)$

c) $ao(b+c) = (aob) + (aoc)$

$$(\lambda\in K,\ a,b,c\in X)$$

are satisfied, then X is called an algebra Moreover if X satisfies.

ao(boc)=(aob)oc

then X is called an associative algebra

Example:3.1.1

Let $C^{\infty}(M)$ = Denote the set of all real valued functions of class C^{∞} on the manifold.

Define the sum and the product of these functions belonging to $C^{\infty}(M)$ then it is an associative algebra.

Definition:3.1.2

A Lie algebra is a vector space L on which an operation called a Lie bracket (denoted as [,]) is defined, which associates with a pair u, v of elements of L, the element [u,v] of L such that, for arbitary vectors u,v, w in L and arbitary scalars (i.e. real or complex numbers depending on whether L is a real or complex vector space) a, b, we have

(a) $[au+bv,w] = a[u,w]+b[v,w]$

(b) $[u,v] = -[v,u]$

(c) $[[u,v],w]+[[v,w],u]+[[w,u],v]=0$

Example:3.1.2

Let X be an associative algebra over K for a,b, $\in X$ set$[a,b]=ab-ba$ then X becomes a Lie - algebra.

Example:3.1.3

The set $\mathcal{X}(M)$ of all C^{∞} vector fields on a manifold M is a Lie-algebra with repect to commutator product

Example:3.1.4

The set of all n x n real matrices constitutes a real Lie algebra of dimension n^2 with the Lie bracket given by

$[A,B]=AB-BA$

Definition:3.1.3

Let $\mathfrak{g}$ be a Lie-algebra over a field F. A sub set $\mathfrak{h}$ of $\mathfrak{g}$ is called a Lie-sub algebra of $\mathfrak{g}$ if it has the following two properties.

(a) The set h is a sub space of $\mathfrak{g}$
i.e. if $X,Y\in \mathfrak{h}$, and $\lambda,\mu\in F$, then$\lambda X + \mu Y\in \mathfrak{h}$
(b) If $X,Y \in \mathfrak{h}$, then $[X,Y] \in \mathfrak{h}$.

Example:3.1.5

The set of $n \times n$ real antisymmetric matrices constitute a Lie subalgebra [of dimension $n(n-1)/2$] of the real Lie algebra of Example 3.1.4 above.

3.2. : Lie -algebras over a Lie Groups :

Let **G** be a Lie -group, define a mapping $L_g : G \to G$ if $L_g(x) = xg$ then it is left translation if $L_g(x) = gx$ then it is right translation by x.

Let L_g and R_g denote the left and right translations, respectively by an element g of Lie group G.

$$L_g(x) = xg \text{ and } R_g(x) = gx$$

we have $L_g.L_h = L_{gh},\qquad R_g.R_h = R_{hg}$

$$L_{g^{-1}} = L_g^{-1},\ R_{g^{-1}} = R_g^{-1}$$

$$L_g.R_h = R_h.L_g$$

For $g \in G$, set $A_g = L_g.R_{g^{-1}}$

the transformation A_g is a diffeomorphism of G and by difinition we have

$$A_g(x) = gxg^{-1}$$

Hence for two element's x,y of G, we have $A_g(xy) = A_g(x).\ A_g(y)$. we call A_g the inner automorphism of G by the element g of G.

If a vector field X on a Lie group G satisfies $(L_g)_* X = X$ for all $g \in G$, then X is called a left invarient field. If instead X satisfies $(R_g)_* X = X$ for all $g \in G$, then X is called a right invariant vector field.

Let g be the set of all left invarient vector field's on G. If $X, Y \in g$ and $\lambda, \mu \in R$ then $\lambda X + \mu Y$, $[X,Y]$ also belong to g

$$(L_g)_* (\lambda X+\mu Y) = \lambda\ (L_g)_* X + \mu\ (L_g)_* Y = \lambda X + \mu Y$$

$$(L_g)_* [X,Y] = [\ (L_g)_* X,\ (L_g)_* Y] = [X,Y]$$

$\mathfrak{g}$ becomes a lie algebra of vector field's with respect to commutator product

$[X,Y]$

Definition. 3.2.1 :

The Lie algebra $\mathfrak{g}$ formed by the set of all left invariant vector field's on G is called the Lie algebra of the Lie group G.

Theorem :

If a Lie group G has dimension **n**, then the Lie algebra of G is also of dimension **n**

Definition 3.2.2 :

Let $\mathcal{g}_1$, $\mathcal{g}_2$ be Lie -algebra's over a field F, Define a linear map. $\propto : \mathcal{g}_1 \rightarrow \mathcal{g}_2$ by $\propto([X, Y]) = [\propto(X), \propto(Y)]$ for arbitrary X,Y of $\mathcal{g}$, then $\propto$ is called a homomorphism from $\mathcal{g}_1$ to $\mathcal{g}_2$. If $\propto(\mathcal{g}_1) = \mathcal{g}_2$, then $\propto$ is called a homomorphism from $\mathcal{g}_1$ onto $\mathcal{g}_2$. If the homomorphism $\propto$ is a one-one map, then $\propto$ is called an isomorphism from $\mathcal{g}_1$ into $\mathcal{g}_2$. If there is an isomorphism from $\mathcal{g}_1$ onto $\mathcal{g}_2$ then $\mathcal{g}_1$ and $\mathcal{g}_2$ are said to be isomorphic and we write $\mathcal{g}_1 \cong \mathcal{g}_2$.

An isomorphism from a Lie algebra $\mathcal{g}$ onto itself is called an automorphism of $\mathcal{g}$.

3.2.3: One parameter sub groups and the Exponentional map :

Let G be a Lie group and $a : t \to a(t)$ a differentiable curve of G defined on $(-\infty,\infty)$. If for any $s,t \in R$, we have

$a(s)\, a(t) = a(s+t)$........... (1)

then $\{a(t)/t \in R\}$ is called a one - parameter sub group of G. By (1) we have $a(0)\, a(t) = a(t)$ so that multiplying by the inverse element of a(t) on the right, we have

$a(0) = e$

also, since $a(t)\, a(-t) = a(-t)\, a(t) = a(t-t) = a(0) = e$, we have

$a(t)^{-1} = a(-t)$

further more, since $a(s)\, a(t) = a(s+t) = a(t+s) = a(t)a(s)$

a(s) and a(t) commute hence a one parameter sub group of G is a commutative sub group of G :$\{L_{a(t)} : t \in R\}$, $\{ R_{a(t)} : t \in R\}$ are both one-parameter groups of transformations of G, and the orbits of the identity element e by these transformation groups coincide with a(t). The following lemma holds.

Lemma 3.2.4 :

Let X be the infinitiesimal transformation of $R_{a(t)}$ and let Y be the infinitesimal transformation of $L_{a(t)}$, then X is left invariant and Y is right invariant and $Xe = Ye = a^1(0)$ holds. Here $a^1(t)$ denotes the tangent vector to the curve a at a(t)

Proof:

If f is a c^{∞} function on a neighborhood of a point h of G, then

$$(L_gX)_hf = X_{g\text{-}1h}\,(f0L_g)$$

On the other hand by the definition of X and by the commutativity of L_g and $R_{a(t)}$, we have

$$X_{g\text{-}1h}(f0L_g) = \lim_{t\to 0} 1/t\,[\,(f0L_g)\,(R_{a(t)\ g\text{-}1h}) - (f0L_g)\,(_{g\text{-}1h})]$$
$$= \lim_{t\to 0} 1/t\ \ [\,f(R_{a(t)h}) - f(h)] = X_hf$$

Hence $(L_gX)_h = X_h$ holds at each point h of G and X is left invariant, Similarly we can show that Y is right invariant, since $R_{a(t)}\,(e) = a(t)$ we have that a(t) is an integral curve of X, and hence $X_{a(t)} = a^1(t)$ holds in particular, we have $X_e=a^1(0)$ similarly we have $Y_e = a^1(0)$.

Lemma 3.2.5 :

Let $\{\varnothing_t : t \in R\}$ be a one-parameter group of transformations of G, and set $\varnothing_t\,(e) = a\,(t)$ if $\varnothing_t.\ L_g = L_g\,\varnothing_t$ holds for all $g \in G$ and for all $t \in R$, then a(t) is a one parameter sub group of G, and $\varnothing_t = R_{a(t)}$ holds for all $t \in R$. If $\varnothing_t R_g = R_g \varnothing_t$ holds for all $g \in G$ for all $t \in R$, then a(t) is a one - parameter sub group of G, and $\varnothing_t = L_{a(t)}$ holds for all $t \in R$.

Proof:

The map $t \rightarrow a(t)$ is differntiable and more over, $a(s+t) = \varnothing_{s+t}(e)$ and $(\varnothing_{s+t}(e)) = \varnothing_t(\varnothing_s(e)) = \varnothing_t(L_{a(s)}(e)) = L_{a(s)}(\varnothing_t(e)) = a(s)a(t)$.

Hence a(t) is a one-parameter sub group of G. On the other hand for any $g \in G$ we have $\varnothing_t(g) = \varnothing_t(L_g(e)) = L_g(\varnothing_t(e)) = g.\ a(t) = R_{a(t)}(g)$.

Hence $\varnothing_t = R_{a(t)}$

We can argue similarly for the case $\varnothing_t R_g = R_g \varnothing_t$

By lemma 3.2.2 and 3.2.3, we see that there is a one-one correspondence between the one-parameter sub groups of G and the left invariant vector fields on G as fallows.

If a(t) is one- parameter sub group of G, then there is an $X \in G$ Such that a(t) = (Exptx) (e) conversly, for an arbitary $X \in G$. (Exptx)(e) = a(t) a one - parameter sub group of G and Expt = $R_{a(t)}$.

Definition 3.2.6:

For $X \in \mathfrak{g}$. set

$\exp tX = (\operatorname{Exp} tX)(e)$, the map $X \rightarrow \operatorname{Exp} X$ is a map from $\mathfrak{g}$ to G, and is

called the exponential map by definition. exptX is a one parameter sub group

of G and we have

$\exp(t+s)X = \exp(tx), \exp(sx)$

$R_{\exp tX} = \operatorname{Exp} tx$

Then for $X,Y \in \mathfrak{g}$, we have

$[X,Y]_g = \lim_{t\to 0} 1/t \{Y_g - ((R_{\exp t}x) Y)_g\}$ .... (2)

Since for an arbitary element g of G, we have $Ag = R_{g^{-1}} L_g$ it fallows that for

$Y \in g$, we have

$A_g Y = R_{g^{-1}} Y = R_{g^{-1}} (L_h Y) = R_{g^{-1}} Y \in g$. The map $Y \rightarrow A_g Y$ is a linear

transformation of the vector space g and we denote this linear transformation

by Ad(g) - i.e.,

$Ad(g)Y = A_g Y = R_{g^{-1}}Y \; (g \in G, Y \in g)$

Since $A_{gh} = A_g . A_h$. we have

$Ad(gh) = Ad(g) Ad(h)$

for any two elements g, h of G, in paraticular it is clear from the definition of Ad (e) is the indentify transformation 1 of the vector space $\mathfrak{g}$ hence we have $Ad(g)^{-1}$. Ad(g)=1. Hence we have $Ad(g)^{-1}$. Ad(g)=1. Hance Ad(g) is a nonsingular linear transformation of g and

$$Ad(g^{-1}) = Ad(g)^{-1} \text{ holds}$$

The map g$\rightarrow$ Ad(g) is called the adjoint representation of the Lie group G.

Since $A_g [X,Y] = [A_gX, A_gY]$, we have

$$Ad(g) [X, Y] = [Ad(g) X, Ad(g) Y], (X, Y \in g)$$

i.e Ad(g) is an automorphism of the Lie - algebra g

if we let $A_x(t)$=Ad (exptx) $(X\in g)$.

Then we have $A_x (t+s) = A_x(t). A_x(s)$ that is $A_x(t)$ is a one-parameter group of linear transformations of the vector space g. If we set

$$C_x =[d/dt\, A_x(t)]$$

then we have $A_x (t) = \text{expt}\, C_x$

in fact from $A_x(ts) = A_x(t), A_x(s)$, we obtain

$$\frac{d}{dt} A_x (t) = C_x A_x (t)$$

$$A_x (0) = 1$$

This shows that $A_x(t)$ a solution to a system of differential equations and satisfies a given initial condition. How ever, clearly expt C_x satisfies the same system of differential equations and the same initial condition hence by uniquenes of solutions we conclude that $A_x(t) = \text{expt } C_x$ On the other hand

from (2) we get $[X,Y] = \lim_{t \to 0} 1/t \{Y - Ad(\exp(-tx), Y\}$

$$= -[\, d/dt\, Ax(-t)\,]_{t=0},\, Y = Cx.\, Y$$

Hence C_x is equal to the linear transformation ad(X) of g defined by $Y \to [X,Y]$ i.e. if we set

$$ad(X), Y = [X,Y] \quad (X,Y \in \mathfrak{g})$$

then Ax(t) = Expt ad (X). Hence

$$Ad(\text{Expt } X) = \text{Expt } ad(X) \text{ holds} \quad \text{-------------(3)}$$

and the map $X \to ad(X)$ is called the adjoint representation of the Lie algebra $\mathfrak{g}$. From the Jacobi identity for Lie algebras we have

$$ad(X)\,[Y,Z] = [\, ad(X)\, Y, Z] + [Y,\, ad(X)\, Z]$$

is the derivation of the Lie algebra $\mathfrak{g}$. Then the equation (3) gives the relation between the automorphism Ad(Expt X) $\mathfrak{g}$ the Lie algebra $\mathfrak{g}$ and the derivation ad(X) of $\mathfrak{g}$.

3.3.1 : Lie algebras of GL(n,R) and GL(n,C) :

Let A(t) be a one parameter sub group of GL(n,R)

we can write A(t) = exptC.

where C is determined uniquely by

C= [dA(t)/dt]t=0

Conversly. If C is an arbitary nxn real matrix, then the exponential function exptC is a one - parameter sub group of GL(n,R)

Now let $\mathfrak{g}$ be the Lie-algebra of GL(n,R) for X $\in \mathfrak{g}$, consider the one-parameter sub group exptX of GL(n,R), then there is an nxn matrix C(X) such that exptx = exptC(X)

applying $X_a f = \lim_{t \to 0} 1/t$ [f Expt X (a)) -f (a)]

and using Exptx = $R_{exptx} = R_{exptc(x)}$, we get for

the matrix $(X_a X^i_j) = \lim_{t \to 0} 1/t$ [aexptc (X-a)]

$= a.c\,(X)$

i.e. if we set

$C(x) = (c^i_j\,(x)),\quad a = (a^i_j)$

then we have

$$Xa\, X^i_j = \sum_{K=1}^{n} a^{I}{}_{k}\, C^k_j\,(X)$$

Hence the vector field X is expressed as

$$X = \sum_{i,j=1}^{n} \left(\sum_{K=1}^{n} (x^i_K\, C^k_j\,(X))\right) d/dx^i{}_j \quad \ldots\ldots\ldots\ldots\ldots(1)$$

with respect to the Co-ordinate system (X^i_j) again if we compute $[X,Y]\, X^i_j$, we

see that it is equal to

$$\sum_{k=1}^{n} X^i_k \left(\left(\sum_{t=1}^{n} C^k_t\,(X)\, C^t_j\,(Y) - C^k_t(Y)\, C^t_j\,(X)\right]\right.$$

hence we have

$$[X,Y] = \sum_{i,\,j=1}^{n}\left(\sum_{k=1}^{n} X^i_k \left(\sum_{t=1}^{n} C^k_t\,(X)\, C^t_j\,(Y) - C^k_t\,(Y)\, C^t_j\,(X)\right)\right) d/dx^i_j$$

and we obtain

$$C^i_j([X,Y]) = \sum_{i=1}^{n}(C^i_t(X)C^t_j(Y) - C^i_t\,(Y)\, C^t_j\,(X))\ldots\ldots\ldots\ldots.(2)$$

Now we define the commutator product [A,B] in the associative algebra g all

nxn real matrices to be

[A,B] = AB-BA

then we obtain a Lie algebra, which will be denoted by gl(n,R), the formula (2) then becomes

$C([X,Y]) = [C(X), C(Y)]$

from (1) it is clear that the correspondence $X \to c(X)$ is one-one and onto and that $C(\lambda x) = \lambda C(X)$ for $a \in R$, and $C(X+Y) = C(X)+C(Y)$ hence the map $X \to C(X)$ is an isomorphism from the Lie algebra g of GL(n,R) onto the Lie algebra gl(n,R), we shall identify of and gl(n,R) by this isomorphism from now on then the Exponential map.

$$\text{Exp} : gl(n,R) \to GL(n,R)$$

is nothing but the exponential function. Which assigns to each matrix X belonging to gl(n,R) the value expX, we have

$\text{expt}\,(Ad(a)X) = a\,(\text{exptx})a^{-1}$

$(X \in gl(n,R),\ a \in GL(n,R))$

Differentiating both sides with respect to t and setting t=0, we obtain

$$Ad(a)\,X = aXa^{-1} \quad \text{.................}(3)$$

i.e if we consider gl(n,R) to be the Lie algebra of GL(n,R) then the adjoint representation of GL(n,R) given by (3) similarly the set gl(n,C) of all nxn complex matrices is a Lie algebra with respect to the commutator product [A,B]= AB-BA as in the case of GL(n, R) we can prove that gl(n,C) is isomorphic to the Lie -algebra of GL(n, C).

3.4: Some results concerning Lie groups and Lie algebras

Definition:3.4.1

If for any two elements X, Y in a Lie algebra $\mathfrak{g}$,[X,Y]=0 then $\mathfrak{g}$ is called a **commutative Lie algebra** or an **abelian Lie algebra**.

Theorem:3.4.1

If a Lie group G has dimension n, then the Lie algebra of G also has dimension n.

Theorem:3.4.2

Let G be a connected Lie group, and $\mathfrak{g}$ its Lie algebra. Then G is commutative if an only if $\mathfrak{g}$ is commutative.

PROPOSITION

A compact connected complex Lie gorup G is commutative.

Theorem:3.4.3

Let G be a Lie group, and H a sub group of G if H and G/H are both connected, then G is connected.

Theorem:3.4.4

If the Lie group G is a complex Lie group then the Lie algebra of a real Lie group G is a complex Lie algebra, then G has the structure of a complex Lie group.

Theorem:3.4.5

Let G be a Lie group, $\mathfrak{g}$ the Lie-algebra of G . If H is a Lie sub group of G, then the Lie-algebra $\mathfrak{h}$ of H can be regarded as a Lie sub algebra of $\mathfrak{g}$ conversly if h is a Lie sub algebra of $\mathfrak{h}$, then there is a unique connected Lie sub group of G whose Lie algebra is $\mathfrak{h}$.

Theorem:3.4.6

Let $\mathfrak{g}$ be the Lie algebra of a complex Lie group G. If H is a Lie sub group of G, then H is a complex Lie sub group of G if and only if the Lie algebra of H is a complex Lie sub algebra of $\mathfrak{g}$.

Remark:

If a subset H of G satisfies the two conditions

(a) H is a closed sub set of G and

(b) H is a sub group of G. then it will be shown that H is a closed Lie group of G.

Theorem:3.4.7

If H is a connected normal Lie sub group of a Lie group G, then the Lie algebra $\mathfrak{h}$ of H is an ideal of the Lie algebra of G. conversly, If G is a connected Lie group with Lie algebra $\mathfrak{g}$, and if $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. Then the connected Lie subgroup H of G correspoding to $\mathfrak{h}$ is a normal sub group of G.

Theorem :3.4.8

Let G be a Lie group with Lie algebra $\mathfrak{g}$. If $H \subset G$ is a Lie subgroup, its Lie algebra is a subalgebra of $\mathfrak{g}$. Conversely, if $\mathfrak{h}$ is any s-dimensional subalgebra of $\mathfrak{g}$, there is a unique connected s-parameter Lie subgroup H of G with Lie algebra $\mathfrak{h}$.

Theorem:3.4.9

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then $\mathfrak{g}$ is isomorphic to a subalgebra of GL(n) for some n.

Theorem:3.4.10

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then there exists a inique connected, simply-connected Lie group G* having $\mathfrak{g}$ as its Lie algebra. Moreover, if G is any other connected Lie group with Lie algebra $\mathfrak{g}$, then$\prod: G^* \to G$ is the simply-connected covering group of G.

Theorem:3.4.11

Let H be a Lie subgroup of a Lie group G. If the topology of H is the induced topology then H is closed.

Theorem:3.4.12

The automorphism group of a finite-dimensional algebra a is a Lie group, and its Lie algebra is the Lie algebra formed by all the derivations of a

CHAPTER-4

APPLICATIONS OF LIE-GROUPS

4.0:Introduction: In this chapter we give application of Lie-groups. It is explained how relativity group can be constructed by Galilean transformations. This group is nothing but Lie-group and hence it can be seen that how Lie-groups are used in relativity theory, similarly Lorentz group which is also a Lie-group is used in quantam theory.

Reference frames and relativity groups:

In physics we are interested in the description of dynamics of various systems. This involves change of various quantities with time. A proper mathematical definition of these quantities which could be translated into observations requires the introduction of a reference frame. All observations reduce in the ultimate analysis to measurements of positions of objects which can only be done relative to some reference objects and times of events (which, again, can only be relative to some reference event). By a reference frame, we mean a coordinate frame (generally chosen as fixed to some object: Earth, Sun, laboratory etc.) relative to which the position coordinates may be assigned to point objects, and a clock to assign time coordinates to events. The position and time-coordinate assignments in two different reference frames must be related by an invertible transformation so that descriptions in various frames can be unambiguously translated into each other. We shall give examples of such transformations below.

Every physical theory takes (explicitly or implicitly) some distinguished class of reference frames as mutually equivalent for the description of physical phenomena (covered by the theory): in mathematical terms, this means that basic equations of the theory take the same form in all reference frames of the distinguished class. This choice of distinguished frames is made by philosophical considerations and reflects the stand taken by the theory regarding the nature of space and time. It is easily seen that the transformations relating the position-and time-coordinate assignments in pairs of distinguished frames form a group. (If a transformation T_1 takes the description of the frame S to S^1 and T_2 from S^1 to S^{11}, then T_2T_1 takes S to S^{11}, T_1^{-1} takes S^1 to S, the identity transformation connects S to S itself, etc.) This group is called the relativity group.

In Newtonian mechanics, the distinguished frames are the so-called inertial frames (or Galilean frames) which, by definition, are those in which Newton's first law (the law of inertia) is valid. Recall that, according to

Newton's first law, a free particle (i.e. one not acted upon by anything external) either remains at rest or moves with a constant speed in a straight line. This law obviously cannot be valid in all reference frames (if it is valid in some frame S, then it cannot be valid in a frame S^1 moving relative to S with nonzero acceleration); it, therefore, implicitly serves to define the class of reference frames which are to be adopted for the standard formulation of Newtonian mechanics.

Let S and S^1 be two inertial frames having their axes parallel, their origins coinciding at a time taken to be zero in both frames and S^1 moving relative to S in the positive x-direction with a speed V. Let the space-time coordinates in the two frames be (x,y,z,t) and (x^1,y^1,z^1,t^1). [The meaning of these coordinates is this: if an event (something hapening at some point of space at some point of time- it is supposed to be taken as a frame-independent entity)is assigned position coordinates (x,y,z) and time coordinate t in S, then its corresponding coordinates in S^1 are (x^1,y^1,z^1,t^1).] They are related by special Galilean transformations.

$x^1=x-vt,\ y^1=y,\ z^1=z,\ t^1=t$......................................(1)

These transformations are easily seen to form a group which, calling the group element corresponding to the transformation (1) g (v), is given by

$g(v^1)g(v)=g(v+v^1),\ g(v)^{-1}=g(-v),\ g(0)=e.$ ....(2)

If we allow for arbitrary relative orientation of axes, initial positions of origins and relative shift in the zero of time, the transformations (1) are generalized to the following transformation (employing matrix notation):

$$x^1=Ax+vt+a,\ t^1=t+b \quad \text{...} \quad (3)$$

where A is a 3 x 3 real orthogonal matrix. We have changed v of Eq. (1) to - v to avoid an unpleasant minus sign. These transformations constitute a ten-parameter Lie group called the Galilean group. Denoting the group element corresponding to the transformation (3) by (A,v,a,b), we have

$$(A^1,v^1,a^1,b^1)\,(A,v,a,b) = (A^1A,\ A^1v+v^1,\ A^1a+bv^1+a^1,\ b+b^1) \text{.................} \quad (4)$$

$$e=(I,o,o,o),\ (A,v,a,b)^{-1} = (A^{-1},\ -A^{-1}v,\ -A^{-1}a+bA^{-1}v,-b).$$

If both the frames are assumed to have, say, right-handed systems of axes, then we must have det A = 1. Such transformations constitute a subgroup of the Galilean group; it may be called the proper Galilean group.

In practical applications, it is often convenient to restrict the choice of frames to those fixed relative to some object (say, the centre of mass of a system of particles) and the zero of time by the initial conditions of a problem (or some arbitrary convention). Tlhis restricts the group of equation (4) to elements of the form (A,o,a,o) which are easily seen to form a group isomorphic to the Euclidean group E_3.

In special relativity, Newton's first law is taken to be valid and, therefore, the distinguished frames are again the inertial frames. However, now the absoluteness of time represented by Eq. (1) is given up and the transformations (1) are replaced by the special Lorentz transformations.

$x^1=\gamma(x-vt)$, $y^1=y$, $z^1=z$, $t^1=\gamma(t-vx/c^2)$ .... (5)

where $\gamma=(1-v^2/c^2)^{-1/2}$ and c is the velocity of light in vacuum.

These transformations form a one-parameter Lie group given by [compare Eq.(2)]

$$g(v^1)g(v)=g(v^{11}), \text{ with } v^{11}=\frac{v+v^1}{1+\frac{vv1}{c^2}} \qquad \text{............................(6)}$$

$e=g(0)$, $g(v)^{-1}=g(-v)$.

The transformations replacing the general Galilean transformations (4) in special relativity are the inhomogeneous Lorentz transformation written in the matrix notation is given by $x^1=Ax+A$

where A is a 4 x 4 real matrix belonging to the group o(3,1) [the (3+1)-dimensional Lorentz group]:

$$\Lambda^T\eta\Lambda=n, \text{ where } \eta= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad \ldots\ldots\ldots\ldots(7)$$

The transformations (6) constitute a ten-parameter Lie group called the inhomogeneous Lorentz group (or the Poincare group) with composition rule etc., given by

$$(\Lambda^1,a^1)\,(\Lambda,a) = (\Lambda^1\Lambda,\Lambda^1a+a^1) \qquad \ldots\ldots\ldots\ldots(8)$$

$$e=(I,o),\ (\Lambda,a).^{-1} = (\Lambda^{-1},\ -\Lambda^{-1}a).$$

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